A SEMI-STRONG PERFECT GRAPH THEOREM

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Abstract

Perfect Graphs were defined by Claude Berge in 1961. Since that time this class of graphs has been intensely studied. Much of the work has been directed towards proving Berge's Strong and Weak Perfect Graph Conjectures. L. Lovasz finally proved the Weak Perfect Graph Conjecture in 1972. The Strong Perfect Graph Conjecture has not been resolved.

Václav Chvátal, in 1982, proposed the Semi-Strong Perfect Graph Conjecture, which falls between these two conjectures. This conjecture suggests that the perfection of a graph depends solely on the way that chordless paths with three edges are distributed within the graph. The main result in this thesis is a proof of Chvátal's conjecture.
Résumé


Václav Chvátal, en 1981, a proposé la Conjecture Demi-Forte des Graphes Parfaits qui s'intercale entre ces deux conjectures. Cette conjecture suggère que la perfection d'un graphe dépend seulement de la manière dont les chemins induits de longueur trois sont distribués dans le graphe. Le résultat principal dans cette thèse est une preuve de la Conjecture Demi-Forte.
To Kirsten
Acknowledgements

My thesis was written partially in Paris, partially in Waterloo, and partially in Montreal.

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Table of Contents

Abstract ......................................................................................................................

Resume .....................................................................................................................

Acknowledgements

1 Introduction 1

2 Building Blocks and Machinery 11

3 Alpha and Omega 61

4 The SPGC for Special Classes of Graphs 76

5 The Semi-Strong Perfect Graph Conjecture 91

6 The Semi-Strong Perfect Graph Theorem 101

Appendices 125

Glossary 141

References 149
CHAPTER 1

INTRODUCTION
A Cycle

A Hole

A Path: $P_6$

Figure 1.1
1.0 Non-Standard Notation

This section defines terms used in this thesis which are not in standard usage and are not defined the first time they appear. All graph theory terms used in the thesis are defined in the glossary. For an introduction to graph theory see [Bo76] or [Be73]. The reader is assumed to have some familiarity with computational complexity theory and algorithm design. For an introduction to complexity theory see [GarJ79]. For an introduction to algorithms on graphs see [Go80].

If \( H \) is a graph we use \( H \) and \( V(H) \) interchangeably. By a subgraph we mean an induced subgraph. If the subgraph is not induced we use partial subgraph. For vertices \( x \) and \( y \) in a graph \( G \), we say \( x \) sees \( y \) if \( xy \) is an edge of \( G \). If \( xy \) is not an edge of \( G \) then \( x \) misses \( y \). We say \( x \) disagrees on two vertices of \( G - x \), if \( x \) sees one of these vertices and misses the other. If \( x \) disagrees on \( a \) and \( b \) then \( a \) and \( b \) disagree on \( x \).

A cycle in a graph is a set of distinct vertices \( \{v_0, v_1, \ldots, v_k\} \) such that \( v_i \) sees \( v_{i+1} \) when \( 0 \leq i \leq k-1 \) and \( v_0 \) sees \( v_k \). A hole is an induced cycle. We denote the hole on \( k \) vertices, \( C_k \). A graph is a disc if it is either a hole of length at least 5 or the complement of such a hole.

The parity and length of a path refer to the number of edges, not the number of vertices. However, we denote the path on \( k \) vertices as \( P_k \) (not \( P_{k-1} \) as might be expected).
1.1 A Perspective on Perfect Graphs

Graph Colouring

Optimization version: Given a graph $G$, determine the minimum number of colours necessary to colour the vertices of $G$ so that no two adjacent vertices receive the same colour (this number is known as the chromatic number of $G$ and denoted $\chi(G)$).

Decision version: Given a graph $G$ and an integer $k$ decide whether or not $\chi(G) \leq k$.

Clique Number

Optimization version: Given a graph $G$, determine the size of the largest clique in $G$ (this is known as the clique number of $G$ and denoted $\omega(G)$).

Decision version: Given a graph $G$ and an integer $k$ decide whether or not $\omega(G) \geq k$.

The two above problems have been extensively studied. They are known to be difficult. In fact, the decision versions of both problems are NP-complete [Ka72].

The two problems are also closely related. Note first that $\chi(G) \geq \omega(G)$. Thus the clique number is a lower bound on the chromatic number and the chromatic number is an upper bound on the clique number. In order to explore this relationship further, we first formulate the two problems as integer linear programs.

Definition: Let $G$ be a graph on $n$ vertices, $\{v_1, v_2, \ldots, v_n\}$. The stable set matrix of $G$, $A=A(G)$, has $n$ columns each of which corresponds to a vertex of $G$. The rows of $A$ correspond to the stable sets of $G$, we will enumerate these as $S_1, \ldots, S_p$. The row corresponding to a stable set $S$ will have a 1 in column $i$ if $v_i$ is in $S$ and a 0 if $v_i$ is not in $S$.

The vertices of any given colour in a colouring form a stable set. Let us associate a variable $y_i$ with each stable set $S_i$. Then for each colouring we can set $y_i$ to 0 or 1 depending on whether or not $S_i$ is the stable set corresponding to one of the colours in our colouring.

Thus graph colouring can be formulated as follows.
(1) Graph Colouring ILP:
\[
\text{minimize } \sum_{i=1}^{n} y_i \text{ subject to }
\]
\[
yA \geq 1
\]
\[
y \geq 0
\]
for all i, y_i is 0 or 1.

A clique is a set of vertices which intersects each stable set in at most one point. Associating the variable z_i with vertex v_i we obtain the following formulation of the clique number problem:

(2) Clique Number ILP:
\[
\text{maximize } \sum_{i=1}^{n} x_i \text{ subject to }
\]
\[
Ax \leq 1
\]
\[
z \geq 0
\]
for all i, z_i is 0 or 1.

The linear programming relaxations of these two problems are:

(3) Fractional Graph Colouring LP:
\[
\text{minimize } \sum_{i=1}^{n} y_i \text{ subject to }
\]
\[
yA \geq 1
\]
\[
y \geq 0
\]

(4) Fractional Clique Number LP:
\[
\text{maximize } \sum_{i=1}^{n} x_i \text{ subject to }
\]
\[
Ax \leq 1
\]
\[
z \geq 0
\]
For a graph $G$, the optimal solution to (3) is known as the fractional colouring number, $\chi'(G)$. Similarly, the optimal value of a solution to (4) is known as the fractional clique number, $\omega'(G)$. Note that (3) and (4) are duals. Thus $\omega(G) \leq \omega'(G) = \chi'(G) \leq \chi(G)$.

It seems possible that a study of those graphs for which $\chi(G) = \omega(G)$ may prove fruitful. In particular, we might be able to solve the graph colouring and clique number problems in polynomial time on these graphs. These graphs are called *good*. Note that a graph is good if and only if (4) has an optimal solution in which each $x_i$ is either 0 or 1 (these are called 0-1 valued solutions). Unfortunately, good graphs seem to be an unnatural class. For example, determining if a graph is good is NP-complete. Further, given a graph we can transform it into a good graph simply by adjoining a sufficiently large clique.

Following Claude Berge[Be61], we call a graph *perfect* if each of its induced subgraphs is good. Chvátal[Ch75] and Fulkerson[Fu71] developed, independently, a polyhedral characterization of perfect graphs. Recall that a graph $G$ is good if the LP

\[
\begin{align*}
\text{(5)} & \quad \text{maximize } \sum_{i=1}^n c_i x_i \text{ subject to } \\
& \quad Ax \leq 1 \\
& \quad x \geq 0
\end{align*}
\]

has a 0-1 valued optimal solution when $c = 1$. Thus, a graph is perfect if (5) has a 0-1 valued optimal solution for any 0-1 valued objective function $c$. The result obtained by Chvátal and Fulkerson is that a graph is perfect if and only if (5) has a 0-1 valued optimal solution for any choice of the objective function. (Actually, Fulkerson, in 1971, showed that this result was implied by an unsettled conjecture. This conjecture was proved in 1972 by Lovász. We shall say a bit more about this in the next section). Now, let $P(G)$ be the polytope defined by $Ax \leq 1$ and $x \geq 0$. Clearly, the result of Chvátal and Fulkerson is equivalent to:

\[
\text{(6) } G \text{ is perfect if and only if } P(G) \text{ has only 0-1 valued extreme points.}
\]
This beautiful characterization suggests that perfect graphs are indeed a natural class. Furthermore, it suggests that our two optimisation problems may possibly be easily solvable for perfect graphs. In fact, this turns out to be the case. In [GrLS84], Grotschel,Lovász and Schrijver describe a polynomial time algorithm to solve these two problems for perfect graphs. Their algorithm uses the ellipsoid method of Khachian[Kh79].
1.2 Another Perspective on Perfect Graphs

In 1960, Claude Berge[Be61] defined the class of perfect graphs. Berge also made the following two conjectures:

(1) A graph is perfect if and only if it contains no odd disk.

(2) A graph is perfect if and only if its complement is perfect.

Trivially (1) implies (2). For this reason, (1) and (2) were known as the Strong and Weak Perfect Graph Conjectures (SPGC and WPGC) respectively. The WPGC was proved by Lovász[Lo72a] and is now known as the Perfect Graph Theorem (PGT). The SPGC is still unresolved. These two conjectures have been the focus of most of the work on perfect graphs. For instance, Fulkerson showed that the polyhedral characterization discussed in the last section was implied by the WPGC while attempting to prove that conjecture.

A graph is Berge if it contains no odd disk. It is easy to see that no odd disc permits a good colouring (for $k \geq 2, \omega(C_{2k+1}) = 2, \chi(C_{2k+1}) = 3, \omega(C_{2k+1}) = k, \chi(C_{2k+1}) = k + 1$). It follows that every perfect graph is Berge. Thus, the difficult part of the SPGC is showing that all Berge graphs are perfect.

One method of attacking the SPGC is to study the properties of minimal imperfect graphs. If a sufficiently extensive list of these properties were built up, it might be possible to show that only the odd disks satisfied them all. If $G$ is good then, since each colour class in a good colouring of $G$ is a stable set, $|G| \leq \alpha(G)\omega(G)$. Thus, if $G$ is perfect then, for every subgraph $H$ of $G$, $|H| \leq \alpha(H)\omega(H)$. In fact, Lovász[Lo72b] has shown that a graph $G$ is perfect if and only for each subgraph $H$ of $G$:

$$|H| \leq \alpha(H)\omega(H).$$

Since $\alpha(H) = \omega(H)$, this result implies the perfect graph theorem. Lovász's work prompted a whole slew of results concerning the properties of maximum cliques and stable sets in minimal imperfect graphs. For example, Padberg[Pad74] showed that any minimal imperfect graph contains precisely $a$ cliques of size $\omega$ and $a$ stable sets of size $\alpha$. We discuss results of this type in chapter 3.
The study of minimal imperfect graphs has not been limited to their cliques and stable sets. For example, Chvátal[Ch85a] demonstrated that a minimal imperfect graph cannot contain a star cutset. A star cutset is a set of vertices $C$ such that $G - C$ is disconnected ($C$ is a cutset) and some vertex $x$ of $C$ is adjacent to all the other vertices of $C$. As a final example of this approach we mention the following result of Meyniel[Me86a]. A pair of vertices, $x,y$, is said to be even if all the induced paths from $x$ to $y$ have an even number of edges. Meyniel showed that a minimal imperfect graph cannot contain an even pair of vertices.

Another method of attacking the Strong Perfect Graph Conjecture is to extend the set of graphs known to be perfect. Classes of graphs which have been shown to be perfect include: comparability graphs[Ga67,Gh62,GiH70,Du84], triangulated graphs[HaJ58,Du61,Du84], parity graphs[BuU84,Sa70], 1-triangulated graphs[Ga62], weakly triangulated graphs[Hay86], Meynien graphs[Me76,BuF84], $P_4$-free graphs[Se74,CoPS85], perfectly orderable graphs[Ch84a], alternation graphs[Ho85b], line graphs of bipartite graphs[Kô35], and quasi-parity graphs[Me86a]. Often a family of perfect graphs is closely linked with a property of minimal imperfect graphs. For example, consider triangulated graphs and clique cutsets. Triangulated graphs are those graphs which contain no holes of length greater than three. A clique cutset is simply a cutset $C$, such that the vertices of $C$ induce a clique. Dirac[D161] showed that every triangulated graph has a clique cutset. It is well known and easy to see that a minimal imperfect graph cannot contain a clique cutset. Since triangulated graphs are a hereditary class it follows that triangulated graphs are perfect. We shall see more examples of this type of relationship in chapter 2.

A third way of attacking the SPGC is to prove the conjecture on restricted classes of graphs. For example, Tucker[Tuc73] has shown that a planar graph is perfect if and only if it contains no odd disk. We shall discuss results of this type in Chapter 4.
A few years ago, Chvátal suggested another way of looking at perfect graphs. This new perspective was motivated by the Perfect Graph Theorem. Since the Perfect Graph Theorem states that perfect graphs are a self-complementary class, it seems natural to study properties of perfect graphs which hold on a graph if and only if they hold on its complement. This consideration motivated the following definition and conjecture.

Graphs $G$ and $H$ with the same set of vertices are called $P_4$-isomorphic if the following condition holds:

A set of four vertices induces a chordless path (called a $P_4$) in $G$ if and only if it induces a $P_4$ in $H$.

This notion was first introduced by Vasek Chvátal in 1982 (see [Ch84b]). At the same time, he made the following conjecture:

\[(1.5) \text{ If a graph } H \text{ is } P_4\text{-isomorphic to a perfect graph then } H \text{ is perfect.}\]

Since $P_4$ is a self-complementary graph, (1.5) implies (2). The less straightforward implication that (1) implies (1.5) has been demonstrated by Chvátal. Thus (1.5) falls between the Strong and Weak Perfect Graph Conjectures and, for this reason, is known as the Semi-Strong Perfect Graph Conjecture (SSPGC). The main result in my thesis is a proof of this conjecture. This proof is presented in chapter 6. Chapter 5 contains some earlier results related to the conjecture.
CHAPTER 2

BUILDING BLOCKS AND MACHINERY
2.0 Constructing Classes of Perfect Graphs.

In this chapter, we discuss a number of classes of perfect graphs and a variety of properties of minimal imperfect graphs. The classes discussed have more in common than their perfection. In particular, the same paradigm can be used to show that each of the classes is a class of perfect graphs. Before describing the general paradigm, we shall consider some example instances. In Section 2.8 we give a formal specification of the paradigm.
2.1 Bipartite Graphs and Even Vertex Addition

Berge graphs are those which contain no odd disc of order five or greater. Clearly the complement of a hole of length at least six contains a triangle. Thus a natural restriction of the class of Berge graphs is the class of graphs which contain no odd holes whatsoever. These are the \textit{bipartite graphs}.

\textit{Theorem 2.1([K555])}: A graph is bipartite if and only if it can be partitioned into two \textit{stable sets}.

Proof: No odd hole can be two-coloured, so if a graph can be partitioned into two stable sets, it must be bipartite.

It remains to show that any bipartite graph can be partitioned into two stable sets. Clearly a graph can be partitioned if each of its connected components can be. Thus, we need only consider connected graphs. Let \( G \) be a connected bipartite graph. Since \( G \) contains no odd hole, it contains no odd cycle. Now, let \( z \) be a vertex of \( G \). If \( y \) is a vertex of \( G-z \) then every path from \( z \) to \( y \) has the same parity; Otherwise, let \( P_1 \) and \( P_2 \) be paths of different parity from \( z \) to \( y \), clearly \( P_1 + P_2 \) contains an odd hole, a contradiction. So, we can partition \( G \) into \( A = \{ y \mid \text{the paths from } z \text{ to } y \text{ are even} \} \) and \( B = \{ y \mid \text{the paths from } z \text{ to } y \text{ are odd} \} \) (note that \( z \) is in \( A \)). If \( w \) and \( u \) are in \( A \) and \( uv \) is an edge of \( G \) then there is an odd path from \( z \) to at least one of \( u \) or \( v \), a contradiction. Thus, \( A \) is a stable set. Similarly, \( B \) is a stable set. So, we have partitioned \( G \) as required. \( \Box \)

\textit{Corollary 2.2}: Bipartite graphs are perfect.

Proof:

Observation: Bipartite graphs are good.

Proof: Clearly if a bipartite graph has no edge it is one colourable. Otherwise, \( \omega(G) = 2 \). Since \( G \) is bipartite, it can be partitioned into two disjoint stable sets. This partitioning provides a two-colouring of \( G \). \( \Box \)
Since a minimal imperfect graph cannot be good, there are no minimal imperfect bipartite graphs. Since the class of bipartite graphs is hereditary, it follows that all bipartite graphs are perfect.

In Appendix A, we present an algorithm which finds a spanning tree for a connected graph in $O(n + m)$ time. This algorithm grows the tree by adding an edge at a time. We can partition the graph as in 2.1 by labelling each vertex with the parity of its path to the root (in the tree). If we grow the tree by adding edge $uv$ and vertex $v$ then $v$ will have opposite parity from $u$. Now, to check if the graph is bipartite, we need only ensure that there are no edges between vertices with the same label. This can be done by checking each edge in turn. Thus, we can determine if a graph is bipartite in $O(n + m)$ time.

In the proof of corollary 2.2, we used the fact that bipartite graphs are good to show that there are no minimal imperfect bipartite graphs. We shall now discuss another property of bipartite graphs which cannot hold on a minimal imperfect graph. We call a vertex $x$, in a graph $G$, *even* if $x$ is in no odd hole in $G$. *Even vertex addition* is the process of adding a vertex $x$ to a graph $G$ in such a way that $x$ is even in $G + x$. Trivially, every vertex in a bipartite graph is even. In fact, it is straightforward to show that a graph is bipartite if and only if it can be constructed from a single vertex through a sequence of even vertex additions. Thus, even vertex addition, given a bipartite graph, creates a new bipartite graph. We now demonstrate the analogous result for perfect graphs.

**Theorem 2.8: No minimal imperfect graph contains an even vertex**

Proof: Assume 2.3 fails. Then there is a minimal imperfect graph $H$ with an even vertex $x$. Since $H - x$ is perfect, it permits a good colouring; colour $H - x$ with a good colouring in which $N_H(x)$ has the minimal possible number of vertices coloured with colour 1. If no vertex in $N_H(x)$ has colour 1 then colouring $x$ with colour 1 gives a good colouring of $H$, contradicting the fact that it is minimal imperfect. If $\chi(H - x) = 1$ then we can use a second colour on $x$ to obtain a good colouring of $H$. Otherwise, let $y$ be a vertex of $N_H(x)$ with colour 1 under this colouring. Consider the graph $F$ induced by
vertices with colours 1 and 2. Switch these two colours in the component of $F$ containing $y$. This new colouring is still good. Since $y$ changes from colour 1 to colour 2, by the minimality property of our colouring, some vertex $z$ of $N_H(z)$ changes from colour 2 to colour 1. Clearly, $y$ and $z$ are connected in $F$ and furthermore, since the $yz$ path alternates colours, they are connected by an odd path. This path together with $x$ is an odd cycle in $H$. Clearly, there is an odd hole containing $z$ within this cycle, contradicting $z$'s evenness.

Corollary 2.4: Any graph constructed by adding an even vertex to a perfect graph is perfect.

Proof: Consider a perfect graph $G$. Assume $G+\varepsilon$ arises from $G$ through even vertex addition. If $G+\varepsilon$ were imperfect then it would contain a minimal imperfect graph $H$. Since $G$ is perfect, $H$ must contain $\varepsilon$. Clearly, $\varepsilon$ is even in $H$, contradicting 2.3.

Thus $G+\varepsilon$ must be perfect.

Unfortunately, determining if a vertex is even seems to be difficult. In particular, it is at least as difficult as determining if a graph is Berge. We shall say more about this in Section 2.11.
Figure 2.1: A $P_4$ and Its Complement
2.2 Union, Complementation and \( P_4 \)-Free Graphs

That complementation preserves perfection is simply a restatement of the Perfect Graph Theorem. We shall show that union also preserves perfection and use these two operations to build a new class of perfect graphs.

**Theorem 2.5:** The union of two perfect graphs is perfect.

**Proof:** Consider two perfect graphs \( G \) and \( H \). Let \( S \) be a subgraph of their union.

\[
\chi(S) = \chi((G \cap S) \cup (H \cap S)) = \max(\chi(G \cap S), \chi(H \cap S)) = \max(\omega(G \cap S), \omega(H \cap S)) = \omega(S)
\]

This shows that any such \( S \) is good and thus \( G \cup H \) is perfect. \( \square \)

**Corollary 2.6:** Every minimal imperfect graph is connected.

A \( P_4 \) is an induced path with three edges and four vertices. We call a graph \( P_4 \)-free, if it contains no \( P_4 \). We claim that the following conditions are equivalent:

1. \( G \) is \( P_4 \)-free
2. Every subgraph of \( G \) with at least two vertices is either disconnected or disconnected in the complement.

**Proof:** To see that (2) implies (1), we note that a \( P_4 \) is both connected and connected in the complement. That (1) implies (2) was initially noted by Seinsche[Se74] and one proof runs as follows.

Since every subgraph of a \( P_4 \)-free graph is \( P_4 \)-free, we need only show that every \( P_4 \)-free graph is disconnected or disconnected in the complement. Assume there exists a \( P_4 \)-free graph which is both connected and connected in the complement. Let \( G \) be such a graph of minimal cardinality. Note that the complement of a \( P_4 \) is a \( P_4 \), so \( \overline{G} \) must also be \( P_4 \)-free. Let \( x \) be an arbitrary vertex of \( G \). By the minimality of \( G \), \( G-x \) is disconnected or disconnected in the complement. Without loss of generality, we may assume that \( G-x \) is disconnected. Since \( \overline{G} \) is connected, there exists a vertex \( y \) of \( G \) such that \( x \) and \( y \) are not adjacent in \( G \). Since \( G-x \) is disconnected, there
exists a vertex $z$ of $G - x$ such that $y$ and $z$ are in different components of $G - x$. Since $G$ is connected, there is a $y z$ path in $G$. Take a $y z$ path $P$ of minimum length, clearly it contains $z$, $y$ and $z$. Now, $x$ and $y$ are not adjacent in $G$. Thus, $P$ is of length at least 4, contradicting our assumption that $G$ is $P_4$-free. 

We note that condition (2) implies that $P_4$-free graphs are precisely those graphs built up from singletons using complementation and union. The perfection preserving properties of these operations imply

**Theorem 2.7:** All $P_4$-free graphs are perfect.

Cornell, Perl, and Stewart [CoPS85] have presented an algorithm to recognize $P_4$-free graphs in $O(n + m)$ time. A simple $O(n^4)$ algorithm is to check, for every set $S$ of four vertices in the graph, whether or not $S$ induces a $P_4$. 


True Twins

False Twins

Figure 2.2
2.3 Domination and Graphs with Small Dilworth Number

To *duplicate* a vertex \( z \) in a graph \( G \) we add a vertex \( y \) to \( G \) such that \( N_{G+y}(y) = N_G(z) \). In a graph \( G \), two vertices \( z \) and \( y \) are *twins* if they see precisely the same vertices of \( G - z - y \). If \( z \) sees \( y \) then we say \( z \) and \( y \) are *true twins*, otherwise they are *false twins*. Clearly duplicating a vertex produces a pair of false twins.

*Theorem 2.8: No minimal imperfect graph contains a pair of twins.*

Proof: We show first that no minimal imperfect graph contains a pair of false twins. Assume the contrary and let \( G \) be a minimal imperfect graph containing false twins \( z \) and \( y \). Since \( G \) is minimal imperfect, \( G - y \) has a good coloring. This coloring can be extended to a good coloring of \( G \) by giving \( y \) the same colour as \( z \). This contradicts the minimal imperfection of \( G \).

Clearly if \( z \) and \( y \) are true twins in \( G \) then they are false twins in \( G \). Thus, no minimal imperfect graph contains true twins, as otherwise its complement would contain false twins, contradicting the above.

*Corollary 2.9: Duplication preserves perfection.*

Proof: Let \( G \) be a perfect graph. Consider the graph \( G + y \) obtained by duplicating a vertex \( z \) of \( G \). Assume \( G + y \) is imperfect. Thus, it contains a minimal imperfect graph \( H \). Since \( H \) is not isomorphic to a subgraph of \( G \), it must contain both \( z \) and \( y \). Clearly, \( z \) and \( y \) are true twins in \( H \), contradicting 2.8. Thus, \( G + y \) is perfect.

If \( z \) and \( y \) are twins in \( G \) then their neighborhoods are equal in \( G - z - y \). A generalization of this idea is to consider vertices \( z \) and \( y \) of a graph \( G \) such that \( N_{G-z-y}(z) \) is contained in \( N_{G-z-y}(y) \). If \( N_{G-z-y}(z) \subseteq N_{G-z-y}(y) \) then we say \( y \) dominates \( z \) and \( z \) is a dominated vertex.

*Theorem 2.10: No minimal imperfect graph contains a dominated vertex.*

Proof: The proof is analogous to that of (2.8).
Two vertices \( x \) and \( y \) in a graph \( G \) are said to have incomparable neighborhoods if 
\[ N_{G-x-y}(x) \] is not contained in \( N_{G-x-y}(y) \) and \( N_{G-x-y}(y) \) is not contained in \( N_{G-x-y}(x) \).

The Dilworth number of a graph, \( D(G) \), is defined as the maximum cardinality of a set of vertices of \( G \) in which any pair of vertices have incomparable neighborhoods. It follows from 2.10 that if \( G \) is a minimal imperfect graph then \( D(G) = |V(G)| \). Clearly, if \( H \) is a subgraph of a graph \( G \) then \( D(H) \leq D(G) \). Since all graphs of order less than five are perfect, the above remarks imply:

**Theorem 2.11 (Payan/Pay88):** All graphs with Dilworth number less than five are perfect.
2.4 Innocuous Vertices and Weakly Bipartite Graphs

We have seen that the addition of a vertex in no odd cycle preserves perfection. The SPGC suggests that we are really only interested in odd cycles of size at least five. We call a vertex innocent if it is in no odd cycle of length greater than 3. If \( y \) is innocent in \( G+y \), for some graph \( G \), \( G+y \) is said to arise from \( G \) by innocent vertex addition.

**Theorem 2.12:** No minimal imperfect graph contains an innocent vertex.

**Proof:** In order to prove this theorem we will need the following lemma.

**Lemma 2.15:** No minimal imperfect graph contains a cutpoint.

**Proof:** Assume there is a minimal imperfect graph \( G \) which contains a vertex \( x \) whose removal disconnects the graph. For any component \( U \) of \( G-x \), we can produce a good colouring of \( U+x \). We can rename the colours so that \( x \) has colour 1 in each colouring. But combining these colourings gives a good colouring of \( G \), a contradiction.

Assume there exists an minimal imperfect graph \( G \) containing an innocent vertex \( x \).

Assume \( \overline{N_G(x)} \) is disconnected. If any component of \( \overline{N_G(x)} \) consists of one vertex \( y \), then \( N_G(x) \) is contained in \( N_G(y) \). By (2.10), this contradicts the minimal imperfection of \( G \). Thus \( \overline{N_G(x)} \) has at least two components \( U \) and \( V \), each with at least two vertices. Choose two vertices \( u_1 \) and \( u_2 \) from \( U \) and two vertices \( v_1 \) and \( v_2 \) from \( V \). Now, \( \{u_1,v_1,u_2,v_2,x\} \) forms a circuit of length five, contradicting the innocuousness of \( x \). Thus \( \overline{N_G(x)} \) is connected. Trivially, \( N_G(x) \) consists of at least two vertices. Also \( N_G(x) \) contains no \( P_4 \), as the four vertices and \( x \) would form an odd cycle. It follows that \( N_G(x) \) is disconnected (see section 2.2). Assume some component, \( U \), of \( N_G(x) \) consists of more than one element. Let \( u_1 \) and \( u_2 \) be two adjacent elements of \( U \), and let \( v \) be an element of some other component of \( N_G(x) \). By (2.13), \( G-x \) is connected.

Consider the shortest path \( P \) from \( v \) to \( u_1 \) in \( G-x \), \( P = \{p_1=v,p_2,\ldots,p_n=u_1\} \).

Clearly \( n \) is not even or \( P+x \) would induce an odd cycle of length at least five. Furthermore \( u_2 \) is not on \( P \) because otherwise it would be \( P_{n-1} \) (by the minimality of \( P \)
and then \( P - p_n + x \) would induce an odd cycle of length at least five in \( G \). But then \( P + u_g + x \) induces an odd cycle of length at least five in \( G \). This contradiction implies that \( N_G(x) \) must be a stable set. In this case, \( x \) is in no triangles and is therefore an even vertex. This contradicts (2.3). \( \square \)

**Corollary 2.14:** *Innocuous vertex addition preserves perfection.*

In [Th85], C. Thomassen describes a method, due to J. Edmonds, for finding the shortest odd length path between two vertices of a graph. This algorithm runs in time bounded by a polynomial function of the number of vertices in the graph. The following algorithm uses Edmonds' idea to determine if a vertex \( x \) is innocuous in a graph \( G \).

(Actually, we do not need to use Edmonds's algorithm. We would be satisfied with an algorithm which finds any odd length path between two vertices of a graph. In Appendix B, we discuss an algorithm which does this.)

**Algorithm 2.1:** Checking If a Vertex Is Innocuous in a Graph.

**Input:** A graph \( G \) represented using adjacency lists and \( x \), a vertex of \( G \).

**Output:** Yes or No.

**Step 1:** Set \( y \) to be the first element on the adjacency list for \( x \). Set \( x \) to \( y \).

**Step 2:** If \( y \) is the last element on the adjacency list for \( x \) go to Step 4. Otherwise set \( x \) to its successor.

**Step 3:** Determine, using Edmonds's algorithm, if there is a path with an even number of vertices from \( y \) to \( x \) in \( G - x - y \). If there is, this path with \( x \) forms an odd cycle, so return No and stop. Otherwise go to step 2.

**Step 4:** If \( y \) is the last element on the adjacency list for \( x \) return Yes and stop. Otherwise, set \( y \) to its successor, set \( x \) to \( y \), and go to Step 2.

We recall that bipartite graphs were precisely those which contained no odd cycles. We define *weakly bipartite graphs* as those which contain no odd cycles of length greater than five. Every vertex in a weakly bipartite graph is innocuous and it is trivial to show that these graphs are precisely those built up from singletons by innocuous vertex addition.
Thus weakly bipartite graphs are perfect.

Weakly bipartite graphs have also been studied by Trotter, de Werra, and Maffray.

Trotter proved that the line graph of $G$ is perfect if and only if $G$ is weakly bipartite.

This generalized a result of König.

*Theorem 2.15([Tr75]):* The line graph of $G$ is perfect if and only if $G$ is weakly bipartite.

*Corollary 2.16([Kö85]):* If $G$ is bipartite then the line graph of $G$ is perfect.

De Werra[DeW78] gave an algorithmic proof of Trotter's result. Since the line graphs of weakly bipartite graphs are perfect, Trotter called the weakly bipartite graphs "line perfect graphs.

Trotter mentioned that weakly bipartite graphs are perfect but did not give a proof. As shown above, this result is a trivial corollary of theorem 2.12. A different characterization of these graphs was developed by Maffray[Ma84].
Figure 2.3: Substitution
2.5 Substitution and Comparability graphs

Substitution is one of the most well-known perfection preserving operations. Its prominence is due to the key role it played in the proof of the perfect graph theorem. (Fulkerson had reduced the WPGC to the statement "Substitution preserves perfection". Lovász, unaware of Fulkerson's work, proved the WPGC). Let G and H be graphs and let x be a specified vertex of G. A new graph, G', arises from substituting H for x in G in the following manner.

To obtain G' take the disjoint union of H and G - x and for every pair of vertices y and z with y in G - x and z in H add the edge yz if and only if yz is an edge of G.

A homogeneous set in a graph G is a proper subset H of the vertices of G containing at least two vertices and such that V(G) - H can be partitioned into A = \{x | x sees every vertex of H\} and B = \{x | x misses every vertex of H\}. We note that if G has a homogeneous set, H, then G arises by substituting the graph induced by H into the graph induced by G - H + y, where y is any vertex of H. Conversely, if G' arises by substituting H for x in G then the vertices of H are a homogeneous set in G' (with A = N_G(x), B = N_G(x)).

Theorem 2.17(Lo72a): No minimal imperfect graph contains a homogeneous set.

Proof: The following easy observations are necessary to prove the theorem.

Lemma 2.18: No stable set in a minimal imperfect graph contains a vertex from every maximum clique of the graph.

Lemma 2.19: Let x be a specified vertex of some perfect graph G. Then, there exists a stable set in G containing x which contains a vertex from all maximum cliques of G.

Assume 2.17 is false. Let G be a counterexample to the theorem, with homogeneous set H. Let x be any vertex of H. Clearly H and G - H + x are perfect. Choose a stable set, S_1, in G - H + x which contains x and meets every maximum clique in G - H + x.

Choose a stable set, S_0, in H which meets every maximum clique in H. We claim
$S_3=S_1-x+S_2$ is a stable set which meets every maximum clique of $G$. Since $x$ misses every element of $S_1-x$, $S_3$ is clearly a stable set. Hence, we need only show that it meets all maximum cliques of $G$.

Let $C$ be a maximum clique of $G$. We want to show that $C$ meets $S_3$. We consider, first, the case where $H \cap C = \emptyset$. In this case, $C$ must be a maximum clique of $G-H+x$. Since $C$ is a maximum clique of $G-H+x$, $C$ meets $S_1$ and therefore $S_1-x$.

Thus $C$ meets $S_3$.

Assume now that $C \cap H \neq \emptyset$. Let $K$ be a maximum clique in $H$, clearly $K \cup (G-C \cap H)$ is a clique in $G$. Also $|C \cap H| \leq |K|$. Now, by the maximality of $C$, equality holds and $C \cap H$ is a maximum clique of $H$. Thus, $C \cap H$ meets $S_2$ and therefore $C$ meets $S_3$. We have shown that $S_3$ is a stable set of $G$ which meets all its maximum cliques, contradicting Lemma 2.18.

**Corollary 2.20:** If $G$ arises by substituting one perfect graph in another perfect graph then $G$ is perfect.

In Appendix C, we present an algorithm which determines, in $o(n^4)$ time, if a graph has a homogeneous set.

We recall that a **partially ordered set** is a set $P$ of elements and a binary relation $<$ on $P$ such that:

1) for $a,b,c$ elements of $P$ if $a < b$ and $b < c$ then $a < c$ (is **transitive**).

2) there do not exist $a,b$ elements of $P$ such that $a < b$ and $b < a$ (is **antisymmetric**).

A comparability graph is a graphic representation of a partially ordered set. That is, $G$ is a **comparability** graph if there exists a mapping $f$ from the vertices of $G$ to the elements of a partially ordered set $P$ such that $x$ and $y$, vertices of $G$, are adjacent if and only if $f(x) < f(y)$ or $f(y) < f(x)$.
Theorem 2.21: A graph $G$ is a comparability graph if and only if it permits an orientation $U$ on its edges such that if $xy$ and $yz$ are elements of $U$ then $zx$ is an element of $U$ (we call this a transitive orientation).

Proof: Consider a comparability graph $G$ and the corresponding partially ordered set $(P, \prec)$ and mapping $f$. Construct an orientation $U$ of $G$ in the following manner. For each edge $ab$ of $G$, if $f(a) \prec f(b)$ then select $\overrightarrow{ab}$ as an element of $U$, otherwise select $\overrightarrow{ba}$ as an element of $U$. By the transitivity of $P$, this orientation will have the desired properties.

Conversely, given a graph $G$ with a transitive orientation, define an ordering $<$ on the vertices of $G$ by making $a < b$ if and only if $\overrightarrow{ab}$ is an element of $U$. Clearly, the vertices of $G$ form a partially ordered set under $<$ and $G$ is the graphical representation of this set. Thus, $G$ is a comparability graph.\[\]We note that comparability graphs form a hereditary class. Thus, to show they are perfect, we need only show that they are good.

Theorem 2.22 (Berge/Be60): Every comparability graph is good.

Proof: Let $G$ be a comparability graph. Consider an transitive orientation $U_G$ of $G$. Label each vertex of $G$ with the number of vertices in the longest directed path which starts at the vertex. Clearly, there are no edges between vertices with the same label. Thus, the labels give a colouring of $G$. Furthermore, by the transitivity of $U_G$, the vertices of any directed path of $U_G$ induce a clique. It follows that the colouring given by the labels is good.\[\]

Corollary 2.23: Every comparability graph is perfect.

The following result of Ghoula-Houri[Gh62] is the key to a polynomial time algorithm for recognising comparability graphs.

Theorem 2.24: $G$ is a comparability graph if and only if it permits an orientation, $U$, of its edges such that if $\overrightarrow{ab}$ and $\overrightarrow{bc}$ are elements of $U$ then either $\overrightarrow{ca}$ or $\overrightarrow{ca}$ is an element of.
Proof: A transitive orientation is semi-transitive, thus every comparability graph permits a semi-transitive ordering. To see that the converse is true we will require the following lemmas.

Lemma 2.25: If \( G' \) arises by substitution, from two comparability graphs \( G \) and \( H \), then \( G' \) is a comparability graph.

Proof: Let \( G \) and \( H \) be comparability graphs and consider \( G' \) which arises by substituting \( H \) for some vertex \( z \) in \( G \). Clearly \( G \) and \( H \) both permit transitive orientations, \( U_G \) and \( U_H \) respectively. We form an orientation \( U_{G'} \) of \( G' \) as follows. Orient the edges of \( H \) as in \( U_H \), and the edges of \( G - z \) as in \( U_G \). For \( y \) in \( G - x \) and \( z \) in \( H \), \( yz \) is in \( U_{G'} \) if and only if \( yz \) is in \( U_G \). Similarly \( xz \) is in \( U_{G'} \) if and only if \( xz \) is in \( U_G \). It is an easy task to verify that \( U_{G'} \) is a transitive orientation of \( G' \) and thus \( G' \) is a comparability graph.

Lemma 2.26: If three vertices form a directed triangle in the semi-transitive orientation of a graph, then no vertex sees precisely one of these three vertices.

Proof: Consider \( \overline{ab}, \overline{bc}, \overline{ca} \), a directed triangle in a semi-transitive orientation \( U_G \) of a graph \( G \). Assume some vertex \( z \) sees \( b \) but misses \( a \) and \( c \). Since \( \overline{bc} \) is in \( U_G \), \( \overline{zb} \) cannot be in \( U_G \). Since \( \overline{ab} \) is in \( U_G \), \( \overline{zb} \) cannot be in \( U_G \) but, since \( z \) sees \( b \) in \( G \), one of these two possibilities must hold, a contradiction. Thus no vertex sees \( b \) and misses \( a \) and \( c \). By symmetry, no vertex in \( G \) sees exactly one of the three vertices in the triangle.

We show now that if a graph has a semi-transitive orientation and no homogeneous set then it is a comparability graph. If a graph has a homogeneous set it arises from substitution of two proper subgraphs, thus this result combined with Lemma 2.25 completes the proof of the theorem.

Consider a graph \( G \) with some semi-transitive orientation \( U_G \). If there is no directed triangle in \( U_G \) then \( U_G \) is a transitive orientation and \( G \) is a comparability graph. We
will show that if $U_G$ has a directed triangle then it has a homogeneous set. Let $ab, bc, cb$ be a triangle in $U_G$. By 2.26, no vertex is adjacent to exactly one of these three vertices. If there were no vertices which saw precisely two vertices of the triangle then it would be a homogeneous set. Thus some vertex $z$ sees precisely two vertices in the triangle; we can say $z$ sees $a$ and $c$ and misses $b$. We let $F = \{ y \mid y$ sees both $a$ and $c \}$. Consider the graph $F'$ induced by $F$ in $\overline{G}$, let $H$ be the component of $F'$ containing $b$ and $z$.

We show first that every vertex in $H$ forms a directed triangle with $a$ and $c$ in $U_G$. We prove this by induction on the number of edges in a minimal length path from a vertex to $b$. For any vertex $z$ in $H$, let $P = \{ p_0 = b, p_1, \ldots, p_k = z \}$ be a minimal length path from $z$ to $b$. If $k = 0$ then $z$ is $b$ and there is nothing to prove. Assume that $k > 0$. By the induction hypothesis $p_{k-1}$ forms a directed triangle with $a$ and $c$ in $U_G$. It follows that $ap_{k-1}$ and $pc_{k-1}$ are in $U_G$ (see Fig. 2.4). Also $z$ does not see $p_{k-1}$ in $G$.

Thus, $za$ and $zc$ cannot be in $U_G$ so $az$ and $cz$ must be. Thus $\{ a, z, c \}$ is a directed triangle as required.

We claim $H$ is a homogeneous set. $H$ contains both $b$ and $z$ and thus has at least two elements; trivially $H \neq G$. Every vertex of $F - H$ sees all of $H$, thus we need only show that no vertex of $G - F$ disagrees on two elements of $H$. Assume some vertex $d$ of $G - F$ sees $y$ in $H$ and misses $z$ in $H$. Since $d$ cannot see only one vertex of $\{ a, c, y \}$, by 2.26, $d$ sees precisely one of $a$ and $c$. But then $d$ sees precisely one vertex in $\{ a, c, z \}$ contradicting 2.26. Thus no vertex of $G - H$ disagrees on $H$ and $H$ is a homogeneous set.

Now, by 2.24, to decide if $G$ is a comparability graph we need only determine if it permits a semi-transitive orientation. This is relatively straight-forward. In Appendix D, we present an algorithm to determine if a graph has a semi-transitive orientation. This algorithm runs in $O(\Delta m)$ time (where $\Delta$ is the maximum vertex degree).
Figure 2.5: Clique Identification
2.6 Clique Cutsets and Triangulated Graphs

A clique cutset is a cutset whose vertices induce a clique.

Definition 2.27: $G'$ is said to arise from graphs $G$ and $H$ by clique identification in the following manner. Select cliques $C_G$ and $C_H$ contained in $H$ such that $|C_G|$ equals $|C_H|$. Consider a bijection $f: C_G \rightarrow C_H$. We construct $G'$ by taking a copy of $G \cup (H - C_H)$ and adding an edge $xy$ for $x \in C_G, y \in H - C_H$ if and only if $f(x)y$ is an edge in $H$.

It is easy to see that $G$ arises by clique identification if and only if $G$ has a clique cutset.

Theorem 2.28: No minimal imperfect graph contains a clique cutset.

Proof: Assume 2.28 is false. Let $G$ be a minimal imperfect graph, with clique cutset $C$.

Let $A$ be a component of $G - C$ and set $B = G - C - A$. Both $C + A$ and $C + B$ have good colourings. In both colourings, each colour appears at most once in the clique $C$.

Thus, we can rename the colours in one of the colourings so that the two colourings agree on $C$. Then, by taking the union of the two colourings, we obtain a good colouring of $G$, a contradiction.

Corollary 2.29([Haj58]): If $G$ arises from perfect graphs $G$ and $H$ through clique identification then $G$ is perfect.

Whitesides[81] has presented an $O(nm)$ algorithm for determining if a graph has a clique cutset.

We motivated the discussion of bipartite graphs in section 2.1 by noting that all complements of cycles of length at least six contain a triangle. Actually $\overline{C_6}$ is $C_6$ and, for $k$ at least six, $\overline{C_k}$ contains a $C_4$. Thus, a graph with no hole of length greater than three is clearly Berge. These graphs are called triangulated.
Theorem 2.30 (Dirac/Dir61): The following characterizations are equivalent:

1. \( G \) is a triangulated graph

2. Every subgraph \( H \) of \( G \) has a simplicial vertex (that is a vertex whose neighborhood in \( H \) forms a clique)

3. Every subgraph of \( G \) which is not a clique has a clique cutset.

We show first that (1) implies (3). We note that, since every subgraph of a triangulated graph is triangulated, we need only show that every triangulated graph, which is not a clique, has a clique cutset. Consider any two non-adjacent vertices, \( x \) and \( y \), in a triangulated graph \( G \) that is not a clique. By a separator of \( x \) and \( y \), we shall mean any cutset \( S \) such that \( x \) and \( y \) are in different components of \( G - S \). A minimal separator is a separator all of whose proper subsets are not separators. Consider a minimal separator \( S \) of \( x \) and \( y \). Let \( U \) and \( V \) be the components of \( G - S \) containing \( x \) and \( y \) respectively. If \( S \) is a single vertex, it is a clique cutset, otherwise let \( a \) and \( b \) be two vertices of \( S \). Since \( S \) is a minimal separator, there is a path from \( x \) to \( y \) in \( G - S + a \). Thus, there is a path from \( a \) to \( x \) in \( a + U \) and a path from \( a \) to \( y \) in \( a + V \). Similarly, there is a path from \( b \) to \( x \) in \( b + U \) and a path from \( b \) to \( y \) in \( b + V \). It follows that there are paths from \( a \) to \( b \) in both \( a + b + U \) and \( a + b + V \). Thus, \( a \) and \( b \) are adjacent, as otherwise the union of these two paths is a cycle of length at least 4. This implies that \( S \) is a clique cutset in \( G \), as desired.

\((2) \Rightarrow (1):\) This follows immediately from the fact that cycles of length greater than three do not contain simplicial vertices.

\((3) \Rightarrow (2):\) We actually show that any graph satisfying (3) is either a clique or contains two non-adjacent simplicial vertices. Assume this statement is false and let \( G \) be a minimal counter-example. \( G \) cannot be a clique so, by (3), \( G \) has a clique cutset \( S \). Partition \( G - S \) into \( U \) and \( V \) in the usual way. \( S + U \) and \( S + V \) are both proper subgraphs of \( G \) with at least two vertices. We claim that \( U \) contains a simplicial vertex of \( G \), clearly it is enough to show that \( U \) contains a vertex which is simplicial in \( S + U \). If
$S + U$ is a clique we can select any vertex of $U$. Otherwise, by the minimality of $G$, $S + U$ contains two non-adjacent simplicial vertices. Since $S$ is a clique, only one of these vertices can be in $S$ so at least one vertex is in $U$. Similarly, $V$ contains a simplicial vertex of $G$. Thus $G$ contains two non-adjacent simplicial vertices, a contradiction.

Note that if $x$ is a simplicial vertex of a graph $G$ then the only holes of $G$ which contain $x$ are triangles. Now, by 2.30, we know that if $x$ is simplicial in $G$ then $G$ is triangulated if and only if $G - x$ is. This observation suggests the following algorithm for recognizing triangulated graphs.

**Algorithm 2.2: Determining If a Graph is Triangulated.**

**Input:** A graph $G$.

**Output:** Yes or No.

**Step 1:** $H < G$.

**Step 2:** If $H$ has no vertices, return Yes and stop.

**Step 3:** Determine if $H$ has a simplicial vertex.
   - If $H$ has no simplicial vertex return No and stop.
   - Otherwise find a simplicial vertex $x$ of $H$.

**Step 4:** $H < H - x$. Go to step 2.

We can determine if a vertex is simplicial by counting the edges in its neighborhood (we want $|E(N(x)| = \binom{|N(x)|}{2}$). Thus, by checking each vertex in turn, we can execute Step 3 in $O(nm)$ time. It follows that algorithm 2.2 runs in $O(n^2m)$ time. Rose, Tarjan and Leuker[71] have developed an algorithm to recognize triangulated graphs which runs in $O(n + m)$ time.
Figure 2.6: A Clique Cutset Tree
2.7 Clique Separable Graphs and Clique Cutset Trees

A graph is *clique separable* if and only if each of its subgraphs is either bipartite, complete multi-partite, or contains a clique cutset. Gavril introduced the class of clique separable graphs.

*Theorem 2.31 ([Gav77]):* Clique separable graphs are perfect.

*Proof:* Recall that there are no minimal imperfect bipartite graphs. Note that there are also no minimal imperfect multi-partite graphs. Since no minimal imperfect graph has a clique cutset, this implies that there are no minimal imperfect clique separable graphs. It follows that there are no imperfect clique separable graphs.

Gavril developed a polynomial-time algorithm to recognize clique separable graphs. We shall present a much simpler algorithm which is due to Whitesides [W84a].

Note first that if we construct $G'$ from two clique separable graphs $G_1$ and $G_2$ through clique identification then $G$ is also clique separable. It follows that:

(2.32) A graph is clique separable if and only if it can be built up from a set of bipartite and complete multi-partite graphs through a sequence of clique identifications.

Whitesides's algorithm attempts to reverse this procedure using a clique cutset tree.

*Definition 2.33:* A clique cutset tree of $G, T(G)$, is a tree each of whose nodes is labelled by a subgraph $H$ of $G$. We recursively define the tree as follows. The root of the tree is labelled $G$. If $G$ has no clique cutset then we are done. Otherwise find a clique cutset $C$ of $G$. Now, $G-C$ splits into components $H_1, H_2, ..., H_k$. The $k$ children of $G$ in $T(G)$ will be labelled by $C \cup H_1, C \cup H_2, ..., C \cup H_k$. Now, the subtree of $T(G)$ rooted at the child of the root with label $C \cup H_i$ is a clique cutset tree of $C \cup H_i$ (see Fig. 2.6 for an example of a clique cutset tree).

Now, let $G$ be a graph and consider $T(G)$ a clique cutset tree of $G$. By 2.32, if every leaf of $T(G)$ is labelled by a bipartite or complete multi-partite graph then $G$ is clique-separable. Conversely, if a leaf of $G$ is neither bipartite nor complete multi-partite then
clearly $G$ is not clique separable. Basically, Whiteside's algorithm consists of building such a tree and then ensuring the leaves are of the proper type. However, the subgraphs are put onto a list rather than into a tree.

Algorithm 2.3: Determining If a Graph Is Clique Separable.

Input: A graph $G$.

Output: Yes or No.

Data Structures: A list $L$ of graphs.

Step 1: Put $G$ onto $L$.

Step 2: If $L$ is empty return Yes and stop. Otherwise take the first graph $H$ from the beginning of the list.

Step 3: If $H$ is bipartite or complete multi-partite return to Step 2.

Step 4: If $H$ has no clique cutset then return No and stop. Otherwise find a clique cutset $C$ of $H$. Let $U_1, U_2, \ldots, U_k$ be the components of $H - C$. Append $C \cup U_1, C \cup U_2, \ldots, C \cup U_k$ to the list. Return to Step 2.

Recall that Whitesides\[W81\] has developed an $O(nm)$ algorithm which given a graph $G$ determines whether or not $G$ has a clique cutset. Furthermore, if $G$ has a clique cutset then the algorithm will find one. Using this algorithm, each iteration of step 4 can be executed in $O(nm)$ time. An iteration of Step 3 takes even less time. We have already seen an $O(n + m)$ algorithm to determine if a graph is bipartite. The complement of a complete multi-partite graph is the disjoint union of cliques. It is easy to design an $O(n + m)$ algorithm to check for this property. Thus, each iteration of our algorithm can be executed in $O(nm)$ time.

We will now show that we execute steps 3 and 4 at most $n^2 + 1$ times. This implies that the algorithm runs in $O(n^3m)$ time. Clearly the number of times we execute steps 3 and 4 is bounded by the number of graphs which pass through our list $L$. Furthermore, this is bounded by the maximum number of nodes in a clique cutset tree of $G$.
Theorem 2.34: If \( G \) is a graph with \( n \) nodes and \( T(G) \) is a clique cutset tree of \( G \) then \( T(G) \) has at most \( n^2 + 1 \) nodes.

Proof: Consider a node of \( T(G) \) other than the root. This node is labelled by some subgraph \( H \) of \( G \). Clearly, this node has a sibling with some label \( F \). \( H \) and \( F \) intersect in a clique \( C \) (\( C \) is a clique cutset in their common parent). Let \( u \) be a vertex of \( H - C \) and let \( v \) be a vertex of \( F - C \). Associate with the node labelled \( H \) the ordered pair \( (u, v) \). It is not difficult to verify that no two nodes in the tree are associated with the same pair, thus the tree contains no more than \( n^2 + 1 \) elements. 

Corollary 2.35: Algorithm 2.3 takes \( O(n^3 m) \) time.

Tarjan[Ta85] has developed a much faster algorithm for building a clique cutset tree. His algorithm runs in \( O(nm) \) time. It builds a tree in which the internal nodes are a path. Furthermore, the tree his algorithm constructs has at most \( 2n \) nodes. Clearly, by checking if each leaf in this tree is of the appropriate type, we can determine if a graph is clique separable in \( O(nm) \) time.
2.8 Paradigms and Examples

In this chapter, we have been discussing instances of the following paradigm:

Perfection Proving Paradigm I

A hereditary class of graphs $A$ conforms to this paradigm if we can associate with it a base class $B \subseteq A$ and a property $P$ such that:

1. $P$ does not hold on minimal imperfect graphs,
2. Every element of $A - B$ has property $P$, and
3. Every graph in $B$ is perfect.

Clearly if (1), (2), and (3) hold then every graph in $A$ is perfect.

For example, let $A$ be the class of triangulated graphs, let $B$ be the class of cliques and let $P$ be the property "$G$ has a clique cutset". To take another example, let $A$ be the class of bipartite graphs, let $B$ be empty, and let $P$ be the property "$G$ has an even vertex".

Sometimes, the graphs in $A$ can be built up from the base class $B$ using some perfection preserving operation $P'$. In fact, we often have the somewhat stronger situation described below.

Perfection Proving Paradigm II

A hereditary class of graphs $A$ conforms to this paradigm if we can associate with it a base class $B \subseteq A$ and an operation $P'$ such that:

1. $P'$ is perfection preserving,
2. Every graph in $A - B$ can be constructed from a set of smaller graphs in $A$ using $P'$,
3. Every graph in $B$ is perfect, and
4. If $G'$ is constructed by applying $P'$ to a set of graphs in $A$ then $G'$ is in $A$.

Clearly, if (1), (2), (3), and (4) hold then every graph in $A$ is perfect.

For example, every clique separable graph which is neither a bipartite graph nor a complete multi-partite graph can be formed from two smaller clique separable graphs through clique...
identification. Furthermore, if we form \( G \) from two clique separable graphs through clique identification then \( G \) is clearly clique separable. \( P_\ell \)-free graphs provide another instance of this paradigm. Every \( P_\ell \)-free graph is either the union of two smaller \( P_\ell \)-free graphs or is the complement of the union of two smaller \( P_\ell \)-free graphs (see section 2.2). Furthermore, if \( G \) is the union of two \( P_\ell \) free graphs then both \( G \) and \( \overline{G} \) are \( P_\ell \)-free.

In the last section, we presented an algorithm to recognize clique separable graphs. This algorithm can be generalized to recognize any class of graphs which conforms to Paradigm II.

Algorithm 2.4: Determining If a Graph Is in Class \( A \).

\( A \) is some class conforming to paradigm II)

Input: A Graph \( G \).

Output: Yes or No.

Data Structures: A list \( L \) of graphs.

Step 1: Put \( G \) onto \( L \).

Step 2: If \( L \) is empty return Yes and stop. Otherwise take the first graph \( H \) off the list.

Step 3: If \( H \) is in \( B \) return to Step 2.

Step 4: If \( H \) does not arise through an application of \( P^* \) then return No and stop. Otherwise find a set of graphs, \( H_1, H_2, \ldots, H_k \) such that \( H = P^*(H_1, H_2, \ldots, H_k) \). Append \( H_1, H_2, \ldots, H_k \) to \( L \), return to Step 2.

Of course, we must provide algorithms which:

(1) Determine if a graph is in Class \( B \) (Step 3), and

(2) Determine if a graph can be built up from smaller subgraphs using \( P^* \) and if so find a set \( H_1, H_2, \ldots, H_k \) of graphs such that the graph is formed from this set using \( P^* \) (Step 4).

If we want an instance of Algorithm 2.4 to run in polynomial time, we need to ensure that our two algorithms run in polynomial time. In addition, we require that for a graph \( G \), the number of graphs passing through \( L \) is bounded by a polynomial function of the order of \( G \). Assume that we can perform (1) in \( O(n^i) \) time and (2) in \( O(n^j) \) time. Then, if the number of graphs passing through \( L \) is bounded by \( O(n^k) \), this instance of Algorithm 2.4
Figure 2.7 A Graph and Its Decomposition Tree.
runs in $O(n^{\max(i,j)+2})$ time.

For example, assume that $A_r$ is the class of $P_r$-free graphs. Then, in Step 3, we want to determine if either $H$ or $\overline{H}$ is disconnected. If $H$ is disconnected, we append the components of $H$ to our list. If $\overline{H}$ is disconnected, we append the components of $\overline{H}$ to our list. There are well-known algorithms which given a graph find its components in $O(n+m)$ time. Thus, each iteration of Step 4 can be executed in $O(n^2)$ time. The base class for $P_r$-free graphs is simply the graph with one vertex. Thus, we can determine if $H$ is in $B$ in constant time. So, each iteration of our algorithm takes $O(n^2)$ time.

Now, as with clique cutsets, we can build a tree which corresponds to our decomposition procedure (see Fig. 2.7). Clearly, the leaves of this tree are disjoint. Thus, there are at most $n$ leaves in the tree and therefore at most $2n$ nodes in the tree. It follows that we append at most $2n$ graphs to our list $L$. This means that our recognition algorithm runs in $O(n^4)$ time. This is slightly better than the straightforward algorithm which runs in $O(n^4)$ time but worse than the $O(n+m)$ algorithm discussed in [CoPS85].
2.9 Amalgam and Meyniel Graphs

A graph is Meyniel if every odd cycle of length at least five in the graph has at least two chords. In 1976, Meyniel showed that these graphs are perfect. Hoang recently developed an elegant characterization of these graphs, strengthening Meyniel's result.

**Definition 2.36:** A dominating stable set is a stable set which meets all maximum cliques.

A good stable set is a stable set which meets all maximal cliques.

Clearly, in any good colouring of a graph, each colour class is a dominating stable set. Thus, every perfect graph has a dominating stable set (this is Lemma 2.19). If G is minimal imperfect, it is not difficult to see that G cannot have a dominating stable set (this is Lemma 2.18). Thus, we can characterize perfect graphs in terms of dominating stable sets.

**Observation 2.37:** A graph is perfect if and only if each of its induced subgraphs has a dominating stable set.

**Definition 2.38 (Berge and Duchet [Be84]):** A graph is strongly perfect if each of its induced subgraphs has a good stable set.

Since a good stable set is dominating, strongly perfect graphs are clearly perfect.

**Definition 2.39 (Hoang [Ho85b]):** A graph is vachement fortement parfait (bloody strongly perfect) if in each of its induced subgraphs each vertex is in a good stable set.

**Theorem 2.40 (Hoang [Ho85b]):** A graph is Meyniel if and only if it is vachement fortement parfait.

**Corollary 2.41 ([Me76]):** Meyniel graphs are perfect.

We are going to discuss a polynomial time algorithm to recognize Meyniel graphs. This algorithm was developed by Burlet and Fonlupt [BuF84]. They showed that Meyniel graphs conform to Paradigm II and thus can be recognised by Algorithm 2.4. We shall now describe the base class and operation associated with Meyniel graphs. These were designed specifically for the algorithm and so may seem somewhat artificial.
Figure 2.8 Amalgam

Figure 2.9: A Basic Meyniel Graph
Definition 2.42: $G$ arises from $G_1$ and $G_2$ by amalgamation in the following manner. Choose a vertex $x_1$ of $G_1$ and a vertex $x_2$ of $G_2$. Choose a clique $K_1$ in $N_{G_1}(x_1)$ and a clique $K_2$ in $N_{G_2}(x_2)$ such that $K_1$ and $K_2$ have the same size. Further, we require that each vertex of $K_i$ sees every vertex of $N_{G_i}(x_i)-K_i$, for $i=1,2$. We obtain $G$ from $G_1$ and $G_2$ by

(i) identifying the vertices of $K_1$ with those of $K_2$.
(ii) adding all the edges between $N(x_1)-K_1$ and $N(x_2)-K_2$.
(iii) deleting $x_1$ and $x_2$.

An amalgam is proper if both $G_1$ and $G_2$ have fewer vertices than $G$.

Definition 2.43: A graph is basic Meyniel if its vertices can be partitioned into three sets $A,K$ and $S$ so that:

(i) $K$ is a clique, $S$ is a stable set, and $A$ is a 2-connected bipartite graph with at least four vertices.
(ii) Each vertex of $K$ sees every vertex of $A$.
(iii) Each vertex of $S$ sees at most 1 vertex of $A$ (thus each vertex of $S$ is simplicial).

The following four observations imply that Meyniel graphs can be shown to be perfect using paradigm II.

Observation 2.44: A basic Meyniel graph is perfect.

Theorem 2.45: The amalgam of two perfect graphs is perfect.

Theorem 2.46: The amalgam of two Meyniel graphs is Meyniel.

Theorem 2.47: If $G$ is Meyniel then either $G$ is basic Meyniel or $G$ is the proper amalgam of two Meyniel graphs.

Now, algorithm 2.4 can be applied, using basic Meyniel graphs as the base class and amalgam as the building operation. The following three non-trivial facts imply that the algorithm runs in $O(n^7)$ time.
Fact 1: There is an algorithm which determines if a graph is basic Meyniel in $O(n^4)$ time.

Fact 2: There is an algorithm which, given a graph $G$ which is not basic Meyniel, will return with one of the two following outputs in $O(n^4)$ time:

(i) $G$ is not Meyniel

(ii) Two graphs $G_1$ and $G_2$ such that $G$ is the proper amalgam of $G_1$ and $G_2$.

Fact 3: If we apply algorithm 2.4 using the two procedures given above as subroutines then at most $n^5$ graphs will pass through the list $L$. 
2.10 Star Cutsets and Weakly Triangulated Graphs.

We turn now to a natural generalization of a clique cutset. A star cutset is a cutset with some specified vertex, the centre, which sees all the other vertices in the cutset. Thus, a clique cutset is simply a star cutset in which every vertex can be the centre. Fig 2.10 is an example of a graph with a star cutset. We shall call a graph \( G \) unbreakable if neither \( G \) nor \( \overline{G} \) contains a star-cutset. We could define star-identification in a manner similar to clique identification, however this operation is not perfect preserving as demonstrated in Fig. 2.11. Nevertheless, as we shall now prove, star-cutsets cannot occur in minimal imperfect graphs.

**Theorem 2.48 (Chvátal/Ch85a):** No minimal imperfect graph contains a star-cutset.

Proof: Assume \( G \) is minimal imperfect and has a star-cutset \( S \) with centre \( x \). We can partition \( G-S \) into two non-empty disjoints sets \( U \) and \( V \) so that there are no edges between \( U \) and \( V \). Now \( S+U \) is perfect so, by 2.18, there is a stable set \( S_1 \) containing \( x \) which meets all the maximum cliques of \( S+U \). Similarly there is a stable set \( S_2 \) containing \( x \) which meets all the maximum cliques of \( S+V \). Since \( x \) sees all the vertices of \( S-x \), \( S_1-x \) is entirely in \( U \) and \( S_2-x \) is entirely within \( V \). It follows that \( S_1=S_1 \cup S_2 \) is a stable set. Now, since there are no edges between \( U \) and \( V \), every clique in \( G \) is either entirely in \( S+U \) or entirely in \( S+V \). It follows that \( S_1 \) meets all the maximum -cliques of \( G \). By 2.18, this contradicts the minimal imperfection of \( G \).

Having described a natural extension of clique cutsets, we turn now to a natural extension of triangulated graphs. We have already mentioned that every complement of a hole of length at least six contains a \( C_4 \). Thus, triangulated graphs can be described as those graphs which contain no hole of length four and no disk. We shall now discuss the class of graphs which arise when we drop the first condition. **Weakly triangulated graphs** (Hayward/Hay85), are those which contain no disk. These graph are clearly Berge as we simply exclude even disks as well as odd ones. To show that weakly triangulated graphs are perfect we need the following theorem, which we shall state without proof:
Theorem 2.49 ([Hay85]): Every weakly triangulated graph with at least three vertices has a star cutset or a star cutset in the complement.

It follows immediately from 2.49 that no minimal imperfect graph is weakly triangulated. Thus all weakly triangulated graphs are perfect. We would like to use algorithm 2.4 to recognize weakly triangulated graphs. Although star-cutset identification does not preserve perfection, we can construct a decomposition tree using star cutsets.

Observation 2.50: Let $C$ be a star-cutset, with centre $x$, in a graph $G$. Let $U$ be any component of $G-C$. Let $G_1 = G-x$; let $G_2 = C + U$; let $G_3 = G - U$. Then, $G$ is perfect if and only if $G_1$, $G_2$, and $G_3$ are perfect.

Proof: $G_1$, $G_2$, and $G_3$ are all subgraphs of $G$. Thus, if $G$ is perfect so are they. We show now that if these three graphs are perfect then so is $G$. Assume $G_1$, $G_2$, and $G_3$ are perfect but $G$ is not. Let $H$ be a minimal imperfect subgraph of $G$. Since $G_1$ is perfect, $H$ must contain $x$. Since $G_3$ is perfect, $H$ must contain some $y$ in $U$. Since $G_2$ is perfect, $H$ must contain some $z$ in $G-C-U$. Now, $H \cap C$ is a star cutset of $H$.

This contradicts the minimal imperfection of $H$. $lacksquare$

Unfortunately, Chvátal [Chv86a] has shown that the star cutset trees which arise from this decomposition procedure can be exponentially large. Thus, we cannot use this decomposition to recognize weakly triangulated graphs in polynomial time.

Fortunately, as pointed out by Hayward [Hay86], it is relatively straightforward to determine if a graph has a hole of length five or greater by simply checking if any path of length three extends into such a hole. Algorithm 2.5 will do this in $O(n^3m)$ time. To determine if a graph $G$ is weakly triangulated, we simply apply Algorithm 2.5 to both $G$ and $\overline{G}$ (This takes $O(n^3)$ time). Clearly $G$ is weakly triangulated if and only if the output is no in both cases.
Algorithm 2.5: Determining If a Graph Contains a Large Hole.

Input: A Graph $G$.
Output: Yes or No.

Data Structures: A list $L$ each element on the list is a set of three vertices which induce a $P_3$ of $G$.

Step 1: Set $L = \emptyset$.

Step 2: For each set $S$ of three vertices in $G$, if $S$ induces a $P_3$ then add $S$ to $L$ (this takes $O(n^2)$ time).

Step 3: If $L$ is empty return No and stop.

Step 4: Take the first triple $S$ from $L$.
Let $x$ be the midpoints of the $P_3$ induced by $S$ and let $y$ and $z$ be the endpoints of this $P_3$.

Step 5: Set $G' = G - (N(x) + (N(y) \cap N(z)) - y - z)$.
If $y$ and $z$ are in the same component of $G'$ return Yes and stop.
Otherwise remove $S$ from $L$ and go to step 3.
(Step 5 takes $O(n + m)$ time).
2.11 Even Pairs and Quasi-Parity Graphs

In an odd hole, any two non-adjacent vertices are connected by both an odd induced path and an even induced path. In the complement of an odd hole of length at least five, any two non-adjacent vertices are connected by an induced path of length 2 and an induced path of length 3. Thus, the following statement is implied by the SPGC.

(2.51) In a minimal imperfect graph, any two non-adjacent vertices are connected by an even induced path and an odd induced path.

A pair of vertices, in a graph $G$, is an even pair if every induced path connecting them has an even number of edges. Similarly, two non-adjacent vertices form an odd pair if every induced path connecting them has an odd number of edges.

Theorem 2.52 (Meyniel [Me86a]): No minimal imperfect graph contains an even pair.

Conjecture 2.53 (Meyniel [Me86b]): No minimal imperfect graph contains an odd pair.

Note that 2.52 and 2.53 taken together imply 2.51.

Definition 2.54: A graph $G$ is a strict quasi-parity graph (SQPG) if, for each induced subgraph $H$ of $G$, either $H$ is a clique or $H$ contains an even pair. A graph $G$ is a quasi-parity graph (QPG) if, for each induced subgraph $H$ of $G$, either $H$ is a singelton, $H$ has an even pair, or $\overline{H}$ has an even pair.

Observation 2.55: Strict quasi-parity and quasi-parity graphs are perfect.

Proof: This is an immediate corollary of 2.52.

There is no known polynomial time algorithm to recognize QPG or SQPG. If a graph $G$ has an even pair $\{x, y\}$ then $G$ is QPG(SQPG) if and only if both $G - x$ and $G - y$ are QPG(SQPG). Thus, we can use algorithm 2.4 to recognize QPG and SQPG. Given a graph $G$ with an even pair $\{x, y\}$ we would decompose it into $G - x$ and $G - y$.

Unfortunately, there is no known polynomial-time algorithm to determine if a graph has an even pair. Moreover, this problem seems to be difficult. In particular, if we could determine whether or not two specific vertices form an even pair then we could recognize Berge
Observation 2.58: If we can determine in polynomial time whether or not an arbitrary pair of vertices in a graph is an even pair then we can recognize Berge graphs in polynomial time.

Proof:

Fact: Let $G$ be a graph and let $xy$ be an edge of $G$. Construct $G^+$ from $G$ by adding a vertex $z'$ which sees only $x$ in $G$. Set $G' = G^+ - xy - N(x) \cap N(y)$. Then $xy$ extends to an odd cycle of length at least five in $G$ if and only if $\{x', y\}$ is not an even pair in $G'$.

By the above fact, if there is a polynomial time algorithm $A$ to determine if some specified pair of vertices in a graph is even then we can determine if a graph contains an odd cycle of length at least five in polynomial time using $m$ calls to $A$. Now, a graph $G$ is Berge if and only if neither $G$ nor $\overline{G}$ contains a long odd cycle and the result follows.

Corollary 2.57: If we can determine if a graph has an even vertex in polynomial time, we can determine if a graph is Berge in polynomial time.

Proof: Let $G$ be a graph and let $x$ and $y$ be a specified pair of non-adjacent vertices of $G$. Obtain $G'$ from $G$ by adding a vertex $z$ which sees $x$ and $y$ and misses $G-x-y$.

Then $\{x, y\}$ is an even pair in $G$ if and only if $z$ is an even vertex of $G'$.

Furthermore, even if we had a polynomial-time algorithm to find an even pair in a graph, our decomposition algorithm would not be polynomial. This is because the number of graphs added to our list can be exponential (2 graphs of size $n-1$, 4 graphs of size $n-2$, ...).

Problem: Is there a polynomial time algorithm to recognize QPG graphs?

Problem: Is there a decomposition algorithm similar to 2.4 which determines in polynomial time if a graph is in QPG?
Problem: Is there a polynomial time algorithm to recognize SQPG graphs?

Problem: Is there a decomposition algorithm similar to 2.4 which determines in polynomial time if a graph is in SQPG?
2.12 A Global View

In this chapter, we have been studying various properties of minimal imperfect graphs and related classes of perfect graphs. We list below those properties of minimal imperfect graphs which we have discussed.

(1) No minimal imperfect graph is good.

(2) No minimal imperfect graph contains an even vertex.

(3) No minimal imperfect graph is disconnected or disconnected in the complement.

(4) No minimal imperfect graph contains a dominated vertex.

(5) No minimal imperfect graph contains an innocuous vertex.

(6) No minimal imperfect graph contains a homogeneous set.

(7) No minimal imperfect graph has a clique cutset.

(8) No minimal imperfect graph arises through amalgalm.

(9) No minimal imperfect graph contains a dominating stable set.

(10) Every minimal imperfect graph is unbreakable.

(11) No minimal imperfect graph contains an even pair.

Of course, this list is somewhat redundant. For example, (2) follows trivially from (5) and (1) follows just as easily from (9). In fact, the following observation of Chvátal [Ch85a] shortens the list considerably.

Theorem 2.58: If $G$ is an unbreakable graph with at least 9 vertices then $G$ does not contain a homogeneous set or a dominated vertex. In addition, $G$ has no clique cutset and $G$ does not arise through amalgalm.

Corollary 2.59: (10) implies (8),(4),(6) and (7).

Observation 2.60: No unbreakable graph with more than 9 vertices contains an innocuous vertex.

Proof: This is easy to show using the techniques used in the proof of Theorem 2.12.
Figure 2.12
Corollary 2.8: (10) implies (5) (which implies (2)).

The above remarks suggest that we can restrict our attention to the properties on the following list.

(12) No minimal imperfect graph contains a dominating stable set.

(13) Every minimal imperfect graph is unbreakable.

(14) No minimal imperfect graph contains an even pair.

We have just discussed how the properties of minimal imperfect graphs developed in this chapter are related. We turn now to a comparison of the various classes of perfect graphs.

The classes of graphs we have shown to be perfect are:

- Bipartite graphs, $P_4$ free graphs, graphs with Dilworth number at most 4,
- weakly bipartite graphs, comparability graphs, triangulated graphs,
- clique separable graphs, strongly perfect graphs, Meyniel graphs,
- weakly triangulated graphs, and quasi-parity graphs.

The relationships between these classes of graphs are presented in Fig. 2.12. Note that quasi-parity graphs are known to contain all the other classes except for the class of strongly perfect graphs.

Problem: Is every strongly perfect graph a quasi-parity graph?
2.13 Star Cutsets and Even Pairs

In the previous section, we summarized the results discussed in this chapter. We noted that almost all the structures which cannot occur in a minimal imperfect graph cannot occur in unbreakable graphs. This suggests that star cutsets are an important tool. We also noted that almost all the classes of graphs we discussed were contained in QPG. This suggests that even pairs are an important tool. In this section, we will examine these two tools in more detail and present some related problems.

2.13.1 Skew Partitions

Definition 2.62: A skew partition of a graph $G$ is a partition of the vertices of $G$ into four disjoint non-empty sets, $A, B, C,$ and $D,$ such that:

(i) If $x$ is in $A$ and $y$ is in $B$ then $xy$ is not an edge of $G.$

(Equivalently $B \cup C$ is a cutset of $G$).

(ii) If $x$ is in $B$ and $y$ is in $C$ then $xy$ is an edge of $G.$

(Equivalently $A \cup D$ is a cutset of $G$).

Note that a graph has a skew partition if and only if it has a cutset $S$ such that $S$ is disconnected. Thus, a star-cutset is a special case of a skew partition. A skew partition is a particularly interesting generalization of a star cutset because a skew partition of $G$ is also a skew partition of $G$.

Conjecture 2.63(Chud{al/Ch85a}): No minimal imperfect graph permits a skew partition.

2.13.2 Closure Classes

Definition 2.64: Let $B$ be a class of graphs. We define the star-closure of $B,$ $B^*,$ as follows. A graph $G$ is in $B^*$ if every unbreakable subgraph of $G$ is in $B.$ A graph $G$ is in the Meyniel closure of $B,$ $B^M,$ if for every subgraph $H$ of $G$ either:

(i) $H$ has an even pair,
Figure 2.13

Figure 2.14
(ii) $H$ has an even pair, or

(iii) $H$ is in $B$.

Observation 2.65: If $B$ is a class of perfect graphs than so are $B^*$ and $B^M$.

One interesting class of graphs is $Bip^*$, the star-closure of the bipartite graphs. Chvátal [Ch85a] has shown that every Meyniel graph is in $Bip^*$. Also, clearly every weakly triangulated graph is in $Bip^*$ (see section 2.10). The graph in Fig. 2.13 is in $QPG$ but not $Bip^*$.

Problem: Is $Bip^*$ contained in $QPG$?

The graph $G$ in Fig. 2.14 is perfect but is not quasi-parity. In fact, neither $G$ nor $\overline{G}$ has an even pair. Note that $G$ is the line graph of a bipartite graph ($K_{2,3}$).

Problem: Find perfect graphs which have no even pair and whose complements have no even pair and which are minimal with this property.

The only such graphs we know of are either line graphs of bipartite graphs or the complements of line graphs of bipartite graphs. We denote by $LGBG$ the line graphs of bipartite graphs. Recall that König has shown that $LGBG$ is a class of perfect graphs (see Corollary 2.15).

Definition 2.66: we define the class of Burrs graphs as follows. A graph $G$ is a Burrs graph if for every subgraph $H$ of $G$ one of the following holds.

(i) $H$ or $\overline{H}$ contains an even pair.

(ii) $H$ or $\overline{H}$ has a star cutset.

(iii) $H$ or $\overline{H}$ is in $LGBG$.

Problem: Is every perfect graph a Burrs graph?

(Note that every Burrs graph is perfect.)
CHAPTER 3

ALPHA AND OMEGA
3.0 Lovász’s Theorem and Checkerboard Graphs

In 1972, Lovász proved the Perfect Graph Theorem. A few months later, he strengthened this result and obtained the following characterization of perfect graphs.

**Theorem 3.1 ([Lov72]):** A graph, $G$, is perfect if and only if for each subgraph $H$ of $G$, $\alpha(H)\omega(H) \leq |H|$.

**Proof:** see Appendix E.

**Corollary:** If a graph is perfect, so is its complement.

**Proof:** Note that $\alpha(H) = \omega(\overline{H})$. Thus, $\alpha(\overline{H})\omega(\overline{H}) = \alpha(H)\omega(H)$. The result follows.

It follows immediately from this theorem that if $G$ is minimal imperfect then:

$$(3.2) |G| = \alpha(G)\omega(G) + 1.$$ 

In fact, consider a minimal imperfect graph, $G$, and let $z$ be any vertex of $G$. Since $|G - z| \leq \alpha(G - z)\omega(G - z)$, clearly $\alpha(G - z) = \alpha(G)$ and $\omega(G - z) = \omega(G)$. Also, since $G - z$ is perfect, $G - z$ can be $\omega$-coloured. This implies that $G - z$ can be partitioned into $\omega$ stable sets each of size $\alpha$. Since $\overline{G - z}$ can be $\alpha$-coloured, we can partition $G$ into $\alpha$ cliques each of size $\omega$. Clearly, a clique can intersect with a stable set in at most one point. In fact, by the pigeon-hole principle, every clique intersects with each stable set in precisely one point. Thus, if $G$ is a minimal imperfect graph then:

$$(3.3) \text{for any vertex, } z, \text{ of } G, \ G - z \text{ can be partitioned into } \omega(G) \text{ stable sets of size } \alpha(G) \text{ and } \alpha(G) \text{ cliques of size } \omega(G).$$

We shall call graphs which satisfy (3.2) and (3.3), checkerboard graphs (note that if $G$ is a checkerboard graph, so is $\overline{G}$). In section 3.1 we shall discuss all the graphs which are known to be checkerboards.

Clearly, all checkerboard graphs are imperfect. As we have just seen, all minimal imperfect graphs are checkerboard graphs. Thus:
(3.4) the minimal imperfect graphs are precisely the minimal checkerboard graphs.

In section 3.2, we shall try to find properties which separate the minimal imperfect graphs from general checkerboard graphs. We close this section by describing how 3.4 can be used to show that perfect graph recognition is in Co-NP.

In [BiHT79], Bland, Huang, and Trotter defined partitionable graphs.

Definition 3.5: A graph $G$ is partitionable if there are integers $r$ and $s$, with $r,s \geq 2$, such that for each vertex $x$ of $G$, $G-x$ can be partitioned into $r$ cliques of size $s$ and $s$ stable sets of size $r$.

Clearly, every checkerboard graph is partitionable. Bland et al observed that every partitionable graph is a checkerboard. To see this note that if $G$ is partitionable then for each vertex $x$ of $G$, $\alpha(G-x) = r$ and $\omega(G-x) = s$. Since $r$ and $s$ are both at least two, it follows that $\alpha(G) = r$ and $\omega(G) = s$.

By this observation and 3.4, a graph is imperfect if and only if it contains no partitionable graph. Clearly, partitionable graph recognition is in NP. (We can certify a partitionable graph $G$ by giving, for each vertex $x$ of $G$, the partition of $G-x$.) Cameron and Edmonds[Ca82] pointed out that this implies that perfect graph recognition is in Co-NP.

(The certificate of imperfection of a graph $G$ consists of a partitionable subgraph $H$ of $G$ along with a certificate that $H$ is partitionable).
Figure 3.1: A Wreath Checkerboard Graph

Figure 3.2: Two Other Checkerboard Graphs
3.1 Known Checkerboards

We begin by describing a simple family of checkerboard graphs.

Definition 3.1: For positive integers $k$ and $n$ with $k < n/2$, $C^k_n$ is the graph with vertices $v_1, v_2, \ldots, v_n$ where $v_i$ sees $v_j$ if and only if $|i-j| \leq k$ (and $i \neq j$) where arithmetic is modulo $n$.

(Note that $C_{2n+1}^1 = C_{2n+1}^1$ and $C_{2n+1}^n = C_{2n+1}^n$. The complement of $C_{2n+1}^n$ is the odd hole: $v_1, v_{n+2}, v_2, v_{n+3}, \ldots, v_n, v_{2n+1}, v_{n+1}$).

It is easy to see that $\omega(C^k_n) = k + 1$ while $\alpha(C^k_n) = \left\lceil \frac{n}{k} \right\rceil$. Chvátal[Ch76] noted that for any integers $\alpha, \omega$, both greater than one, $C_{\omega+1}^{\omega-1}$ is a checkerboard graph with clique number $\omega$ and stability number $\alpha$. To see this note that $C_{\omega+1}^{\omega-1} - v_{\omega+1}$ can be partitioned into cliques $\{v_1, v_2, \ldots, v_\omega\}, \{v_{\omega+1}, v_{\omega+2}, \ldots, v_{2\omega}\}, \ldots, \{v_{(\omega-1)\omega+1}, v_{(\omega-1)\omega+2}, \ldots, v_{\omega^2}\}$ and stable sets $\{v_1, v_{\omega+1}, \ldots, v_{(\omega-1)\omega+1}\}, \{v_2, v_{\omega+2}, \ldots, v_{(\omega-1)\omega+2}\}, \ldots, \{v_\omega, v_{2\omega}, \ldots, v_{\omega^2}\}$. The result follows by symmetry.

Chvátal also pointed out that we can obtain new checkerboard graphs from $C_{\omega+1}^{\omega-1}$ by adding edges which do not create any new maximum cliques or destroy any maximum stable sets. Since such edges do not change the structure of the maximum cliques and stable sets in the graph, we shall call them ornamental. If $H$ arises from $G$ through the addition of ornamental edges then we say $H$ is a decoration of $G$. If no edge of $G$ is ornamental, we say that $G$ is undecorated. Thus, Chvátal proved that, for $\alpha$ and $\omega \geq 2$, every decoration of $C_{\omega+1}^{\omega-1}$ is a checkerboard graph. We shall call these graphs the wreath checkerboard graphs.

Chvátal, Graham, Perold and Whitesides [ChGPW79] and Bland, Huang, and Trotter [BlHT70] independently discovered the two checkerboard graphs depicted in Fig. 3.2. Neither of these graphs is a wreath checkerboard graph. Chvátal et al also gave a method for constructing an infinite family of checkerboard graphs different from the wreath checkerboard graphs. In addition, they discussed an operation which extends some checkerboard
Figure 3.3: Some More Checkerboard Graphs
graphs into larger checkerboard graphs.

Whitesides [W84b] provided a catalogue of all the checkerboard graphs with clique number three and stability number four. Every such graph is a decoration of one of five basic subgraphs. We have already seen two of these graphs; one is $C_{12}$ (see Fig. 3.1), another is depicted in Fig. 3.2. The other three graphs are depicted in Fig. 3.3.

Lam, Swiercz, Thiel, and Regener [LaSTR79] used a computer to search for checkerboard graphs of low order. They show that there are precisely 12 undecorated checkerboard graphs with $\alpha=4$ and $\omega=4$ and precisely 21 undecorated checkerboard graphs with $\alpha=5$ and $\omega=3$. 
3.2 Which Checkerboards Are Minimal Imperfect.

In the last section, we presented a variety of examples of checkerboard graphs. We also mentioned a procedure which, given, a checkerboard graph, constructs a new larger checkerboard graph. These results suggest that the catalogue of checkerboard graphs may be neither short nor simple. Thus, we may need to use fairly sophisticated tools to winnow out the minimal imperfect graphs from this catalogue. In this section, we divide the known properties of minimal imperfect graphs into two types, those which will help us to separate the minimal imperfect graphs from general checkerboard graphs and those which will not.

3.2.1 Properties which Do Not Separate

Padberg [Pad74] has shown that, in addition to being a checkerboard graph, a minimal imperfect graph, \( G \), must have the following properties:

(3.7) \( G \) contains precisely \( n (= \vert G \vert ) \) cliques of size \( \omega \) and \( n \) stable sets of size \( \alpha \).

Further, we can enumerate the cliques as \( C_1, C_2, \ldots, C_n \) and the stable sets as \( S_1, S_2, \ldots, S_n \) such that \( C_i \cap S_j = 0 \) and \( \vert C_i \cap S_j \vert = 1 \) for \( i \neq j \). (Thus every maximum clique is disjoint from exactly one maximum stable set).

Every vertex of \( G \) is in precisely \( \omega \) cliques of order \( \omega \) and \( \alpha \) stable sets of order \( \alpha \).

Later, Bland, Huang, and Trotter [BIHT79] demonstrated that, in fact, every checkerboard graph satisfied the list of properties in 3.7. They also showed that:

(3.8) If \( G \) is a checkerboard graph and \( S_1 \) and \( S_2 \) are two stable sets of size \( \alpha (G) \) in \( G \) then \( S_1 \cup S_2 - S_1 \) is connected.

In particular, 3.8 implies that if \( S_1 \) and \( S_2 \) are disjoint maximum stable sets then \( S_1 \cup S_2 \) is connected. The latter result was proved first by Tucker [78].

Olaru [O169] was another researcher who studied the properties of minimal imperfect graphs. He developed a substantial list of properties of minimal imperfect graphs before Lovász had proved the PGT. Most of the properties he discovered are subsumed by 3.7.

For example, he showed that every minimal imperfect graph has at least \( n \) maximum cliques and \( n \) maximum stable sets. He also proved that in a minimal imperfect graph:
(3.9) Every vertex has degree at least $2\omega - 2$ and at most $n - 2\alpha + 2$.

We shall show that 3.9 is true for all checkerboard graphs.

**Theorem 3.10:** Let $G$ be a checkerboard graph. Every vertex of $G$ has degree at least $2\omega(G) - 2$ and at most $n - 2\alpha(G) + 2$.

**Proof:** We need only show that every vertex of $G$ has degree at least $2\omega - 2$. Since this is also true for $\overline{G}$, this implies every vertex of $G$ has degree at most $n - 2\alpha(G) - 2$.

**Fact 3.11:** By 3.7, we can enumerate the cliques of $G$ as $C_1, C_2, \ldots, C_n$ and the stable sets of $G$ as $S_1, S_2, \ldots, S_n$ so that $C_i \cap S_j = \emptyset$ if and only if $i = j$. Then, a vertex $x$ belongs to $C_i$ if and only if $S_i$ is one of the colour classes in an $\omega(G)$-colouring of $G - x$.

**Proof:** To establish the "if" part colour $G - S_i$ by assigning $v$ one colour and letting the other colour classes be as in the $\omega(G)$-colouring of $G - v$. Note that $C_i$ must include one vertex of each colour.

To establish the "only if" part, consider an $\omega(G)$-colouring of $G - v$. Since precisely $\omega(G) - 1$ colours intersect $C_i - v$, the remaining colour class must be distinct from $C_i$.

**Fact 3.12:** If $G$ is a checkerboard graph and $x$ is a vertex of $G$ then $N(x)$ is $(\omega(G) - 1)$-colourable.

**Proof:** Take any $S_i$ which includes $x$. Since $N(x)$ contains no clique of size $\omega(G)$, there is a vertex $v$ in $C_i - N(x)$. By the "only if" part of Fact 3.10, $G - v - S_i$ is $\omega(G) - 1$-colourable.

Now, let $x$ be any vertex of $G$. By fact 3.12, $N(x)$ permits an $\omega(G) - 1$ colouring; we need only show that each colour class contains at least two vertices. For this purpose consider an arbitrary vertex $y$ of $N(x)$. Since $G - y$ can be covered by cliques of size $\omega(G)$, some clique $C$ of size $\omega(G)$ contains $x$ but not $y$. Since $C - x \subseteq N(x) - y$, some vertex of $C - x$ must have the same colour as $y$.■
In chapter 2, we saw that the three properties listed below are useful in determining which graphs are minimal imperfect.

(I) No minimal imperfect graph has a dominating stable set.

(II) No minimal imperfect graph contains an even pair.

(III) Every minimal imperfect graph is unbreakable

Unfortunately, as demonstrated by the following three observations, these properties are of absolutely no value in the present context.

**Theorem 3.13:** No checkerboard graph has a dominating stable set.

**Proof:** Let $G$ be a checkerboard graph and $x$ be any vertex of $G$. $G-x$ has a partition into a maximum cliques. Any dominating stable set must hit each of these cliques and thus has size $\alpha$. But, by 3.7, any maximum stable set misses precisely one maximum clique and therefore cannot be dominating. \(\square\)

**Theorem 3.14:** No checkerboard graph contains an even pair.

**Proof:** Let $G$ be a checkerboard graph. Consider a pair of non-adjacent vertices $x$ and $y$ of $G$. Clearly, $x$ is in a maximum stable set $S_1$ in the partitioning of $G-y$. Similarly, $y$ is in a maximum stable set, $S_2$ of $G-x$. Clearly, $y$ is in $S_2-S_1$ and $x$ is in $S_1-S_2$. Now, by 3.8, $S_1-S_2 \bigcup S_2-S_1$ is connected. Thus, there must be an induced $x$ to $y$ path in this subgraph. Since this path alternates between vertices of $S_1$ and vertices of $S_2$, it must be even. \(\square\)

**Theorem 3.15:** Every checkerboard graph is unbreakable.

**Proof:** Assume there are checkerboard graphs with star cutsets. Let $G$ be such a graph. Let $S$ be a star-cutset of $G$ with centre $x$. If $x+N(x)$ is not a star cutset of $G$ then some vertex $y$ in $N(x)$ is dominated by $x$. But then $x$ is in every maximum clique $y$ is in, a contradiction. Thus $G-N(x)-x$ splits into components $U_1, U_2, \ldots, U_k$. To obtain a maximum stable set containing $x$, we select a maximum stable set from each of these.
components and add $x$ to their union. If, for some $i$, $U_i$ had only one maximum stable set then each vertex of this stable set would be in every stable set that $x$ is in, a contradiction. Thus, $U_1$ contains two distinct maximum stable sets, $A_1$ and $A_2$, and $U_2$ contains two distinct maximum stable sets, $B_1$ and $B_2$. We can extend $A_1 + B_1 + x$ into a maximum stable set, $S_1$ of $G$ by appending maximum stable sets from each $U_i$ for $i \geq 3$. Similarly, we can extend $A_2 + B_2 + x$ into a maximum stable set $S_2$ of $G$. Now, by 3.8, $S_1 - S_2 \cup S_2 - S_1$ is connected. But this set contains vertices from $U_1$ and $U_2$ and no vertices of $N(x)$, a contradiction. Thus, no checkerboard graph contains a star cutset and, since the complement of a checkerboard graph is a checkerboard graph, the result follows.

3.2.2 Properties which Separate

The above remarks imply that many of the properties which hold on minimal imperfect graphs are properties of checkerboard graphs and not directly related to minimal imperfection. There are, however, three properties of minimal imperfect graphs which do not hold on all checkerboard graphs. The first was discovered by Chvátal[Ch76] when he was studying wreath checkerboard graphs.

Definition 3.16: A hitting set of a graph $G$ is a subset of its vertices which meets all maximum cliques and all maximum stable sets. A hitting set of $G$ is small if it contains no more than $\alpha(G) + \omega(G) - 1$ vertices.

Observation 3.17: No minimal imperfect graph has a small hitting set.

Proof: Let $G$ be a checkerboard graph with a small hitting set $S$. Let $H$ be the graph induced by $V - S$. Since $S$ is a hitting set, $\alpha(H) \leq \alpha(G) - 1$ and $\omega(H) \leq \omega(G) - 1$. Since $S$ is small,

$$|H| \geq |G| - \alpha(G) - \omega(G) + 1$$

$$\geq \alpha(G) + \omega(G) - \alpha(G) - \omega(G) + 2$$

$$\geq (\alpha(G) - 1)(\omega(G) - 1) + 1$$
Figure 3.4: A Checkerboard Graph with Anti-twins
Thus, $H$ is imperfect and $G$ is not minimal imperfect.

Chvátal used the above observation to show that the only minimal imperfect wreath checkerboard graphs are the odd disks.

**Theorem 3.18:** If $G$ is a decoration of $G_\infty^{\omega_1}$, for some $\alpha \geq 3, \omega \geq 3$, then $G$ has a small hitting set.

**Corollary 3.19:** The only minimal imperfect wreath checkerboard graphs are the odd disks.

The second property of minimal imperfect graphs which does not hold on all checkerboard graphs was defined by Olariu [Ol86].

**Definition 3.20:** Two vertices $x$ and $y$ in a graph $G$ are anti-twins if every vertex of $G - x - y$ sees precisely one of them.

**Theorem 3.21** ([Ol86]): No minimal imperfect graph contains a pair of anti-twins.

**Observation 3.22:** The checkerboard graph in Fig. 3.4 contains anti-twins.

In [ChS86], Chvátal and Sbihi defined homogeneous pairs. In the same paper, they showed that no minimal imperfect graph contains a homogeneous pair.

**Definition 3.23:** A homogeneous pair, in a graph $G$, is a pair of disjoint subsets, $Q_1$ and $Q_2$, of $V$ such that:

(i) If $x$ is an element of $G - Q_1 - Q_2$, then $x$ either sees all of $Q_1$ or misses all of $Q_1$.

(ii) At least one of $Q_1$ or $Q_2$ has two elements, and

(iii) there are at least two vertices in $G - Q_1 - Q_2$.

Note that if $x_1$ and $x_2$ are anti-twins, in a graph $G$ with at least five vertices, then $(Q_1 = N_{G-x_1}(x_2), Q_2 = N_{G-x_2}(x_1))$ is a homogeneous pair. Thus the checkerboard graph in Fig. 3.4 contains a homogeneous pair.
Figure 3.5: A Checkerboard Graph with a Skew Partition
The three above properties are, to my knowledge, the only (natural) properties of minimal imperfect graphs which do not hold on all checkerboard graphs. We would very much like to find more such properties. We conclude this section by suggesting two possibilities. As we mentioned in section 2.13, Chvátal conjectured that no minimal imperfect graph permits a skew partition.

Observation 3.24: The checkerboard graph in Fig. 9.5 has a skew partition.

Meyniel conjectured that no minimal imperfect graph contains an odd pair.

Observation 3.25: If two non-adjacent vertices are anti-twins they are an odd pair. Thus, the checkerboard graph in Fig. 9.4 contains an odd pair.
CHAPTER 4

THE SPGC FOR SPECIAL CLASSES OF GRAPHS
4.0 Special Classes

As we saw in chapters 2 and 3, perfect graph theorists have developed a substantial list of properties of minimal imperfect graphs. Unfortunately, this list building process has not yet led to a proof of the SPGC. It has, however, allowed researchers to prove the SPGC in restricted settings.

Definition 4.1: said to hold for a class of graphs $A$ if the following statement is true:

A graph in $A$ is perfect if and only if it contains no odd disc.

In this chapter, we discuss two of the classes of graphs on which the SPGC has been shown to hold. These are the planar graphs and the claw-free graphs.

Since the SPGC holds on the planar graphs, planar Berge graphs are perfect. Similarly, for each class of graphs on which the SPGC holds, we obtain a new class of Berge graphs which are perfect. In Chapter 2, we proved classes of graphs were perfect by considering properties of minimal imperfect graphs. In this chapter, we shall prove that classes of Berge graphs are perfect by considering properties of minimal imperfect Berge graphs. Any property of minimal imperfect graphs is also a property of minimal imperfect Berge graphs. However, there are some properties of minimal imperfect Berge graphs which do not hold on all minimal imperfect graphs. A trivial example of such a property is "has clique number at least 3". We mention two other such properties below. (Of course, if the SPGC is true then minimal imperfect Berge graphs have all conceivable properties simply because they do not exist.)

Theorem 4.2(Tucker/Tuc88): No minimal imperfect Berge graph has a cutset which induces a stable set.

Observation 4.3: In an odd hole of length at least five, every stable set is a cutset.

Theorem 4.4(Tucker/Tuc77?): If $G$ is a minimal imperfect Berge graph and some edge, $xy$, of $G$ does not extend into a triangle then $G-xy$ is also minimal imperfect and Berge.
Proof: See Appendix F.

Observation 4.5: No edge of a long odd hole extends into a triangle but if we remove any edge of a long odd hole we obtain a perfect graph.

As with all classes of perfect graphs, we would like to be able to recognize the classes of graphs discussed in this chapter in polynomial time. In fact, we chose to discuss planar perfect graphs and claw-free perfect graphs precisely because there are recognition algorithms for these two classes of graphs. These algorithms are based upon algorithm 2.4.
4.1 Planar Graphs

4.1.1 The SPGC Holds

In this section, we provide a proof that all planar Berge graphs are perfect. As we mentioned in the last section, the clique number of any minimal imperfect Berge graph is at least 3. In fact, by the following lemma, the clique number of a minimal imperfect planar Berge graph must be exactly 3.

**Lemma 4.6:** Every planar graph contains a vertex of degree at most 5.

*Proof:* This follows from a classic result due to Euler.

**Corollary 4.7:** A minimal imperfect Berge planar graph has clique number 3.

*Proof:* Recall that, by 3.10, we know that in a minimal imperfect graph with clique number ω every vertex has degree at least 2ω−2.

We shall now consider some properties of minimal imperfect graphs with clique number 3.

**Lemma 4.8:** Let $G$ be a minimal imperfect graph such that $\omega(G)=3$. For any vertex $z$ of $G$, the subgraph induced by $N(z)$ has exactly 9 edges.

*Proof:* Since $G$ is a checkerboard graph, by 3.7, every vertex of $G$ is in precisely 3 triangles.

**Lemma 4.9:** Let $G$ be a minimal imperfect graph with clique number 3. Let $z$ be any vertex of $G$. In any 3 colouring of $G-z$, $N(z)$ contains a $\{1,2\}$ edge, a $\{1,3\}$ edge and a $\{2,3\}$ edge.

*Proof:* Assume $N(z)$ does not contain a $\{1,2\}$ edge. Consider $F=\{\text{vertices in } G-z \text{ with colour 1 or 2}\}+z$. Clearly, $F$ contains no triangle and is therefore bipartite. Thus, we can 2-colour $F$. Colouring the rest of $G$ with a third colour gives a 3-colouring of $G$. This contradiction implies that $N(z)$ contains a $\{1,2\}$ edge. Lemma 4.9 follows by symmetry.
Figure 4.1

Figure 4.2
Lemmas 4.8 and 4.9 apply to all minimal imperfect graphs with clique number 3. We shall now restrict our attention to planar graphs. In this context, the following lemmas will be useful. (We note that we do not really need lemma 4.10. However, it makes the proof which follows much easier to understand).

**Lemma 4.10 (Fáry/1948):** Every planar graph can be embedded in the plane so that edges are straight lines.

**Lemma 4.11:** Let $G$ be a minimal imperfect planar graph embedded in the plane so that the edges of $G$ are straight lines. Consider a vertex $x$ in $G$. Enumerate the neighbors of $G$ as $v_0, v_1, \ldots, v_d(x)$ by taking a ray from $x$ and rotating it 360 degrees and listing the vertices in the order which the ray hit them. Then the only possible edges between neighbors of $G$ are $v_i v_{i+1}$ for $i=0, \ldots, d(x)-1$ and $v_0 v_d(x)$.

Proof: Assume $v_i v_j$ is an edge where $j \neq i+1, j \neq i-1$ (addition modulo $d(x)$). Since edges cannot cross, $\{x, v_i, v_j\}$ is a cutset of $G$. But $\{x, v_i, v_j\}$ is a clique, contradicting the minimal imperfection of $G$. Thus, no such edges can occur.

We are now ready to prove the following theorem. This theorem is originally due to Tucker, and most of the machinery for this proof is drawn from his work ([Tuc73], [Tuc77]).

**Theorem 4.12:** Every planar Berge graph is perfect.

Assume theorem 4.12 is false; that is, there exists a minimal imperfect planar Berge graph. By 4.4, we can remove edges from this graph to obtain a minimal imperfect planar Berge graph $G$ such that every edge of $G$ extends into a triangle. Now, by 4.7, we know that $\omega(G)=3$. By lemma 4.6, some vertex, $x$, of $G$ has degree at most 5. It follows from 4.8 and 4.11 that $x + N(x)$ induces one of the graphs depicted in Fig. 4.1.

Consider a 3-colouring of $G - x$. Combining lemmas 4.9 and 4.11, we see that $x + N(x)$ induces one of the three coloured subgraphs depicted in Fig. 4.2, where the vertices are enumerated as in lemma 4.11. We shall show that none of these three colourings is possible. In doing so, the following definition will be useful.
Definition 4.18: Let $i$ and $j$ be distinct colours in our colouring of $G$. By $G_{ij}$, we mean the subgraph induced by those vertices which are coloured with either $i$ or $j$.

Case 1: $x+N(x)$ induces Fig. 4.2.1

Clearly $v_1$ and $v_0$ must be connected in $G_{12}$. Otherwise we could swap colours in the component of $G_{12}$ containing $v_0$ to obtain a colouring of $G-x$ which contradicts lemma 4.9. Let $P_1$ be a minimal length path from $v_1$ to $v_0$ in $G_{12}$. Clearly $P_1$ is even. Now $P_1$ cannot include $v_2$ as otherwise $P_1-v_1+x$ would be an odd hole. In fact, $v_2$ cannot see a vertex on $P_1-v_1$ for the same reason. Using analogous arguments, we can show there is a $\{1,3\}$ path, $P_2$, from $v_1$ to $v_0$ such that $v_3$ sees no vertex in $P_2-v_0$. Thus, we have the situation depicted in Fig. 4.3. We obtain a new path $P_3$ from $P_1$ and $P_2$ by starting along the inside path and every time we come to an intersection continuing along the inside path. We note that since $P_3$ is the inside path, $v_2$ and $v_3$ can only see the vertices of $P_3$ where $P_1$ and $P_2$ intersect. But since $v_2$ misses $P_1$ and $v_3$ misses $P_2$, we see that there are no edges from $\{v_2,v_3\}$ to $P_3$. Now, we obtain a chordless path $P$ from $P_3$ by replacing parts of the path by chords as necessary. If $P$ is odd then $P+x$ is an odd hole. Otherwise, $P+v_3+v_0$ is an odd hole. In either case we contradict the fact that $G$ is Berge. Thus, this case cannot occur.

Case 2: $x+N(x)$ induces Fig. 4.2.2 or Fig. 4.2.3

In this case, as in case 1, we can find a $\{1,2\}$ path, $P_1$, between the two vertices in $N(x)$ of colour 1. As before, this path does not use any vertex of $N(x)$ as an interior point. Similarly, we can find a $\{1,2\}$ path, $P_2$, between the two vertices in $N(x)$ of colour 2 which uses no vertex of $N(x)$ as an interior point. If these two paths intersected then we would have a $\{1,2\}$ path between two non-adjacent vertices of $N(x)$ of different colours with all its interior points outside $N(x)$. However, this path would form an odd hole with $x$. Thus the two paths do not intersect. In particular this implies that the neighborhood of $x$ cannot be as depicted in Fig. 4.2.2, and we must have the situation depicted in Fig. 4.4. Note that there cannot be any edges between
Now, as in case 1, we can find a \( \{1,3\} \) path \( P_2 \) from \( v_0 \) to \( v_1 \) such that \( v_4 \) misses \( P_2-v_0 \) and construct a path \( P \) from \( P_1 \) and \( P_2 \). Similarly, we can find a \( \{2,3\} \) path, \( P_4 \), from \( v_2 \) to \( v_4 \) such that \( v_4 \) misses \( P_4-v_3 \) and construct a path \( P' \) from \( P_3 \) and \( P_4 \).

Now clearly, \( P \) and \( P' \) do not intersect and there are no edges between these two paths (by the above paragraph and the fact that \( P \) lies inside or on \( P_1 \) while \( P' \) lies inside or on \( P_2 \)). Also, since \( v_4 \) misses \( P_2-v_0 \) and \( P_4-v_3 \), \( v_4 \) misses \( P+P'-v_4-v_0 \).

Now, if \( P \) is odd then \( P+x \) is an odd hole. Similarly, if \( P' \) is odd then \( P'+x \) is an odd hole. However, if \( P' \) and \( P \) are both even then \( P+v_4+P' \) forms an odd hole.

In any case, we contradict the fact that \( G \) is Berge. This final contradiction completes the proof of Theorem 4.12.

4.1.2 Recognition

We have just seen that planar Berge graphs are perfect. Hsu [Hs84a] has developed an algorithm to recognize planar perfect graphs which runs in \( O(n^5) \) time. He showed that planar perfect graphs conform to Paradigm II and thus can be recognized by Algorithm 2.4. He builds up all planar perfect graphs from the planar comparability graphs and planar line graphs of bipartite graphs. To decompose a graph he first finds a cutset of size at most four in it. He then splits the graph up according to the type of cutset.
Figure 4.5: A Claw
4.2 Claw Free Graphs

4.2.1 The SPGC Holds

A claw is simply a $K_{1,3}$. In 1976, Parthasarathy and Ravindra [Par76] showed that every minimal imperfect Berge graph contains a claw. In this section, we present a slightly simpler proof based on their ideas. We also discuss a polynomial time algorithm for recognizing claw-free perfect graphs which was developed by Chvátal and Sbihi [ChS85].

**Theorem 4.14:** Every minimal imperfect Berge graph contains a claw.

Proof: Assume the theorem is false. Then, let $G$ be a claw-free minimal imperfect Berge graph. Let $x$ be any vertex of $G$. Since $G$ is claw-free, $\overline{N(x)}$ contains no triangle; since $G$ is Berge, $\overline{N(x)}$ contains no odd odd cycle of length greater than 3. Thus, $\overline{N(x)}$ is bipartite. It follows that $N(x)$ can be partitioned into two cliques. Recall that in a minimal imperfect graph, every vertex has at least $2\omega - 2$ neighbors. Since any clique in $N(x)$ has at most $\omega - 1$ elements, $N(x)$ must have precisely $2\omega - 2$ elements which can be partitioned into two cliques, $C_1$ and $C_2$, each of order $\omega - 1$.

**Lemma 4.15:** If $C$ is an $\omega - 1$ clique in $N(x)$ then $N(x) - C$ is an $\omega - 1$ clique.

Proof: Let $C$ be an $\omega - 1$ clique in $N(x)$. Assume $N(x) - C$ is not a clique. Then, there exist a pair of non-adjacent vertices, $r$ and $s$ in $N(x) - C$. Now, $C + z$ is an $\omega$-clique. By 3.7, we know there exists a unique $\alpha$-stable set, $S$, such that $S \cap C + z = \phi$. Clearly, $S \cap G - N(z)$ is not empty (since $\alpha(N(z)) = 2$). Choose some $y$ in $S - N(x)$ and colour $G - y$. Since $\alpha(N(z)) = 2$, any colour appears at most twice in $N(x)$. Since $C + z$ is an $\omega$ clique in $G - y$, every colour appears once in $C + z$. It follows that $r$ and $s$ have distinct colours (or their colour would appear three times in $N(x)$). Let $r'$ be the vertex of $C$ with the same colour as $r$ and let $s'$ be the vertex of $C$ with the same colour as $s$. Now $r$ sees $s'$ as otherwise $\{x, r', r, s\}$ would be a claw. Similarly, $s$ sees $r'$ as otherwise $\{x, r', r, s\}$ would be a claw.
Figure 4.6
Remember that \( I = \{ z \mid z \text{ has the same colour as colour } r \} \) is a stable set of size \( \alpha \). If \( s \) missed \( I - r' \), then \( I - r' + s \) would be an \( \alpha \)-stable set disjoint from \( G + x \). This would imply that \( I - r' + s = S \). But this is a contradiction since \( y \) is in \( S \) but not \( I - r' + s \). Thus \( s \) sees some \( r' ' \) in \( I - r' \). Similarly, \( r \) sees some \( s' ' \) in \( \{ z \mid z \text{ has colour } c_s \} - s' ' \). Now, \( G \) is unbreakable so \( G - x - N(x) \) is connected. It follows that there is a path, \( P \), from \( s' ' \) to \( r' ' \) in \( G - x - N(x) \). Thus, we have the situation depicted in Fig. 4.6. It is a tedious but routine matter to verify that since \( G \) is claw free, there is an odd hole in \( P + x + s + r + s' + r' \) (see Appendix G). However, \( G \) is Berge. This contradiction implies that there exists no pair of non-adjacent vertices in \( N(x) - C \) and thus, \( N(x) - C \) is a clique.

Now, by the above lemma, every \( \omega - 1 \) clique in \( N(x) \) is one half of a bipartition of \( N(x) \). Also, since \( G \) is unbreakable, \( N(x) \) is connected. Moreover, it is easy to see that a connected bipartite graph has a unique bipartition. It follows that \( N(x) \) contains exactly two \( \omega - 1 \) cliques. Thus, \( x \) is in precisely \( 2 \omega \) cliques. Now, since \( G \) is a checkerboard graph, \( \omega(G) = 2 \). But this implies that \( G \) is bipartite and hence perfect, a contradiction.

### 4.2.2 Recognition

Chvátal and Sbihi [ChS85] have shown that claw-free perfect graphs can be recognised in polynomial time. Actually, they recognize a class of graphs \( A \), which conforms to Paradigm II. They also show (and this is the core of their result) that every claw-free perfect graph is in \( A \). It follows that \( G \) is a claw-free perfect graph if and only if \( G \) is in \( A \) and \( G \) is claw-free. Clearly, we can check if a graph has a claw in \( O(n^4) \) time. Thus if we can recognize \( A \) in polynomial time, we can recognize claw-free perfect graphs in polynomial time.

We shall now describe \( A \).
Definition 4.18: A graph is called elementary if its edges can be coloured red and blue such that if $xy$ and $yz$ are edges of $G$ and $xz$ is not then $xy$ and $yz$ receive different colours.

Definition 4.17: We define a class $A$ of graphs as follows. A graph $G$ is in $A$ if and only if for each subgraph $H$ of $G$, either $H$ is elementary, $H$ is weakly triangulated, or $H$ has a clique cutset.

As mentioned earlier, the following two statements are true.

Observation 4.18: Every graph in $A$ is perfect.

Proof: Clearly, every elementary graph is claw-free Berge. Thus, by 4.14, every elementary graph is perfect. Now, recall that every weakly triangulated graph is perfect. It follows that every graph in $A$ is perfect.

Theorem 4.19 ([ChS85]): Every claw free perfect graph is in $A$.

Clearly, we can recognize $A$ using algorithm 2.4. The operation used to build this class is clique identification. The bricks used are the elementary graphs and the weakly triangulated graphs. The following observations imply that we can determine if a graph is in $A$ in $O(n^3)$ time.

Observation 4.20: We can determine if a graph is elementary in $O(n^3)$ time.

Observation 4.21: We can determine if a graph is weakly triangulated in $O(n^6)$ time.

Observation 4.22: A clique cutset tree of a graph with $n$ vertices can have at most $n^2+1$ nodes.

Observation 4.23: We can find a clique cutset in a graph or determine that the graph does not have one in $O(n^5)$ time.
CHAPTER 6

A SEMI-STRONG PERFECT GRAPH CONJECTURE
Figure 5.1
5.0 $P_4$-Isomorphism and a Semi-Strong Perfect Graph Conjecture

In 1982, Chvátal [Ch84b] interposed the following conjecture between the WPGC and the SPGC.

**Definition 5.1:** Two graphs $G$ and $H$ on the same vertex set are said to be $P_4$-isomorphic (and $G$ is $P_4$-isomorphic to $H$) if a set of four vertices induces a $P_4$ in $G$ if and only if it induces a $P_4$ in $H$.

(Actually, we have modified the definition somewhat for the sake of simplicity. Chvatal called $G$ and $H$ $P_4$-isomorphic if the vertices of $G$ could be relabelled so that the above situation occurs. In Appendix H, we present his original definition and discuss the complexity of this relabelling process.)

**Conjecture 5.2:** If $G$ is $P_4$-isomorphic to a perfect graph then $G$ is perfect.

Note that the complement of the $P_4$ $v_0v_1v_2v_3$ is the $P_4$ $v_1v_2v_0v_3$. Thus, $\overline{G}$ is $P_4$-isomorphic to $G$ (However there are $P_4$-isomorphic graphs $G$ and $H$ such that $G$ is neither $H$ nor $\overline{H}$; for example see Fig. 5.1). So, 5.2 implies the WPGC.

**Lemma 5.8([Ch84b]):** The only graph $P_4$-isomorphic to an odd disk is an odd disk of the same order.

We shall prove an extension of lemma 5.3 in section 5.1(lemma 5.7). By Lemma 5.3, the SPGC implies 5.2.

Since 5.2 falls between the WPGC and the SPGC, Chvátal [Ch84b] called it a Semi-Strong Perfect Graph Conjecture (SSPGC). The main result in this thesis is a proof of the SSPGC. This will be presented in Chapter 6. In the remainder of this chapter, we outline the progress which had been made on the SSPGC before it was settled.
Figure 5.2
5.1 The SSPGC for Special Classes of Perfect Graphs

Let \( A \) be a class of perfect graphs. Then, the statement,

\[ (5.4) \text{ If } G \text{ is } P_4\text{-isomorphic to a graph in } A \text{ then } G \text{ is perfect,} \]

is implied by the SSPGC. If \( A \) is the class of all cliques, then 5.4 is simply Seinsche's Theorem (see section 2.2). Thus, Seinsche Theorem provides support for the SSPGC. David Avis suggested that additional support would be provided by proving 5.4 for the class of bipartite graphs.

This was done by Chvátal and Hoang [ChH85]. Actually, they proved the following stronger theorem.

**Theorem 5.5:** Let the vertices of a graph \( G \) be coloured in such a way that each induced \( P_4 \) in \( G \) has an even number of vertices of each colour. Then \( G \) is perfect if and only if both of the two subgraphs induced by all the vertices of the same colour are perfect.

**Corollary 5.8:** If \( G \) is isomorphic to a bipartite graph then \( G \) is perfect.

**Proof:** Let \( G \) be a graph which is \( P_4\)-isomorphic to a bipartite graph \( H \). The vertices of \( H \) can be bicoloured so that there are no edges between vertices of the same colour. This implies that any \( P_4 \) in \( G \) has two vertices of each colour. Since \( G \) is \( P_4\)-isomorphic to \( H \), if we colour the vertices of \( G \) in the same manner, every \( P_4 \) in \( G \) will have precisely two vertices of each colour. Thus, by 5.5, \( G \) is perfect if and only if the two subgraphs induced by all the vertices of the same colour are perfect. Now, both these graphs are \( P_4\)-free and therefore perfect. It follows that \( G \) is perfect as required.

Hoang proved that 5.4 holds true when \( A \) is the class of line graphs of bipartite graphs.

The analogous result for triangulated graphs is implied by the following lemma.

**Lemma 5.7 (Hayward/Hay86b):** The only graphs \( P_4\)-isomorphic to a disk \( D \), of size at least 7 are \( D \) and \( \overline{D} \). Further, any graph isomorphic to \( C_6 \) is a \( C_6 \) and the only graphs isomorphic to \( C_6 \) are the four depicted in fig. 5.2.
Proof: Note that if $D$ is a $C_5$ and $H$ is $P_4$-isomorphic to $D$ then $H$ need only be isomorphic to $D$. However, if $D$ is a disk of size at least 7 and $H$ is $P_4$-isomorphic to $D$ then $H$ is $D$ or $\overline{D}$.

The proof relies on the following 8 observations. The routine and tedious proofs of these observations are omitted.

Observation 1: Any graph $P_4$-isomorphic to $C_5$ is a $C_5$.

Observation 2: The only graphs $P_4$-isomorphic to $C_6$ are those depicted in Fig. 5.2.

Observation 3: The only graphs isomorphic to $P_6$ are those depicted in Fig. 5.3.

Observation 4: The only graphs $P_4$-isomorphic to a $P_6$ are those depicted in Fig. 5.4.

Observation 5: The only graphs $P_4$-isomorphic to a $P_7$ are those depicted in Fig. 5.5.

Furthermore, if the $P_7$ is labelled as in Fig. 5.5 then the graphs must be labelled as depicted in Fig. 5.5.

Observation 6: If $H$ is $P_4$-isomorphic to a $C_7$ with vertices $\{1,2,\ldots,7\}$ and edges $\{12,23,\ldots,67,71\}$ then $H$ is this $C_7$ or its complement. If $H$ is $P_4$-isomorphic to a $C_9$ with vertices $\{1,2,\ldots,9\}$ and edges $\{12,23,\ldots,78,81\}$ then $H$ is this $C_9$ or its complement.

Observation 7: Neither $A$ nor $\overline{A}$ (of Fig. 5.5) can be extended to a graph which isomorphic to a $P_9$.

Now, let $D$ be a cycle of length $k \geq 9$, with vertices $\{1,2,\ldots,k\}$ and edges $\{12,23,\ldots,(k-1)k,k1\}$.

Observation 8: Let $H$ be a graph which is $P_4$-isomorphic to $D$. If vertices 1 through 7 in $H$ induce a $P_7$ then $H$ is $D$. If vertices 1 through 7 in $H$ induce a $\overline{P_7}$ then $H$ is $\overline{D}$.

Now, let $H$ be a graph which is $P_4$-isomorphic to $D$. We shall show that $H$ is either $D$ or $\overline{D}$. Assume not. Note first that observations 7 and 8 imply that every set of 7 consecutive vertices in $H$ induce either $B$ or $\overline{B}$. It is easy to see that if vertices 1 through
Figure 5.5

B (two labellings)

\[ B_1 \]

\[ B_2 \]

\[ \overline{B} \] (two labellings)

\[ \overline{B}_1 \]

\[ \overline{B}_2 \]
7 induce $B$ then vertices 2 through 8 induce $B$. Thus we can shift the labels by 1 to ensure that vertices 1 through 7 induce $B$.

**Case 1: Vertices 1 through 7 are labelled as in $B_1$.**

Since $\{8,7,5,3\}$ induces a $P_4$ in $H$ and 1 misses $\{7,6,5\}$ in $H$, if 18 is an edge of $B$ then 1 would form a $P_4$ with 3 vertices of $\{8,7,5,3\}$. But this contradicts the $P_4$ isomorphism of $D$ and $H$. Thus 18 is not an edge of $H$.

We show by induction that:

(*) 1 $j$ is not an edge of $H$ for $j > 2$.

Note that (*) holds for $j$ at most 8. Let $j'$ be the first integer for which (*) fails.

Note that $\{j' - 6, j' - 5, \ldots, j'\}$ induces either a $B_1$, a $B_2$, a $B_1$ or a $B_2$ (where $j' - 6$ plays the role of $j$). Assume first that these seven vertices induce $B_1$. Since 1 misses 1 for $i < j'$, we know that $\{1, j', j' - 1, j' - 3\}$ induces a $P_4$ in $H$. But this contradicts the $P_4$ isomorphism of $D$ and $H$. The other three cases can be handled similarly.

Applying induction again, we note that 2 must see $j$ for $4 \leq j \leq k - 1$. Assume not and let $j'$ be the first counter-example. Now, clearly 2 and $j'$ have a common neighbor, $i$, with $4 \leq i < j'$. But then 1, 2, $i$, $j'$ forms a $P_4$ in $H$ but not $D$ - a contradiction.

Also, since $\{k, 1, 2, 3\}$ induces a $P_4$ in $H$ and $k$ misses 1, we know that 2 sees $k$ in $H$. But, now, 2 sees 1, $k - 1$, and $k$ in $H$ while $\{k - 1, k, 1, 2\}$ induces a $P_4$ in $D$ - a contradiction.

**Case 2: Vertices 1 through 7 induce a $B_2$.**

A $B_2$ is simply the reverse ordering of a $B_1$. Thus, by shifting labels appropriately, we find that this case is equivalent to case 1.

**Corollary 5.8:** If $G$ is $P_4$ isomorphic to a triangulated graph then $G$ is weakly triangulated.

**Corollary 5.9:** If $G$ is $P_4$ isomorphic to a triangulated graph then $G$ is perfect.
5.2 The $P_4$s in Minimal Imperfect Graphs

The SSPGC motivated a number of results concerning the $P_4$s of minimal imperfect graphs. By 5.5, the vertices of a minimal imperfect graph cannot be two-coloured so that every $P_4$ contains an even number of vertices of each colour. Hoang [Ho85b] showed that the vertices of a minimal imperfect graph cannot be two-coloured so that every $P_4$ contains an odd number of vertices of each colour. The following theorem of Chvátal generalizes both these results.

Definition 5.10: Vertices $x$ and $y$ of a graph are siblings if there is a set $S$ of three vertices such that both $S + x$ and $S + y$ induce $P_4$s.

Theorem 5.11 ([Ch85b]): The vertices of a minimal imperfect graph cannot be two coloured so that every two siblings belong to the same class.

Chvátal also proved the following theorem.

Theorem 5.12 ([Ch85b]): Let $x$ and $y$ be vertices of a minimal imperfect graph, then there is a sequence of $P_4$s $S_1, S_2, \ldots, S_k$ such that $x$ is in $S_1$, $y$ is in $S_2$, and, for $1 \leq i \leq k-1$, $S_i$ and $S_{i+1}$ have three vertices in common.

We close this section with a conjecture about the $P_4$s in minimal imperfect graphs which is equivalent to the SPGC.

Conjecture 5.13: A minimal imperfect graph with $n$ vertices contains precisely $n$ $P_4$s.

That 5.13 is equivalent to the SPGC is an easy consequence of a theorem of Cliariu ([Ol86], Theorem 3.2.1).
CHAPTER 6

A SEMI-STRONG PERFECT GRAPH THEOREM
6.0 A Semi-Strong Perfect Graph Theorem

If the SSPGC fails, there must exist $P_4$-isomorphic graphs $G$ and $H$ such that $G$ is perfect and $H$ is minimal imperfect. Trivially, $G$ is not $H$ and, by the Perfect Graph Theorem, $G$ is not $\overline{H}$. We shall prove the SSPGC by showing that a graph $H$, that is $P_4$-isomorphic to a graph which is neither $H$ nor $\overline{H}$, must have one of three properties a minimal imperfect graph cannot have.

First, no minimal imperfect graph with more than five vertices contains a $C_6$. Second, as shown by Chvátal[Ch85a], no minimal imperfect graph contains a star-cutset (Recall that a star cutset is a cutset in which some vertex, the centre, sees all the other vertices of the cutset). An endomorphism of a graph $G$ is a mapping $f$ which maps the set $V$ of vertices of $G$ into itself in such a way that $f(u)$ and $f(v)$ are adjacent whenever $u$ and $v$ are. The endomorphism is proper if the image of $V$ is a proper subset of $V$. It is not difficult to show (as we shall do in section 6.1.3) that no minimal imperfect graph has a proper endomorphism.

We actually prove the following:

**Theorem 6.1**: Let $G$ and $H$ be $P_4$-isomorphic graphs such that $G$ is neither $H$ nor $\overline{H}$. Then at least one of the following holds:

(i) $H$ contains a proper induced subgraph isomorphic to $C_6$.

(ii) $H$ or $\overline{H}$ has a star cutset.

(iii) $H$ or $\overline{H}$ has a proper endomorphism.

There are two key lemmas which are used as stepping stones in the proof. These are worth stating as independent theorems and are proved in Sections 6.2 and 6.3, respectively. Section 6.1 presents some terminology and basic observations which will be used throughout the proof. Section 6.4 explains how the various pieces fit together to provide a proof of theorem 6.1.
6.1 Basics

6.1.1 $P_4$-Isomorphism

We will need some new terminology to deal with the concepts of $P_4$-isomorphism. Consider two vertices, $x$ and $y$, contained in the common vertex set $V$ of two $P_4$-isomorphic graphs $G$ and $H$. If $x$ sees $y$ in one graph and $x$ misses $y$ in the other then we say $x$ is variant with respect to $y$ or that $xy$ is a variant pair. If $x$ misses $y$ in both graphs or $x$ sees $y$ in both graphs then $xy$ is an invariant pair. We extend this terminology to sets. A set $S$ contained in $V$ is called invariant if, for every $x, y$ in $S$, $xy$ is an invariant pair. This is equivalent to requiring that $G$ and $H$ induce exactly the same graph when restricted to the vertices of $S$. We define an invariant path to be an invariant set which induces a path. We can similarly define invariant hole, invariant stable set, and so on.

Recall that Chvátal [Ch84b], in order to demonstrate that the SSPGC was implied by the SPGC, showed that the only graphs $P_4$-isomorphic to a hole of odd length at least five were the hole itself and its complement. Hayward [Hay86b] extended this result to all holes of length at least seven. These results suggest that large holes and their complements play an important part in the determination of $P_4$-isomorphism. This motivates the following definitions and observations.

**Observation 1.1:** A disc with $k$ vertices such that $k > 6$ is $P_4$-isomorphic only to itself and its complement.

This is simply a restatement of Hayward's result (lemma 5.7).

Given a disc $D$ in a graph $G$, we can partition the vertices of $G - D$ into three sets by considering their neighborhoods on $D$. $U_D^G$, the universal vertices, are those which see all the vertices of $D$. $R_D^G$, the remote vertices, are those which miss all the vertices of $D$. $M_D^G$, the mixed vertices, are all the remaining vertices of $G$. Clearly a mixed vertex sees some but not all of the vertices of $D$. Mixed vertices have some special properties.

**Observation 1.2:** Let $D$ be a disc in a graph $G$. If $m \in M_D^G$ then $m$ disagrees on two adjacent vertices of $D$. 
This follows trivially from the fact that a disc is connected.

**Observation 1.3:** Let $D$ be a disc in a graph $G$. If $m \in M^D$ then $m$ disagrees on two non-adjacent vertices of $D$.

The proof of this observation is analogous to that of 1.2.

**Observation 1.4:** Let $D$ be a disc in a graph $G$ and let $x$ be any vertex in $G \cdot D$. Then $x \in M^D$ if and only if $x$ forms a $P_4$ with three elements of $D$.

Clearly if $x$ forms a $P_4$ with three vertices of $D$ then $x$ misses at least one of these three vertices and $x$ sees at least one of these three vertices. It follows that $x$ is mixed with respect to $D$.

Assume now that $D$ is a hole and $x \in M^D$. By (1.2), there exist $c_1$ and $c_2$, adjacent vertices in $D$, such that $x$ sees $c_1$ and misses $c_2$. We extend the numbering of the vertices in the natural fashion around the cycle. If $x$ misses $c_1$ then $x$ forms a $P_4$ with $c_1, c_2, c_3$. If $x$ sees $c_4$ then $x$ forms a $P_4$ with $c_1, c_2, c_4$. If $x$ sees $c_4$ and misses $c_4$ then $x$ forms a $P_4$ with $c_1, c_2, c_4$. In any case, clearly $x$ forms a $P_4$ with three elements of $D$.

Now if $x$ is a mixed vertex with respect to the complement of a hole $D$ then $x$ is also mixed with respect to $\overline{D}$ in $\overline{G}$. Thus $x$ forms a $P_4$ with three vertices of $\overline{D}$ in $\overline{G}$. But since $P_4$ is self complementary, $x$ will form a $P_4$ with the same three vertices in $G$.

As stated in section 6.0, our work will make use of two properties in minimal imperfect graphs. We will now examine these properties in more detail.

### 6.1.2 Star Cutsets

This section presents some definitions and results related to star-cutsets. Some of this material has already been presented in chapter 2. In order to make our proof clearer, we reproduce it here in a slightly different form. We will call a graph $G$ unbreakable if neither $G$ nor $\overline{G}$ contains a star-cutset. If a graph is not unbreakable, it will be called fragile.

Obviously all minimal imperfect graphs are unbreakable. It is a routine matter to verify
that all disks are also unbreakable. There are many different special cases of star-cutsets and we shall mention some here. A dominated vertex is a vertex whose neighborhood is contained in the neighborhood of another vertex. That is, a vertex $x$ is dominated by $y$ if $y$ sees every vertex that $x$ sees in $G - x - y$. A homogeneous set $H$ is a proper subset of $V$ containing at least 2 vertices such that $V - H$ can be partitioned into $A = \{x | x$ sees $y$ for all $y \in H\}$ and $B = \{x | x$ misses $y$ for all $y \in H\}$.

**Observation 1.5:** No unbreakable graph of size at least three contains a dominated vertex.

If $x$ is dominated by $y$ in $G$ and $x$ misses some vertex $z$ of $G - y$, other than itself, then $y + (N(y) - z - x)$ is clearly a star-cutset in $G$ with center $y$. If $x$ sees all of $G - y$, then $y$ is a star-cutset in $\overline{G}$ separating $z$ from $\overline{G} - x - y$.

**Observation 1.6:** No unbreakable graph contains a homogeneous set.

Consider a homogeneous set, $H$, in an unbreakable graph $G$ and a corresponding partition of $G - H$ into sets $A$ and $B$. If $B$ is non-empty then for any $x$ in $H$, $x + A$ is a star-cutset with center $x$ in $G$. If $B$ is empty then $\overline{G}$ is disconnected (with components $H$ and $A$), and so any vertex of $H$ is a star-cutset.

**Observation 1.7:** If $G$ is unbreakable then the neighborhood of any vertex in $G$ induces a connected subgraph of $\overline{G}$.

If $N_G(x)$ induces a disconnected subgraph of $\overline{G}$ then $x + N_G(x)$ is a star-cutset in $\overline{G}$ with center $x$.

We noted that all discs are unbreakable. In fact, Hayward has shown that any non-trivial unbreakable graph must contain a disc.

**Lemma 1.8 ([Hay85]):** Every discless graph with at least three vertices is fragile.

Hayward named the class consisting of graphs with no discs, weakly triangulated.

### 6.1.3 Proper Endomorphisms
We recall that an endomorphism of a graph is simply a mapping from the vertex set $V$ into itself such that if $x$ sees $y$ in $G$ then $f(x)$ sees $f(y)$ in $G$. To see that no minimal imperfect graph has a proper endomorphism, we rely on two easy observations:

(i) A graph $H$ is $\omega(H)$-colourable if and only if it admits an endomorphism $g$ such that $g(H)$ is a clique with $\omega(H)$ vertices,

(ii) the composition of two endomorphisms is an endomorphism.

Assuming that some minimal imperfect graph $H$ has a proper endomorphism $f$, let $G$ stand for the subgraph of $H$ induced by $f(H)$. Since $G$ is perfect, it is $\omega(G)$-colourable; now (i) and (ii) imply that $H$ is $\omega(H)$-colourable, a contradiction.
6.2 Invariant Discs in Variant Graphs

We know that all discs with more than six vertices are $P_4$-isomorphic only to themselves and their complements. Furthermore, we know that graphs without discs are perfect. Thus, if we want to show that a graph $P_4$-isomorphic to a perfect graph is perfect, it makes sense to consider the discs in the two graphs. We shall now investigate what happens if two $P_4$-isomorphic graphs are invariant on some large disc. In particular, we shall demonstrate the following theorem:

**Theorem 6.2:** If $G$ and $H$ are $P_4$-isomorphic graphs which are invariant on some disc of size at least six then either

(i) $G = H$, or

(ii) $H$ or $\overline{H}$ has a star-cutset, or

(iii) $H$ contains a $C_6$ as a proper subgraph.

We will consider an unbreakable graph $H$, which contains no $C_6$, and is $P_4$-isomorphic to some graph $G$. Furthermore, we will assume there exists an invariant disc $D$ of size at least six in their common vertex set $V$. Before demonstrating that $V$ is invariant (that is $G = H$), we make a few elementary observations about these two graphs and the relation between them.

**Observation 2.1:** If $H$ is an unbreakable graph and $D$ is some disc in $H$ then for every vertex $y$ in $R^D_H$ there exists a path from $y$ to some vertex $z$ in $D$ consisting of vertices from $V - U^D_H$.

Otherwise $z + U^D_H$ would be a star cutset for any vertex $z$ in $D$.

The **universal free distance** of a vertex $z$ ($ufd(z)$) in $V - U^D_H$ is defined to be the number of edges in a minimal length path from $z$ to $D$ in $H - U^D_H$. By (2.1) this is a well-defined function on unbreakable graphs.

**Observation 2.2:** If $G$ and $H$ are $P_4$-isomorphic and $D$ is an invariant disc of size at least five in $V$, then $M^D_G = M^D_H$. 
By (1.4), a vertex outside $D$ is an element of $M_D^P$ if and only if it forms a $P_4$ with three elements of $D$ in $G$ and is an element of $M_H^P$ if and only if it forms a $P_4$ with three elements of $D$ in $H$. Since $G$ and $H$ are $P_4$-isomorphic, these two conditions are clearly equivalent.

Since for any invariant disc $D$ the set of mixed vertices is the same in the two graphs, we no longer need to add the subscripts and can refer to this set as $M^D$. We can make the same observation about the universal and remote vertices (in fact, this is precisely Lemma 2.4). We will need the following observation to prove this lemma.

Observation 2.3: If $G$ and $H$ are $P_4$-isomorphic graphs and $D$ is an invariant disc in $V$, then for all $m$ in $M^D$ there exists some element $d$ in $D$ such that $md$ is an invariant pair.

By (1.4), $m$ forms a $P_4$ with a three element subset $S$ of $D$ in $H$. Since $D$ contains $S$, $S$ is invariant. Also, by (2.2), $m + S$ induces a $P_4$ in both graphs. By counting the edges in the subgraph induced by $m + S$ in both graphs, we see that $m$ must be invariant with respect to at least one element of $S$.

Lemma 2.4: $R_H^D = R_G^D$.

Proof: We shall first prove the following statement by induction on $l$:

If $r \in R_H^D$ and $\text{ufd}(r) = i$ then

(i) $r \in R_G^D$

(ii) If $P$ induces a path with $i$ edges in $H - U_H^P$ with one endpoint $r$ and the other some $d$ in $D$, then $P - d$ is invariant.

Clearly if $r \in R_H^D$ then $\text{ufd}(r) \geq 2$. If $\text{ufd}(r) = 2$ then $r$ sees some element of $M^D$ in $H$.

Let $m$ be an arbitrary element of $M^D$ which sees $r$ in $H$. By (1.2), there exist two adjacent vertices $l_1$ and $l_2$ of $D$ such that $m$ is adjacent to $l_1$ but not $l_2$ in $H$. Clearly \{r, m, l_1, l_2\} induces a $P_4$ in $H$. It follows that $r$ is not an element of $U_H^D$ (otherwise these four vertices could not form a $P_4$ in $G$ because \{r, l_1, l_2\} would induce a triangle).

By assumption, $r$ is not in $M^D$, thus $r \in R_G^D$. But then, since \{r, m, l_1, l_2\} induces a $P_4$ in $H$ and thus in $G$, $r$ must be adjacent to $m$ in $G$ and therefore (ii) holds.
Assume now that \( ufd(r) = j \geq 3 \) and that (i) and (ii) hold for all vertices \( x \) with \( ufd(x) < j \). Let \( P = \{ p_1 = r, p_2, \ldots, p_j = m \in M^D, p_{j+1} \} \) be an arbitrary minimal-length path from \( r \) to \( D \) in \( H - U^D_H \). We show first that \( r \in R^D_H \) whenever \( j \geq 5 \). We know that \( \{ r, p_2, p_3, p_4 \} \) induces a \( P_4 \) in \( H \). It follows that these four vertices induce a \( P_4 \) in \( G \) and, in particular, that some set of three of these vertices induces a \( P_3 \) with \( p_1 \) as an endpoint. If \( r \) were universal in \( G \), this \( P_3 \) would extend into a \( P_4 \) with any element of \( D \) in \( G \). Since this would also be a \( P_4 \) in \( H \), we would have a \( P_4 \) consisting of an element of \( D \) and three remote vertices in \( H \), a contradiction.

We show now that \( r \in R^D_H \), even if \( j < 5 \). Assume first that \( r \) misses \( m \) in \( G \). By (1.3), there exist non-adjacent vertices \( k \) and \( k \) in \( D \) such that \( m \) sees \( k \) but misses \( k \) in \( G \). If \( r \) were in \( U^D \) then \( \{ m, k, r, k \} \) would form a \( P_4 \) in \( G \). However, in \( H \), \( \{ r, k, k \} \) induces a stable set of size three. This would contradict the \( P_4 \)-isomorphism of \( G \) and \( H \).

Assume \( r \) is not remote in \( G \) then, by the above, \( r \) must see \( m \) in \( G \). Since \( 3 \leq j < 5 \), \( m \) is \( p_3 \) or \( p_4 \) and \( r \) is in a \( P_4 \) with \( m \) and \( p_{j-1} \) in \( H \), and thus in \( G \). We note that \( p_{j-1} \) misses \( r \) in \( G \) or \( \{ r, m, p_{j-1} \} \) would induce a triangle in \( G \). Let \( c \) be a vertex of \( D \) such that \( m \) misses \( c \) in \( G \). Then \( \{ c, r, m, p_{j-1} \} \) forms a \( P_4 \) in \( G \) and therefore in \( H \). This implies \( j = 3 \). By (2.3), there exists a vertex \( d \) of \( D \) such that \( md \) is an invariant pair. It is easy to see that \( \{ r, p_{j-1} m, d \} \) forms a \( P_4 \) in precisely one of \( G \) and \( H \). This contradicts the fact that the two graphs are \( P_4 \)-isomorphic.

We have shown that \( r \) must be remote in \( G \); we shall now show that \( r \) misses \( m \) in \( G \) each \( p_i \) with \( i \geq 3 \). Assume \( r \) sees some such \( p_i \) and let \( k \) be the highest index for which \( r \) sees \( p_k \). Since \( r \) is remote in \( G \), it follows that \( k \leq j \). Now \( \{ r, p_k, p_{k+1}, p_{k+2} \} \) forms a \( P_4 \) in \( G \) (where, if necessary, we extend \( P \) past \( m \) by an appropriate choice of elements of \( D \); see (1.2)). But then \( r \) sees one of \( p_k, p_{k+1}, p_{k+2} \) in \( H \), contradicting the minimality of \( P \).
FIGURE 6.1
We now know that, in $G$, $r$ misses all $p_i$ with $i \geq 3$. Since \( \{r, p_2, p_3, p_4\} \) forms a $P_4$ in $H$ and therefore in $G$, $r$ must see $p_2$ in $G$. Then, by the induction hypothesis, $P - p_{j+1}$ is an invariant path.

We have shown that $R^p_H$ is contained in $R^p_G$. By applying the same conclusion to $\overline{G}$ and $\overline{H}$ we see that $U^p_H$ is contained in $U^p_G$. But $R^U_H \cup U^p_H = R^p_G \cup U^p_G = V - D - M^p_D$. Thus $R^p_H = R^p_G$ and $U^p_H = U^p_G$.

We can now refer to $U^D$ and $R^D$ with no ambiguity since these are the same sets in the two graphs. Observation 2.4 and Lemma 2.4 can be combined and restated in the following succinct form:

Corollary 2.4a: If an unbreakable graph $H$ is $P_4$-isomorphic to a graph $G$ then $R^p_H = R^p_G, M^p_H = M^p_G$ and $U^p_H = U^p_G$.

We note that if $x \in U^D$ or $R^D$ then trivially $x + D$ is invariant. We now show that this is true for all vertices of $V - D$ by showing that it holds for all vertices in $M^p_D$.

Lemma 2.5: If $m \in M^p_D$ then $m + D$ is invariant.

Proof: We shall prove this lemma for $D$, a hole of length at least six. By considering the complements of the graphs this implies that 2.5 holds for all discs.

Step 0: If $D$ induces an invariant hole of size at least six and, for some $m \in M^p_D$, $m + D$ is not invariant then one of the situations depicted in Fig. 6.1 occurs.

To simplify matters we shall enumerate the vertices of the hole so that the edges are $e_i e_{i+1}$ where addition is modulo $n$ ($n$ is the length of the hole $D$). We note that we can select any vertex in the hole to be $e_1$; with an appropriate choice of this vertex the cases are as described below.

Case 1: $D$ has six vertices and $N_H(m) \cap D = \{e_2, e_3\}, N_G(m) \cap D = \{e_4, e_5\}$.

Case 2: $N_H(m) \cap D = \{e_2, e_3, e_4\}, N_G(m) \cap D = \{e_4, e_5\}$. 
Case 3: \( N_H(m) \cap D = \{e_2, e_3\} \), \( N_G(m) \cap D = \{e_2, e_3, e_4\} \).

The proof of this fact is left to the reader (one method of proving it is to start by considering the \( P_4 \) that \( m \) forms with three elements of \( D \)).

**Step 1: Case 1 does not occur.**

Assume there exists a vertex \( m \) and a disc \( D \) such that this situation occurs. Since \( H \) has no star-cutset, the subgraph \( F \) of \( H \) induced by \( N_H(m) \) is connected. Let \( P = \{p_1 = e_2, p_2, \ldots, p_k = e_3\} \) be a shortest path from \( e_2 \) to \( e_3 \) in \( F \). Furthermore, choose \( P \) of minimum length over all appropriate choices of \( D \) and \( m \).

If \( e_1 \) does not see \( p_2 \) in \( H \) then \( \{e_1, e_2, m, p_2\} \) forms a \( P_4 \) in \( H \) and thus in \( G \). This implies that \( m \) sees \( p_2 \) in \( G \) and \( p_2 \) is adjacent to precisely one of \( e_1 \) and \( e_2 \) in \( G \). If \( p_2 \) sees \( e_2 \) in \( G \) then, by Step 0, \( N_G(p_2) \cap D = \{e_2, e_3\} \). In this case \( \{m, p_2, e_2, e_1, e_3\} \) induces a \( C_5 \) in \( G \). But these five vertices will also induce a \( C_5 \) in \( H \) (see 1.1) contradicting our assumption (iii). On the other hand, if \( p_2 \) sees \( e_1 \) in \( G \) then, by Step 0, \( N_G(p_2) \cap D = \{e_1, e_3\} \) and \( N_H(p_2) \cap D = \{e_4, e_3\} \). But then \( \{e_4, p_2, m, e_2\} \) induces a \( P_4 \) in \( H \) but not in \( G \). These contradictions indicate that \( p_2 \) must see \( e_1 \) in \( H \).

Clearly \( p_2 \) sees \( e_3 \) in \( H \); otherwise \( \{e_3, m, p_2, e_1\} \) would form a \( P_4 \) in \( H \) while \( \{m, e_1, e_3\} \) induces a stable set in \( G \).

Assume that, in \( H \), \( p_2 \) saw some \( e_i \) with \( 4 \leq i \leq 6 \). Then by step 0, \( p_2 + D \) would be invariant; in particular, \( e_2 \) would miss \( p_2 \) in \( G \). But then \( \{e_i, p_2, m, e_2\} \) forms a \( P_4 \) in \( H \) while \( e_2 \) misses \( p_2, e_i, m \) in \( G \) and thus these four vertices do not induce a \( P_4 \) in \( G \).

This contradiction implies that, in \( H \), \( p_2 \) sees precisely \( e_1 \) and \( e_4 \) in \( D - e_2 \) and, by Step 0, \( p_2 \) sees precisely the same two vertices in \( D_G - e_2 \).

We construct a shorter path \( P' \) by letting \( D' = D - e_2 + p_2, e_2' = p_2 \) and \( P' = P - p_1 \), contradicting the minimality of \( P \).
Step 2: Case 2 does not occur.

Assume there exists an m and D such that this situation occurs. By (1.7), since H is unbreakable, the subgraph F of H induced by \( N_H(e) \) is connected. Let P be a shortest path from m to either of \( e_2 \) or \( e_4 \) in F. Furthermore we choose P to be minimal over all possible choices of m and D. By relabelling the vertices of D we can assume \( P = \{ p_1 = e_2, \ldots, p_n = m \} \) is a path from \( e_2 \) to m.

Assume, some vertex of P other than m, is variant with respect to \( e_2 \). Then we let \( m' = p_k \) and \( P' = \{ p_1, \ldots, p_k \} \) contradicts the minimality of P. Thus \( p_k \ e_2 \) is invariant whenever \( k < n \).

Consider now some vertex \( p_k \) of P which is variant with respect to \( e_2 \). By Steps 0 and 1, in both G and H, \( p_k \) sees precisely \( e_1 \) and \( e_3 \) in \( D - e_2 \). If \( D' = D - e_2 + p_k \) and \( P' = \{ p_k, \ldots, p_n \} \), then \( P' \) contradicts the minimality of P. Therefore \( p_k \ e_2 \) is invariant for \( 1 \leq k \leq n \). By a similar argument \( p_k \ e_3 \) is invariant for \( 1 \leq k \leq n \). In fact, by Steps 0 and 1, this implies each \( p_k \) with \( k < n \) is invariant with respect to D.

Assume some vertex \( p_i \) on the path is variant with respect to \( e_2 \). We note by Steps 0 and 1 that \( i < n \) (consider \( D' = D - e_2 + m \)). Now \( p_2 \) must miss \( e_4 \) or \( \{ p_2, e_4, p_i, e_2 \} \) would induce a \( P_4 \) in precisely one of G and H. If both \( p_i \) and \( p_2 \) saw \( e_1 \) then \( \{ e_4, p_i, e_1, p_2 \} \) would induce a \( P_4 \) in precisely one of G and H. Also, if both \( p_i \) and \( p_2 \) missed \( e_1 \) then \( \{ e_1, e_2, p_i, p_2 \} \) would induce a \( P_4 \) in precisely one of G and H. Therefore exactly one of the two vertices \( p_i \) and \( p_2 \) sees \( e_1 \). But then \( \{ e_1, p_2, e_2, p_i \} \) induces a \( P_4 \) in precisely one of G and H. This contradiction implies that \( p_i \ p_2 \) is an invariant pair for every \( p_i \) in P. It follows that P is an invariant set. If not, there exists a \( p_j \) variant with respect to some \( p_i \). Assume \( j < i \). Choose a minimal such \( j \) and the minimal \( i \) for this \( j \). Clearly, by the previous remarks, \( j \geq 3 \) and \( i \geq 4 \). But then \( \{ p_1, p_j, p_{j-1}, p_{j-2} \} \) forms a \( P_4 \) in precisely one of the two graphs.

We now consider the four vertices \( p_n = m, p_{n-2}, e_3, p_{n-1} \). These clearly induce a \( P_4 \) in G but not in H. This contradicts the \( P_4 \)-isomorphism of G and H.
Step 3: Case 3 does not occur.

Assume there exist an \( m \) and a \( D \) such that this situation occurs. Since \( H \) has no star cutset, \( H \) has no dominated vertex; in particular, there must exist an \( x \) in \( N_H(c_3)-N_H(m) \). We know that \( x \) must see \( c_3 \) in \( H \) or \( \{ x, c_3, e_2, m \} \) would induce a \( P_4 \) in \( H \) but not in \( G \). Similarly, \( x \) must see \( c_4 \) in \( H \). Hence by Steps 0, 1, and 2, \( x \) must be invariant on \( D \). If \( x \) saw some \( d \) in \( D-e_3-c_3-c_4 \), then \( x \) would have to be invariant on \( D' = D-e_3+m \). But then \( \{ d, x, c_4, m \} \) or \( \{ d, x, c_3, m \} \) would form a \( P_4 \) in \( G \) and not in \( H \). Thus \( N_H(x) \cap D = \{ e_2, e_3, c_4 \} \). Since \( H \) has no star cutset, \( H-(N_H(x)-c_3) \) must be connected. Let \( P = \{ p_1 = e_3, \ldots, p_n = m \} \) be a minimal length path from \( e_3 \) to \( m \) in \( H-(N_H(x)-c_3) \). Furthermore, let \( P \) be minimal over all appropriate choices of \( D, e_3, m, \) and \( x \).

We note first that \( n \geq 4 \). Otherwise by Steps 0-2, \( P_2 \) would be invariant with respect to \( P_1 \) and \( P_3 \) (consider \( D' = D-e_3+m \)). But then \( P+x \) would induce a \( P_4 \) in \( H \) but not in \( G \).

Now \( P_2 \) sees \( c_3 \) and not \( m \) in \( H \), and as with \( x \) must therefore see exactly \( c_2, e_3, c_4 \) in \( D \). By the minimality of \( P \), each \( p_k \) in \( P-m \) is invariant with respect to \( c_3 \) (consider \( P' = p_1, \ldots, p_k \)). By a similar argument \( P_2 \) is invariant with respect to \( P \) (consider \( c_3' = p_2, P' = p_2, \ldots, p_k, x' = p_1 \)). We claim that \( P-m \) is invariant. If not, let \( p_i \) be the first element of \( P-m \) to be in some variant pair \( p_i, p_j \). Clearly \( i \geq 3 \). But then \( \{ p_j, p_i, p_{i-1}, p_{i-2} \} \) induces a \( P_4 \) in exactly one of \( G \) and \( H \). Thus \( P-m \) is invariant.

We know that \( \{ p_1, p_2, p_{n-1} \} \) is invariant. Furthermore, by Step 2, we know that \( p_{n-1}p_n \) is invariant (consider \( D' = D-e_3+m \)). Thus \( \{ p_1, p_2, p_{n-1}, p_n \} \) induces a \( P_4 \) in precisely one of \( G \) and \( H \), a contradiction.

We can summarize the results so far with the following statement:

**Corollary 2.5a**: For all \( x \in V-D \), \( x+D \) is invariant.
We now make the following definitions. The distance of a vertex \( x \) in \( G \), \( \text{dist}_G(x) \), is the number of edges in a minimal-length path from \( x \) to some vertex of \( D \). We note that 
\[ \text{dist}(x) \leq ufd(x) \]
for all \( x \) in \( V - D - U^D \). Thus \( \text{dist}(x) \) is clearly a well-defined function in both graphs. The \( i \)th level of \( G \), \( L_i^G \), consists of all vertices with distance \( i \) in \( G \). We make analogous definitions in the graph \( H \). We will now show that \( L_i^G = L_i^H \).

**Lemma 2.6:** For every \( x \in V \), \( \text{dist}_H(x) = \text{dist}_G(x) \). Furthermore if \( \text{dist}_H(x) \neq \text{dist}_H(y) \) then \( xy \) is an invariant pair.

**Proof:** We shall proceed by induction on \( \text{mindist}(x) \) defined as the minimum of \( \text{dist}_H(x) \) and \( \text{dist}_G(x) \). We shall actually show that for each vertex \( x \) in \( V \) with 
\[ \text{mindist}(x) = k. \]

(i) \( \text{dist}_H(x) = \text{dist}_G(x) \), and
(ii) \( xy \) is invariant for all vertices \( y \) s.t. \( \text{mindist}(y) < k \).

For \( k = 0 \), \( z \) must be in \( D \) and thus (i) and (ii) hold trivially. For \( k = 1 \), (i) is simply a restatement of 2.4a and (ii) is a restatement of 2.5a.

To show (i) and (ii) for \( k = 2 \), we need only show that each \( z \) in \( R^D \) is invariant with respect to all \( y \in M^D \cup U^D \).

We show first that \( z \in R^D \) is invariant with respect to any \( y \in M^D \). By (1.3) there exist two adjacent vertices \( l_1 \) and \( l_2 \) in \( D \) such that \( y \) sees \( l_1 \) but not \( l_2 \). Clearly \( \{x, y, l_1, l_2\} \) induces a \( P_4 \) if and only if \( xy \) is an edge. Thus, by the \( P_4 \)-isomorphism of \( G \) and \( H \), \( xy \) is an invariant pair. (Note: By looking at \( \overline{G} \) and \( \overline{H} \) we see that \( xy \) is an invariant pair whenever \( z \in U^D \), \( y \in M^D \).)

We show now that if \( z \in R^D \cup M^D \) and \( y \in U^D \) then \( xy \) is an invariant pair. We prove this by induction on the universal free distance of \( z \) in \( H \). If \( ufd(x) = 1 \) then clearly \( z \in M^D \) and we are done by the above note. Now assume that \( ufd(x) > 1 \) and the statement holds true for \( y \) with \( ufd(y) < ufd(x) \). It is easy to see that \( z \) sees (in both graphs) some \( y \) with \( ufd(y) = ufd(x) - 1 \). Furthermore, there exists some \( d \in D \) such that \( y \) misses \( d \) (in both graphs). It is easy to verify that if \( z \) is variant with
respect to some $x$ in $U^D$ then $\{x,y,z,d\}$ induces a $P_4$ in precisely one of $G$ and $H$. This would contradict the $P_4$-isomorphism of $G$ and $H$. Thus (i) and (ii) hold for $k=2$.

Consider now a vertex $x$ with $\text{mindist}(x)=k \geq 3$. Assume furthermore that (i) and (ii) hold for all $y$ with $\text{mindist}(y)<k$. Since $\text{mindist}(x)=k$, clearly $x$ misses all vertices $y$ with $\text{mindist}(y)<k-1$ in both graphs. Consider any vertex $y$ with $\text{mindist}(y)=k-1$. There exist vertices $z$ and $a$ with $\text{mindist}(z)=k-2$, $\text{mindist}(a)=k-3$, such that $yza$ is a $P_3$. If $xy$ were a variant pair then $\{x,y,z,a\}$ would induce a $P_4$ in precisely one of $G$ and $H$. Thus $zy$ is invariant for all vertices $y$ with $\text{mindist}(y)=k-1$. We have now demonstrated that (ii) holds for $x$, and clearly (i) follows. Now we can write $\text{dist}$ instead of $\text{mindist}$.

**Lemma 2.7:** $H$ is a subgraph of $G$.

**Proof:** We need only prove the following:

If $\text{dist}(x)=\text{dist}(y)$ and $xy$ is an edge of $H$ then $xy$ is an edge of $G$.

Since $\text{dist}_G(x)=\text{dist}_H(x)$, $L^D_i=L^H_i$ and we shall denote this set as $L_i$.

**Case 1:** The two vertices $x$ and $y$ have differing neighborhoods in the set of vertices at distance $\text{dist}(x)-1$ from $D$.

In this case we shall prove that $xy$ is an invariant pair. We have already noted (in the proof of Lemma 2.6) that every universal vertex is invariant with respect to every mixed vertex. Thus if $\text{dist}(x)=1$ then we may assume that $x$ and $y$ are both mixed. If we can find two vertices of $D$, $d_1$ and $d_2$, such that $d_1$ sees $x$ but not $y$ and $d_2$ sees $y$ but not $x$ then it follows that if $xy$ is variant then $\{x,y,d_1,d_2\}$ is a $P_4$ in precisely one of $G$ and $H$. Thus, in this case $xy$ must be an invariant pair. If we cannot find two such vertices $d_1$ and $d_2$ then clearly we can rename $x$ and $y$ as $m_1$ and $m_2$ in such a way that $N(m_2) \cap D \subset N(m_1) \cap D$. The vertices of $D$ must either miss both of $m_1$ and $m_2$ (missed vertices), see $m_1$ but miss $m_2$ (varying vertices) or see both $m_1$ and $m_2$ (seen vertices).
If $m_1m_2$ is variant the following observations hold:

(i) No varying vertex is adjacent to a missed vertex
(by the $P_4$-isomorphism of $G$ and $H$).

(ii) Every varying vertex is adjacent to every seen vertex
(by the $P_4$-isomorphism of $G$ and $H$).

(iii) There exists at least one vertex of each type
(since $m_1$ and $m_2$ are mixed).

Clearly (i),(ii) and (iii) imply that if there is more than one varying vertex then the set of varying vertices is homogeneous in $D$. But no disc has a homogeneous set so there must exist precisely one varying vertex $v$. Now (i) and (ii) imply that the neighborhoods of $m_2$ and $v$ on $D-v$ agree. But this means that $D' = D - v + m_2$ is also a disc. However $m_1$ is variant with respect to this disc. This contradicts (2.5). So, any two mixed vertices with differing neighborhoods in $D$ are an invariant pair.

For $k \geq 2$, let $x$ and $y$ be two vertices in $L_k$ with differing neighborhoods in $L_{k-1}$. We can assume that $x$ sees some vertex $z$ in $L_{k-1}$ that $y$ misses. Let $a$ be a vertex in $L_{k-2}$ which $z$ sees. Clearly $\{x,y,z,a\}$ induces a $P_4$ in $G$ (or $H$) if and only if $z$ sees $y$ in $G$ (or $H$). Thus, by the $P_4$-isomorphism of $G$ and $H$, $xy$ must be an invariant pair.

**Case 2:** The two vertices, $x$ and $y$, have exactly the same neighbours in the set of vertices at distance $\text{dist}(x)-1$ from $D$.

In this case we shall prove that if $x$ sees $y$ in $H$ then $x$ sees $y$ in $G$. Let $x$ and $y$ in $L_q$ be a variant pair of vertices, with the same neighbours in $L_{q-1}$, such that $xy$ is an edge of $H$. Clearly $i \geq 1$. Let $x$ be a vertex of $L_{i-1}$ adjacent to $x$ and $y$. Since $H$ is unbreakable, by (1.7), the graph $F$ induced by $N_H(x)$ in $\overline{H}$ is connected. Let $P = \{p_1 = x, p_2, \ldots, p_n = y\}$ be a minimal length path from $x$ to $y$ in $F$. Furthermore let $P$ be of minimum length for all appropriate choices of $x,y,z$. If $p_k$ is in $L_{i-1}$ for some $k$ greater than one then we can replace $z$ by $p_k$, contradicting the minimality of
Thus \( p_1p_k \) is an invariant pair whenever \( k > 1 \). If \( x_p \) is variant for some \( k \) smaller than \( n \) then \( p_k \in L_i \) (and \( k \geq 3 \) by Case 1) so we can replace \( y \) by \( p_k \), again contradicting the minimality of \( P \). Thus \( x_p \) is invariant whenever \( k < n \). If \( k > 2 \) then, since \( p_2 \) misses \( z \) while \( p_k \) sees \( z \), and since \( z \in L_{i-1} \), Case 1 with \( z \) replaced by \( p_2 \) and \( y \) replaced by \( p_k \) assures us that \( p_2p_k \) is an invariant pair. Thus \( p_2p_k \) is invariant whenever \( k 
eq 2 \). We claim now that \( p_jp_k \) is invariant for all \( j \) and \( k \). We can assume that \( j < k \). Take the minimal \( j \) for which this claim is false. Clearly \( j > 2 \). It is not difficult to verify that \( \{p_{j-2}, p_{j-1}, p_j, p_k\} \) induces a \( P_4 \) in precisely one of \( G \) and \( H \). This contradicts the \( P_4 \)-isomorphism between the two graphs. Thus \( P \) is clearly invariant. But then \( \{p_{x_1}, p_{x_2}, p_{x_3}, p_{x_4}\} \) induces a \( P_4 \) in \( G \) but not in \( H \). This contradiction implies that \( x \) must see \( y \) in \( G \).

Finally, Lemma 2.7 with \( G \) replaced by \( \overline{G} \) and \( H \) replaced by \( \overline{H} \) implies that \( G \) is a subgraph of \( H \). This completes the proof of Theorem 6.2.
6.3 Concerning the Number Six.

Most discs are $P_6$-isomorphic only to themselves and their complements. Discs of size six are in fact the only exception to this rule. It is a routine exercise to verify that $G_6$ is $P_6$-isomorphic only to $G_6, \overline{G_6}, F$ of Fig. 6.2, and $\overline{F}$. We will now investigate what happens if a graph containing a set $D$ of vertices inducing a $C_6$ is $P_6$-isomorphic to a graph in which $D$ induces an $F$.

**Theorem 6.8:** Consider an unbreakable graph $H$, containing no $C_6$, that is $P_6$-isomorphic to a graph $G$. If some set $D$ induces a $C_6$ in $H$ and an $F$ in $G$ then $H$ has a proper endomorphism.

Proof: We may enumerate the vertices of $D$ as $d_1, d_2, \ldots, d_6$ in such a way that the $\overline{F}$ induced by $D$ in $G$ is labeled as in Fig. 6.2 and the edges induced by $D$ in $H$ are precisely $d_i d_{i+1}$ with addition modulo six. It is a tedious but routine matter to verify that each vertex outside $D$ is of one of the following fourteen types.

Type 1: $N_H(x) \cap D = \emptyset$ or $D$.

Type 2: $N_H(x) \cap D = \{d_1\}$.

Type 3: $N_H(x) \cap D = \{d_4\}$.

Type 4: $N_H(x) \cap D = \{d_1, d_3\}$ or $\{d_2, d_4, d_6\}$.

Type 5: $N_H(x) \cap D = \{d_2, d_4\}$ or $\{d_3, d_5, d_6\}$.

Type 6: $N_H(x) \cap D = \{d_1, d_3\}$ or $\{d_2, d_4, d_6\}$.

Type 7: $N_H(x) \cap D = \{d_5, d_1\}$ or $\{d_6, d_2, d_4\}$.

$N_G(x) \cap D = \emptyset$ or $D$. 

$N_G(x) \cap D = \{d_1\}$.

$N_G(x) \cap D = \{d_4\}$.

$N_G(x) \cap D = \{d_1, d_3\}$ or $\{d_2, d_4, d_6\}$.

$N_G(x) \cap D = \{d_2, d_4\}$ or $\{d_3, d_5, d_6\}$.

$N_G(x) \cap D = \{d_1, d_3\}$ or $\{d_2, d_4, d_6\}$.

$N_G(x) \cap D = \{d_5, d_1\}$ or $\{d_6, d_2, d_4\}$. 

$N_G(x) \cap D = \{d_6, d_1\}$ or $\{d_8, d_2, d_4\}$.
Type 8: \( N_H(x) \cap D = \{d_8,d_9\} \) or \( \{d_8,d_4,d_8\} \).
\[ N_G(x) \cap D = \{d_9,d_9\} \) or \( \{d_9,d_4,d_9\} \).

Type 9: \( N_H(x) \cap D = \{d_8,d_2\} \) or \( \{d_8,d_4,d_2\} \).
\[ N_G(x) \cap D = \{d_8,d_8\} \) or \( \{d_8,d_8,d_1\} \).

Type 10: \( N_H(x) \cap D = \{d_1,d_8,d_8\} \).
\[ N_G(x) \cap D = \{d_1,d_2,d_4,d_8\} \).

Type 11: \( N_H(x) \cap D = \{d_2,d_4,d_8\} \).
\[ N_G(x) \cap D = \{d_2,d_4,d_2\} \).

Type 12: \( N_H(x) \cap D = \{d_1,d_5,d_4,d_8\} \).
\[ N_G(x) \cap D = \{d_1,d_5,d_5\} \).

Type 13: \( N_H(x) \cap D = \{d_1,d_5,d_4,d_8\} \).
\[ N_G(x) \cap D = \{d_1,d_5,d_5\} \).

Type 14: \( N_H(x) \cap D = \{d_2,d_5,d_8,d_8\} \).
\[ N_G(x) \cap D = \{d_2,d_5,d_6,d_8\} \).

The possible neighborhoods of \( x \) on \( D \) in \( H \) are shown in Fig. 6.3.

We claim there are no vertices of types 4,5,6 or 7. It will suffice to show that there are no vertices of type 4; our claim follows by symmetry. For this purpose assume the contrary: the set \( S \) of vertices of type 4 is non-empty. Since \( H \) contains no homogeneous set, some vertex \( z \) not in \( S \cup \{d_2\} \) must disagree with \( x \), on two vertices of \( S \cup \{d_2\} \).

Thus \( z \) disagrees, in \( H \), on \( d_2 \) and some \( y \) in \( S \). We may assume \( z \) sees \( y \) and misses \( d_2 \) (in the other case replace \( D \) by \( D-d_2+y \) and replace \( y \) by \( d_2 \)). Now \( N_H(x) \cap D \) must be some set \( N \) such that both \( N \) and \( N+d_2 \) are allowed by our fourteen-type classification. Since \( x \) is not of type 4, \( x \) must be of type 3 or type 6. In either case, \( \{d_4,x,y,d_1\} \) induces a \( P_4 \) in \( H \) and thus in \( G \). Since \( d_1 \) misses both \( d_4 \) and \( y \) in \( G \), it must see \( x \) in \( G \), and so \( x \) is of type 6. Now \( \{d_4,x,y,d_8\} \) induces a \( P_4 \) in \( H \) but not in \( G \) (both \( d_8 \) and \( d_8 \) see \( x \) and \( y \) in \( G \)), which is the desired contradiction.
So far we have proved that there are no vertices of Types 4-7. This is tantamount to saying that, in $H$, a vertex outside $D$ sees $d_2$ if and only if it sees $d_4$, and, symmetrically, sees $d_3$ if and only if it sees $d_5$. It follows that the desired proper endomorphism of $H$ may be obtained by mapping $d_4$ to $d_2$, $d_5$ to $d_3$, and all other vertices to themselves.
6.4 Conclusion

The Proof of Theorem 6.1:

Let \( G \) and \( H \) be \( P_4 \)-isomorphic graphs such that \( G \) is neither \( H \) nor \( \overline{H} \). We need only prove that at least one of the conclusions (i), (ii), (iii) of Theorem 6.1 holds true. To begin, note that \( H \) has at least three vertices (else \( G = H = \overline{H} \), a contradiction). If \( H \) contains no disc then (ii) holds by (1.8); if \( H \) contains a disc of size five then (i) holds trivially. Hence we may assume that \( H \) contains a disc of size at least six. Let \( D \) be the set of vertices of this disc inducing the subgraphs \( D_G \) and \( D_H \) of \( G \) and \( H \) respectively. If \( D_H = D_G \) then (i) or (ii) holds by Theorem 6.2. If \( D_H = D_{\overline{G}} \) then (i) or (ii) holds by Theorem 6.2 with \( \overline{G} \) in place of \( G \). Thus, we can assume:

\[ D_H \neq D_G \text{ and } D_H \neq D_{\overline{G}}. \]

By (1.1) and the remark at the beginning of Section 6.3, this assumption implies that \( D_G \) or \( D_{\overline{G}} \) is the graph \( F \) of Fig. 6.3. If \( D_G = F \) then (i), (ii), or (iii) holds by Theorem 6.3; if \( D_{\overline{G}} = F \) then (i), (ii), or (iii) holds by Theorem 6.3 with \( \overline{G} \) in place of \( G \).

It may be possible to improve on Theorem 6.1. In particular, it may be possible to drop conclusion (i) from the statement of the theorem.

Another possible area of research is relating \( P_4 \)-isomorphism to the Reconstruction Conjecture (Bondy [Bo77], Ulam [U60]). This conjecture states that if for two graphs \( G \) and \( H \), the set of subgraphs of \( G \) with one vertex deleted can be put in one-to-one correspondence with the similar subgraphs of \( H \) such that corresponding subgraphs are isomorphic, then \( G \) is isomorphic to \( H \). It would be of interest to show that \( G \) and \( H \) are necessarily \( P_4 \)-isomorphic.
Algorithm A.1: Building a Spanning Tree

Input: A connected graph $G$.

Output: A set of edges of $G$ which form a spanning tree.

Data Structures: A set $S$ of vertices of $G$.
A list $M$ of vertices of $G$.
A set $T$ of edges of $G$.

Step 0: Arbitrarily choose some $z$ in $G$.
Set $S = \{ z \}$, set $T = \emptyset$, set $M = \{ z \}$.

Step 1: If $M$ is empty, return $T$ and stop.
Otherwise take the first vertex $y$ off $M$.

Step 2: For each neighbor $x$ of $y$, if $x$ is not in $S$ then:
(i) add $x$ to $S$ and $xy$ to $T$, and
(ii) append $x$ to $M$.

Step 3: Go to Step 1.

The amount of time taken by the algorithm depends on the time taken by Step 2. If $G$ is represented by an adjacency list then each iteration of Step 2 takes $O(d(y))$ time. Thus the algorithm takes $O(\sum_{z \in V} d(z)) = O(m)$ time. The algorithm can easily be modified to find the components of a graph in $O(n+m)$ time.
Appendix B

In this appendix, we describe how to determine if there is an odd path between two specified vertices of a graph in $O(n + m)$ time.

Observation B.1: Consider two vertices $x$ and $y$ of a connected graph $G$. There is a set of blocks $\{B_1, B_2, \ldots, B_k\}$ such that:

(i) $x$ is in $B_1$, $y$ is in $B_k$,
(ii) every path from $x$ to $y$ contains at least two vertices from each block in the set, and
(iii) every path from $x$ to $y$ passes only through vertices which are contained in $\bigcup_{i=1}^{k} B_i$.

Lemma B.2: Let $x$ and $y$ be vertices of a connected graph $G$. Let $B=\{B_1, B_2, \ldots, B_k\}$ be the associated set of blocks as described in B.1. If $H=\bigcup_{i=1}^{k} B_i$ is bipartite then all the paths from $x$ to $y$ have the same parity. Otherwise, there are paths of both parities from $x$ to $y$.

Proof: By B.1, every path from $x$ to $y$ in $G$ is contained within $H$. As we saw in section 2.1, all the paths between two vertices of a bipartite graph have the same parity. So, if $H$ is bipartite then all the paths from $x$ to $y$ have the same parity.

Now, we need only show that if $H$ is not bipartite then there are paths of both parities from $x$ to $y$. This is an easy consequence of the following theorem which we state without proof.

Theorem B.3: Let $F$ be a two-connected non-bipartite graph. If $x$ and $y$ are vertices of $F$ then there is an even path from $x$ to $y$ and an odd path from $x$ to $y$.

Now let $x$ and $y$ be vertices of a connected graph $G$ such that the associated graph $H$ is not bipartite. Since $H$ is not bipartite some block $B_i$ is not bipartite. Let $P$ be any path from $x$ to $y$. Let $u$ be the first vertex of $B_i$ on this path and let $v$ be the last.
By B.1, \( u \) and \( v \) are distinct. Let \( P' \) be the portion of \( P \) between \( u \) and \( v \). By B.2, there is a path \( P'' \) from \( u \) to \( v \) in \( B_i \) which does not have the same parity as \( P' \).

Now \( P - P' + P'' \) is a path from \( z \) to \( y \) which does not have the same parity as \( P \).

Algorithm B.1: Checking for Odd Paths.

Input: A graph \( G \) and two specified vertices \( z \) and \( y \) of \( G \).

Output: Yes or No.

Step 1: Break \( G \) up into its connected components. (this can be done in \( O(n+m) \) time).

Step 2: If \( z \) and \( y \) are in different components of \( G \) return No and stop.
Otherwise find a path \( P \) from \( z \) to \( y \) in \( G \). (this can be done in \( O(n+m) \) time).

Step 3: Break \( G \) up into its blocks.
Set \( H \) to be the union of all the blocks which contain two or more vertices of \( P \):
(this can be done in \( O(n+m) \) time).

Step 4: Determine if \( H \) is Bipartite.
(this can be done in \( O(n+m) \) time).

Step 5: If \( H \) is not bipartite return Yes and stop.

Step 6: If \( P \) is even then return No and stop.
Otherwise, return Yes and stop.
Appendix C

Algorithm C.1 takes as input a graph $G$ and two specified vertices of $G$. It determines if there is a homogeneous set of $G$ which contains both of the specified vertices. To determine if a graph has a homogeneous set, we need only apply Algorithm C.1 $\binom{n}{2}$ times, initialising it with a different pair of vertices each time. Since Algorithm C.1 runs in $O(n^3)$ time, we can check if a graph has a homogeneous set in $O(n^4)$ time.

Algorithm C.1: Finding a Homogeneous Set,

Input: A graph $G$ and two specified vertices $x$ and $y$ of $G$.

Output: Yes or No. If the answer is Yes, the algorithm also returns a partition of the vertices of $G$ into sets $H$, $A$, and $B$ such that:

1. $H$ is a homogeneous set of $G$ containing $x$ and $y$,
2. $A = \{z | z$ sees all of $H\}$, and
3. $B = \{z | z$ sees none of $H\}$.


Step 1: Set $S = \{x, y\}$. Set $A = N(x) \cap N(y)$. Set $B = V - x - y - (N(x) \cup N(y))$. Set $C = V - A - B - x - y$.

Step 2: If $S = V$ return No and stop.

Step 3: If $C = \phi$ then set $H = S$, return (yes, $H, A, B$), and stop.

Step 4: Set $C' = \phi$. For each vertex $z$ of $C$,

set $C' = C' \cup (N(z) \cap B) \cup (A - N(z))$.


Go to Step 2.

It is easy to see that the algorithm is correct. We shall show that the algorithm takes $O(n^2)$ time. The time consuming part of each iteration is Step 4. For each $z$ in $C$, we have to compute $B \cap N(z)$ and $A - N(z)$. By traversing the adjacency list for $z$, we can do this in $O(n)$ time. Since a vertex passes through $C$ at most once, it follows that C.1 takes $O(n^2)$ time.
Appendix D

By 2.21, a graph is a comparability graph if and only if it permits a semi-transitive orientation. In this appendix, we show how to determine if a graph has a semi-transitive orientation in $O(|\Delta|)$ time. (See [GO80] for a survey of algorithmic results on comparability graphs).

Recall that an orientation $U_G$, of a graph $G$, is semi-transitive, if and only if:

(i) if $\overline{ab}$ and $\overline{bc}$ are in $U_G$ then $ac$ is an edge of $G$.

We now give a necessary and sufficient condition for a graph to be semi-transitive.

Definition: We define a relation $R$ on directed edges of $G$ as follows. If $xy$ and $yz$ are edges of $G$ and $xz$ is not then $xy \ R \ yz$ and $yz \ R \ xz$. Also, $xy \ R \ yz$. $R^*$ is the transitive closure of $R$.

Definition: Let $E(\overline{ab})$ be the equivalence class of directed edges under $R^*$ which contains $\overline{ab}$. Let $E(\overline{ab})$ be the underlying set of undirected edges.

Observation D.1: The edge $xy$ is in $E(\overline{ab})$ if and only if $yz$ is in $E(\overline{ba})$. Thus, $E(\overline{ab})=E(\overline{ba})$.

Observation D.2: $G$ has a semi-transitive orientation if and only if for each edge $ab$ of $G$, $\overline{ba}$ is not in $E(\overline{ab})$.

Proof: It is easy to see that the "only if" part of this statement must hold. We shall prove the "if" part. To this end, consider a graph $G$ such that for each edge $ab$ of $G$, $\overline{ba}$ is not in $E(\overline{ab})$. By D.1, $R^*$ partitions the edges of $G$ into disjoint equivalence classes $E_1=E(x_1 \overline{y_1}), E_2=E(x_2 \overline{y_2}), \ldots, E_k=E(x_k \overline{y_k})$. Clearly, $E(x_1 \overline{y_1}) \cup E(x_2 \overline{y_2}) \cup \ldots \cup E(x_k \overline{y_k})$ is a semi-transitive orientation of $G$.

By D.2, to determine if a graph $G$ has a semi-transitive orientation we need only check if $\overline{ba}$ is in $E(\overline{ab})$ for each edge $ab$ of $G$. We now describe how to do this. We shall first construct a graph $F(G)$. For each edge $xy$ of $G$, $F(G)$ has two vertices $\{x,y\}$ and $\{y,x\}$. Two vertices of $F(G)$, $\{x,y\}$ and $\{a,b\}$, are adjacent if and only if $xy \ R \ ab$. 
Now, the components of $F(G)$ correspond to the equivalence classes under $R^*$. Thus, $\overrightarrow{ab}$ is in $E(\overrightarrow{ab})$ if and only if $\{a, b\}$ and $\{b, a\}$ are in the same component of $F(G)$. To determine if $G$ is semi-transitive, we need only construct $F(G)$ and then break $F(G)$ up into its connected components.

Clearly, the degree of a vertex $\{x, y\}$ in $F(G)$ is at most $d_G(x) + d_G(y)$. Thus $|E(F(G))|$ is at most $2\Delta_G m$. So, once we have constructed $F(G)$, we can break it into its components in $O(\Delta_G m)$ time.

To construct $F(G)$, we consider each of the edges of $G$ in turn. To compute the neighbors of $\{x, y\}$ and $\{y, x\}$ we simply need to traverse the adjacency lists of both $x$ and $y$.

If $z$ is adjacent to $x$ but misses $y$ then $\{x, y\}$ is adjacent to $\{x, z\}$ and $\{y, x\}$ is adjacent to $\{z, y\}$. Similarly, if $z$ is adjacent to $y$ but misses $x$ then $\{x, y\}$ is adjacent to $\{z, y\}$ and $\{y, x\}$ is adjacent to $\{y, z\}$. Thus, we can construct $F(G)$ in $O(\Delta_G m)$ time.
Appendix E

Definition: Let \( x \) be a vertex of a graph \( G \) and let \( k \) be a non-negative integer. Multiplying \( x \) by \( k \) consists of substituting a stable set of size \( k \) for \( x \). \( G' \) arises from \( G \) by multiplication if we can obtain \( G' \) from \( G \) by substituting a stable set for each vertex of \( G \).

Lovász's Theorem: The following three conditions are equivalent.

(1) \( G \) is perfect.
(2) \( \overline{G} \) is perfect.
(3) For every subgraph \( H \) of \( G \), \( |H| \leq \alpha(H)\omega(H) \).

Proof: Assume that the theorem is false and let \( G \) be a minimal counter-example. Clearly (1) implies (3) and (2) implies (3). Thus, (3) must hold on \( G \) and (1) or (2) must fail. Since \( \overline{G} \) is also a minimal counter-example to the theorem, we can assume (3) holds and (1) fails. Note that (1),(2) and (3) all hold on any subgraph of \( G \). Thus \( G \) and \( \overline{G} \) are minimal imperfect.

Lemma E.1: If \( G' \) arises from a perfect graph by multiplication then \( G' \) is perfect.

Proof: This is a trivial corollary of the fact that substitution preserves perfection.

Lemma E.2: If \( G' \) arises from \( G \) by multiplication then (3) holds on \( G' \).

Proof: Assume E.2 is false and let \( G_1 \) be the smallest graph which is a multiple of \( G \) and on which (3) does not hold. By E.1, every vertex of \( G \) was multiplied by at least one. Clearly, some vertex \( z \) of \( G \) was multiplied by at least two. Let \( z_1, z_2, \ldots, z_k \) be the stable set of \( G_1 \) which corresponds to \( z \).

Now, \( |G_1| \geq \alpha(G_1)\omega(G_1) \geq \alpha(G_1-z_k)\omega(G_1-z_k) \).

Also, \( |G_1-z_k| \leq \alpha(G_1-z_k)\omega(G_1-z_k) \).

Thus, \( \alpha(G_1)=\alpha(G_1-z_k) \) and \( \omega(G_1)=\omega(G_1-z_k) \). Furthermore, \( |G_1|=\alpha(G_1)\omega(G_1)+1 \).

Now, set \( G_2=G_1-\{z_1, z_2, \ldots, z_k\} \). Clearly, \( \alpha(G_2)=\alpha(G_1) \). Let \( \alpha'=\alpha(G_1) \). Since \( \overline{G_2} \) arises from \( G-z \) through substitution, \( \overline{G_2} \) is perfect. Thus \( G_2 \) can be covered by \( \alpha' \).
cliques. Note that \(|G_2| \geq \omega(G_1) + 1 - k\). Since \(k \leq \alpha\), at least \(\alpha - k + 1\) of the cliques in the cover of \(G_2\) contain \(\omega(G_1)\) vertices. Arbitrarily choose \(\alpha - k + 1\) cliques in the cover each of which contains \(\omega(G_1)\) vertices. \(G_2\) is the graph induced by the vertices of this clique and \(x_1\). Now \(|G_2| = \omega(G_1)(\alpha - k + 1) + 1\). By the minimality of \(G_1\), \(|G_2| \leq \omega(G_3)\alpha(G_3)\). Since \(G_2\) is a subgraph of \(G_1\), \(\omega(G_2) \leq \omega(G_1)\). It follows that \(G_2\) contains a stable set \(S\) with at least \(\alpha - k + 2\) vertices. Since \(G_3 - x_1\) can be covered by \(\alpha - k + 1\) cliques, \(S\) contains \(x_1\). But, \(S + \{x_2, x_3, \ldots, x_k\}\) is a stable set in \(G_1\) containing \(\alpha + 1\) vertices, a contradiction.

Now, since \(G\) is minimal imperfect, \(G\) does not contain a stable set which meets all its maximum cliques (this is lemma 2.18). Thus, to each stable set \(S\) of \(G\), we can associate a maximum clique \(C_S\) such that \(S\) and \(C_S\) are disjoint. For each vertex \(x\) of \(G\), let \(k(x)\) be the number of \(C_S\)s which contain \(x\). Consider the graph \(G'\) obtained by multiplying each vertex \(x\) by \(h(x)\).

By E.2, we have \(|G'| \leq \alpha(G')\omega(G')\). Let \(F\) be the family of stable sets of \(G\) and let \(f = |F|\). Then, \(|G'| = \sum_{x \in V} k(x) = f \omega(G)\). Thus, \(\alpha(G') \geq f\).

Let \(T\) be a maximum stable set of \(G'\). Clearly \(T\) corresponds to a maximum stable set of \(G\). In particular we have

\[|T| = \sum_{x \in S} h(x) = \sum_{U \in F - S} |\cap U_S| \leq \sum_{U \in F - S} 1 = f - 1.\]

But this contradicts \(\alpha(G') \geq f\).
Appendix F

Lemma F.1([Tuc77]): Consider a minimal imperfect Berge graph $G$. If $xy$ is an edge of $G$ which does not extend into a triangle then $G-xy$ is also a minimal imperfect Berge graph.

Proof: We show first that $G-xy$ is Berge. Since $G$ is Berge and $xy$ is in no triangles of $G$, $G-xy$ contains no $C_{2k+1}$ for $k \geq 2$. If $G-xy$ contained an odd hole of length at least 5, $C$, then in $G$, $C$ has only one chord, $zy$. Now, $zy$ is not a short chord of $C$ because it does not extend into a triangle. But this implies that $G$ contains an odd hole of length at least 5, a contradiction. Thus $G-xy$ is Berge.

We show now that $G-xy$ is minimal imperfect. Let $H$ be a proper subgraph of $G-xy$. If $H$ is a stable set then $H$ is good. Otherwise, $\omega(H+zy) = \omega(H)$. Thus, since $H+zy$ is good, so is $H$. It remains only to show that $G-xy$ is not good. If $x$ and $y$ had different colours in a good colouring of $G-xy$ then this colouring would also be a good colouring of $G$, a contradiction. Now, assume $x$ and $y$ both have colour 1 in a good colouring of $G$. Let $F$ be the graph induced by the vertices of colours 1 and 2 in this good colouring. Let $F_1$ be the component of $F$ containing $y$. Note that there is no even path from $x$ to $y$ in $G-xy$. Thus, $x$ is not in $F_1$, and we can obtain a good colouring of $G$ by swapping colours 1 and 2 in $F_1$. This contradicts the minimal imperfection of $G$. ■
Appendix G

Lemma G.1: Let $G$ be a claw free graph consisting of a cycle $C = \{v_1, v_2, \ldots, v_e\}$, of length at least 7, and a vertex $x$ such that:

(i) $\{v_1, v_2, \ldots, v_e\}$ induces a path
(ii) $\{v_e, v_1, \ldots, v_{e-1}, v_1\}$ induces a path, and
(iii) $N(x) = \{v_2, v_3, v_4, v_5\}$.

Then, $G$ contains an odd hole of length at least 5.

Proof: Assume G.1 is false and let $G$ be a counter-example.

Fact 1: $N(v_2) \cap (C - v_k) = \{v_1, v_3\}$.

Fact 2: Either $N(v_e) \cap (C - \{v_1, v_2, \ldots, v_{e-1}\}) = \{v_i, v_{i+1}\}$ for some $i$, or $N(v_e) \cap (C - \{v_1, v_2, \ldots, v_{e-1}\}) = \emptyset$.

Fact 3: Either $N(v_i) \cap (C - \{v_1, v_2, \ldots, v_{e-1}\}) = \{v_i, v_{i+1}\}$ for some $i$, or $N(v_i) \cap (C - \{v_1, v_2, \ldots, v_{e-1}\}) = \emptyset$.

Fact 4: $N(v_3) \cap (C - \{v_1, v_2, \ldots, v_{e-1}\}) = N(v_3) \cap (C - \{v_1, v_2, \ldots, v_e\})$.

Proof: Assume $v_e$ sees a vertex $v_j$ of $C - \{v_1, v_2, \ldots, v_e\}$ which $v_3$ misses. Note that $v_e$ sees $v_j$ as otherwise $\{v_4, v_5, v_6, v_7\}$ would be a claw. Now, by Fact 1, $j = 7$. 

By Fact 2, \( v_4 \) sees \( v_6 \). Now, \( v_6 \) sees \( v_e \) as otherwise \( \{v_4, v_5, v_e, v_6\} \) would be a claw.

This implies that \( \{x, v_e, v_7, v_8, v_5\} \) is an odd hole, a contradiction. By symmetry, Fact 4 holds.

We obtain a new shorter cycle \( C' \) from \( C \) by removing \( v_6 \) if \( v_6 \) sees \( v_7 \) and removing \( v_1 \) if \( v_6 \) sees \( v_2 \). By Fact 1 and (ii), \( C' - v_2 - v_3 \) is an induced path. If \( C' \) is a hole then either \( C' \) is an odd hole of length at least 5 or \( C' - v_2 - v_3 + x \) is. Otherwise, by Facts 2-4, \( N(v_4) \cap (C' - \{v_1, v_2, \ldots, v_8\}) = N(v_4) \cap (C' - \{v_1, v_2, \ldots, v_8\}) = \{v_i, v_{i+1}\} \) for some \( i \). Then, one of the three following sets of vertices induces an odd hole.

(i) \( C' - v_2 - v_3 + x \).

(ii) \( \{v_4, v_5, \ldots, v_i, v_{i+1}\} \).

(iii) \( \{v_i+1, v_{i+2}, \ldots, v_8, v_1, v_2, v_4, x\} \).
Appendix H

In chapter 5, we defined $P_4$-isomorphism in terms of graphs on the same labelled vertex set. Actually, Chvatal's original definition was as follows.

Definition: Two graphs $G$ and $H$, are $P_4$-isomorphic if there is bijection $f$ from $V(G)$ to $V(H)$ such that a set of four vertices $S$ in $G$ induces a $P_4$ if and only if $f(S)$ induces a $P_4$ in $H$.

This definition parallels that of isomorphism.

Definition: Two graphs, $G$ and $H$, are isomorphic if there is a bijection $f$ from $G$ to $H$ such that $xy$ is an edge of $G$ if and only if $f(x)f(y)$ is an edge of $H$.

For the remainder of this section, we shall use the above definition of $P_4$-isomorphism. Of course, given $G$ and $H$ which obey the above condition, we can obtain graphs which are $P_4$-isomorphic under our original definition by relabelling each vertex of $G$ with the name of its image. Thus, the SSPGC says the same thing under both definitions. In this section, we consider the complexity of the following decision problem.

Graph $P_4$-Isomorphism: Given graphs $G$ and $H$ determine if $G$ is $P_4$-isomorphic to $H$.

The above problem is closely related to the following one.

Graph Isomorphism: Given graphs $G$ and $H$ determine if $G$ is isomorphic to $H$.

Graph Isomorphism has been intensely studied for many years ([ReC77],[Ba81],[Lu80]). It is one of the best-known problems which has neither been shown to be in $P$ nor been shown to be NP-complete. We shall now show that Graph $P_4$-Isomorphism is polynomially equivalent to Graph Isomorphism.

Theorem H.1: Graph $P_4$-Isomorphism is polynomially equivalent to Graph Isomorphism.
Proof: We first show how to transform an instance of Graph $P_r$-Isomorphism into an instance of Graph Isomorphism. Consider an instance of Graph $P_r$-Isomorphism on graphs $G$ and $H$. We have $G = \{v_1, v_2, \ldots, v_n\}$ and $H = \{w_1, w_2, \ldots, w_n\}$. Enumerate the $P_a$ of $G$ as $S_1, S_2, \ldots, S_r$ and the $P_a$ of $H$ as $T_1, T_2, \ldots, T_r$. We shall restrict our attention to cases where $r$ is at least 5. The $P_a$ graph of $G$, $P_a(G)$ is the bipartite graph constructed as follows.

(i) $V(P_a(G)) = \{v_1', v_2', \ldots, v_n'\} \cup \{s_1, s_2, \ldots, s_r\} \cup \{x, y\}$.

(ii) $E(P_a(G)) = \{v_i' s_j \mid v_i \text{ is in } S_j\} \cup \bigcup_{j=1}^r \{z_j\} \cup \{xy\}$

We construct $P_a(H)$ in a similar fashion.

Lemma H.2: $G$ is $P_a$-isomorphic to $H$ if and only if $P_a(G)$ is isomorphic to $P_a(H)$.

Proof: Assume $G$ is $P_a$-isomorphic to $H$ and let $f$ be a $P_a$-structure preserving mapping from $G$ to $H$. We obtain an adjacency preserving mapping $g$ from $P_a(G)$ to $P_a(H)$ as follows.

(i) $g(v_i') = f(v_i')$ for $i = 1$ to $n$.

(ii) Given $S_i$, a $P_a$ in $G$, we know that $f(S_i)$ is a $P_a$ in $H$. Thus $f(S_i) = T_j$ for some $j$, $1 \leq j \leq r$. Set $g(s_i) = t_j$.

(iii) Finally, $g(x_G) = x_H$ and $g(y_G) = y_H$.

It is a routine matter (not even that tedious!!) to verify that $g$ an adjacency preserving map from $P_a(G)$ to $P_a(H)$ and thus $P_a(G)$ and $P_a(H)$ are isomorphic.

Conversely, assume there is an adjacency preserving map $g$ from $P_a(G)$ to $P_a(H)$. Since $G$ contains at least 5 $P_a$'s, $x_G$ sees at least 6 vertices of $P_a(G)$. So, $x_G$ is the unique vertex of maximum degree in $P_a(G)$. Similarly, $x_H$ is the unique vertex of maximum degree in $P_a(H)$. Thus, $g(x_G) = x_H$. Further, clearly $g(P_a(G) - x_G - N_{P_a(G)}(x_G)) = P_a(H) - x_H - N_{P_a(H)}(x_H)$. Thus $g(V_{\text{prime}}) = W'$. If we define $f$ so that $f$ maps $v_i$ to $w_j$ precisely if $g$ maps $v_i'$ to $w_j'$ then $f$ is a bijection from $V(G)$ to $V(H)$. Further if $S_i = \{a, b, c, d\}$ is a $P_4$ in $G$ which
corresponds to the vertex \( s_i \) of \( P_4(G) \) then \( f(S_i) = \{ f(a), f(b), f(c), f(d) \} \) is a \( P_4 \) in \( H \) which corresponds to the vertex \( g(s_i) \) in \( P_4(H) \). Thus, \( f \) is a \( P_4 \)-isomorphism.

Corollary H.3: Graph \( P_4 \)-Isomorphism is polynomially transformable to Graph Isomorphism

Proof: Given \( G \) and \( H \), we can construct \( P_4(G) \) and \( P_4(H) \) in \( O(n^4) \) time by considering every set of four vertices as a possible \( P_4 \). Then, by solving Graph Isomorphism on these two graphs we solve Graph \( P_4 \)-Isomorphism on \( G \) and \( H \). We shall now show how to polynomially transform Graph Isomorphism into Graph \( P_4 \)-Isomorphism. Assume we are given an instance of Graph Isomorphism on graphs \( G \) and \( H \). We may assume that neither of these two graphs have isolated points and that \( n = |G| = |H| \) is at least 5. We construct a new graph, \( G' \), from \( G \) by adding for each edge \( e = xy \) 2n new vertices \( v_{1,e}, v_{2,e}, \ldots, v_{2n,e} \) such that \( \{z, y, v_{1,e}, \ldots, v_{2n,e}\} \) and \( \{x, y, v_{n+1,e}, \ldots, v_{2n,e}\} \) are both holes of length \( n + 2 \) in \( G' \). Further, each new vertex of \( G' \) is adjacent only to two vertices, its neighbors on the hole. We construct \( H' \) in an analagous fashion.

Lemma H.4: \( G \) is isomorphic to \( H \) if and only if \( G' \) is \( P_4 \)-isomorphic to \( H' \).

Proof: Assume there is an isomorphism \( g \) from \( G \) to \( H \). we extend \( g \) to an isomorphism from \( G' \) to \( H' \) by setting \( g(v_{i,e}) = v_{i,g(e)(v)} \). Clearly, an isomorphism is a \( P_4 \)-isomorphism. Thus, if \( G \) is isomorphic to \( H \) then \( G' \) is \( P_4 \)-isomorphic to \( H' \).

Assume there is a \( P_4 \)-isomorphism, \( g \), from \( G' \) to \( H' \). Recall that under \( g \), the image of any disc of order at least 7 is a disc of the same order. Thus, for a vertex \( v \) of \( G' \), \( g(v) \) is in precisely as many discs of order \( n+2 \) as \( v \) is. Now the only discs of order \( n+2 \) in \( G' \) are the \( 2|E| \) holes we have created, 2 of which correspond to each edge of \( G \). An analagous statement holds for \( H' \). Clearly, each
new vertex of $G'$ is in exactly one hole of length $n + 2$. Since no vertex of $G$ is isolated, each vertex of $V(G)$ is in at least two holes of length $n + 2$ in $G'$. Similarly, each vertex of $V(H)$ is in at least two holes of length $n + 2$ in $H'$. It follows that $g(V(G)) = V(H)$. Further, if $x$ and $y$ are vertices of $G$, then some hole of length $n + 2$ in $G'$ contains both $x$ and $y$ if and only if some hole of length $n + 2$ in $H'$ contains both $g(x)$ and $g(y)$. However, some hole of length $n + 2$ in $G'$ contains both $x$ and $y$, if and only if $x$ is adjacent to $y$. A similar result holds for $g(x)$ and $g(y)$ in $H'$. Thus, if we restrict $g$ to $V(G)$ then it is an isomorphism from $G$ to $H$.

**Corollary H.5**: Graph isomorphism is polynomially transformable to Graph $P_n$-Isomorphism.

**Proof**: Given $G$ and $H$ we can construct $G'$ and $H'$ in $O(2nm)$ time in a straightforward fashion.
GLOSSARY
Glossary

adjacent: two vertices, \( x \) and \( y \), are adjacent if \( xy \) is an edge of \( G \).

adjacency list: let \( x \) be a vertex of a graph \( G \). An adjacency list for \( x \) is a list of all those vertices to which \( x \) is adjacent.

amalgamation: \( G \) arises from \( G_1 \) and \( G_2 \) by amalgamation in the following manner. Choose a vertex \( x_1 \) of \( G_1 \) and a vertex \( x_2 \) of \( G_2 \). Choose a clique \( K_1 \) in \( N_{G_1}(x_1) \) and a clique \( K_2 \) in \( N_{G_2}(x_2) \) such that \( K_1 \) and \( K_2 \) have the same size. Further, we require that each vertex of \( K_1 \) sees every vertex of \( N_{G_1}(x_1)-K_1 \), for \( i=1,2 \). We obtain \( G \) from \( G_1 \) and \( G_2 \) by
1. identifying the vertices of \( K_1 \) with those of \( K_2 \).
2. adding all the edges between \( N(x_1)-K_1 \) and \( N(x_2)-K_2 \).
3. deleting \( x_1 \) and \( x_2 \).

antitwins: two vertices, \( x \) and \( y \), are antitwins if every vertex in \( G-x-y \) either sees \( x \) and misses \( y \) or sees \( y \) and misses \( x \).

Berge graphs: a graph is Berge if it contains no odd disc.

bipartite graphs: a graph is bipartite if it can be two-coloured. Equivalently, a graph is bipartite if it contains no odd cycle.

Burra graphs: we define the class of Burra graphs as follows. A graph \( G \) is a Burra graph if for every subgraph \( H \) of \( G \) one of the following holds.
1. \( H \) or \( \overline{H} \) contains an even pair.
2. \( H \) or \( \overline{H} \) has a star cutset.
3. \( H \) or \( \overline{H} \) is in LGBG.

checkerboard graphs: \( G \) is a checkerboard graph if for each vertex \( x \) of \( G \), \( G-x \) can be partitioned into \( \omega(G) \) stable sets of size \( \alpha(G) \) and \( \alpha(G) \) cliques of size \( \omega(G) \).

chord: a chord of a path is an edge between two vertices \( v_i \) and \( v_j \) on the path such that \( i \neq j+1 \) and \( i \neq j-1 \). A chord of a cycle is an edge between two vertices \( v_i \) and \( v_j \) in the circuit such that \( i \neq j+1 \) and \( i \neq j-1 \) (where addition is modulo \( k+1 \)).

chromatic number: the chromatic number of \( G \), \( \chi(G) \), is the minimum number of colours needed to colour the vertices of \( G \) so that no two adjacent vertices receive the same colour.

clique: a clique is a set of pairwise adjacent vertices. A \( k \)-clique is a clique of size \( k \).

clique cover number: a clique cover is a set of cliques of \( G \) such that each vertex is in one of the cliques. The clique cover number of \( G \), \( \theta(G) \), is the number of cliques in the smallest clique cover of \( G \).

clique cutset see cutset
clique cutset tree: a clique cutset tree of a graph \( G \), \( T(G) \), is a tree each of whose nodes is labelled by a subgraph \( H \) of \( G \). We recursively define the tree as follows. The root of the tree is labelled \( G \). If \( G \) has no clique cutset then we are done. Otherwise find a clique cutset \( C \) of \( G \). Now, \( G - C \) splits into components \( H_1, H_2, \ldots, H_k \). The \( k \) children of \( G \) in \( T(G) \) will be labelled by \( C \cup H_1, C \cup H_2, \ldots, C \cup H_k \). Now, the subtree of \( T(G) \) rooted at the child of the root with label \( C \cup H_i \) is a clique cutset tree of \( C \cup H_i \).

clique number: The clique number of \( G \), \( \omega(G) \), is the size of the largest clique in \( G \).

clique separable graph: a graph is clique separable if and only if each of its subgraphs is either bipartite, complete multi-partite, or contains a clique cutset.

colouring: an assignment of colours to the vertices of \( G \) so that no two adjacent vertices receive the same colour. A \( k \)-colouring is a colouring which uses \( k \) colours.

comparability graph: a graph in which the edges can be oriented so that if \( xy \) and \( yz \) are in the orientation so is \( zx \).

complement: the complement of \( G \), which is denoted \( \overline{G} \), has the same vertex set as \( G \). Two vertices are adjacent in \( \overline{G} \) if and only if they are non-adjacent in \( G \).

complete multi-partite graph: a graph is complete \( k \)-partite if it can be split into \( k \) stable sets such that if \( x \) and \( y \) are in different stable sets then \( x \) and \( y \) are adjacent. A graph is complete multi-partite if it is complete \( k \)-partite for some \( k \geq 2 \).

component: a component of a graph \( G \) is a maximal connected subgraph of \( G \).

connected: a graph is connected if there is a path between every two vertices in the graph.

cutpoint: A cutpoint is a one vertex cutset.

cutset: a cutset in a graph \( G \), is a subset \( C \), of the vertices of \( G \) such that \( G - C \) is disconnected. A clique cutset is a cutset whose vertices induce a clique. In a star cutset, \( C \), some vertex \( x \) of \( C \) sees all of \( C - x \).

cycle: a set of vertices \( v_0, v_1, \ldots, v_k \) such that \( v_i v_{i+1} \) is an edge (where addition is modulo \( k+1 \)).

decoration: \( G \) is a decoration of \( H \), if \( G \) has the same maximum cliques and maximum stable sets as \( H \) and has more edges.

degree: the degree of a vertex in a graph is the number of vertices it is adjacent to.

Dilworth number: the Dilworth number of a graph \( G \), \( D(G) \), is defined as the maximum cardinality of a set of vertices of \( G \) in which any pair of vertices have incomparable neighborhoods.

disagrees: a vertex \( x \) disagrees on two other vertices if it sees one and misses the other. If \( x \) disagrees on \( a \) and \( b \) then \( a \) and \( b \) disagree on \( x \).
a graph is a disc if it is a hole of length at least 5 or the complement of such a hole.

disconnected a graph is disconnected if it contains two vertices which are not connected by a path.

dominates: a vertex \( z \) dominates a vertex \( y \) in a graph \( G \) if \( z \) sees every vertex which \( y \) sees. In this case, \( y \) is a dominated vertex.

dominating stable set: a stable set, \( S \), is dominating if every maximal clique contains a vertex of \( S \).

duplication: we duplicate a vertex \( z \) of a graph \( G \) by adding a vertex \( y \) such that \( \hat{N}_{G+\hat{y}}(y) = N_{G}(z) \).

endomorphism: an endomorphism of a graph is a mapping, \( f \), from the vertex set \( V \) into itself such that if \( x \) sees \( y \) in \( G \) then \( f(x) \) sees \( f(y) \) in \( G \). An endomorphism, \( f \), is proper if \( f(V) \) is a proper subset of \( V \).

even pair: a pair of vertices, \( x \) and \( y \), in a graph is even if every induced path from \( x \) to \( y \) has an even number of edges.

even vertex: a vertex of a graph is even if it is in no odd cycles in the graph.

even vertex addition: \( G' \) arise from \( G \) through even vertex addition if there is an even vertex \( y \) in \( G' \) such that \( G' = G + y \).

false twins: see twins.

fragile: a graph \( G \) is fragile if either \( G \) or \( \overline{G} \) has a star cutset.

good colouring: a good colouring of a graph \( G \) is one which uses \( \omega(G) \) colours.

good graph: a graph \( G \) is good if it can be coloured with \( \omega(G) \) colours.

good stable set: a stable set \( S \) in a graph \( G \) is good if every maximal clique of \( G \) contains a vertex of \( S \).

hereditary class: if a class \( A \) of graphs is hereditary then if \( G \) is in \( A \) so is every induced subgraph of \( G \).

hitting set: a hitting set, of a graph \( G \), is a subset \( S \) of the vertices of \( G \) such that every maximum clique and every maximum stable set contains a vertex of \( S \). A hitting set of \( G \) is small if it contains no more than \( \omega(G) + \omega(G) - 1 \) vertices.

hole: a hole is an induced cycle. That is, a cycle with no chords.

homogeneous set: a homogeneous set in a graph \( G \) is a proper subset \( H \) of the vertices of \( G \) containing at least two vertices and such that \( V(G) - H \) can be partitioned into \( A = \{ x \mid x \) sees every vertex of \( H \} \) and \( B = \{ x \mid x \) misses every vertex of \( H \} \).
incomparable neighborhoods: two vertices \(z\) and \(y\) in a graph \(G\) have incomparable neighborhoods if there is a vertex \(w\) which sees \(z\) and not \(y\) and a vertex \(v\) which sees \(y\) and not \(z\).

innocuous vertex: a vertex of a graph is innocuous if it is in no odd cycles of length at least five in the graph.

innocuous vertex addition: \(G'\) arises from \(G\) through innocuous vertex addition if there is an innocuous vertex \(y\) in \(G'\) such that \(G' = G + y\).

internal node: a vertex in a tree is an internal node if it is not a leaf.

invariant: let \(G\) and \(H\) be \(P_5\)-isomorphic graphs on a common vertex set \(V\). A subset \(S\) of \(V\) is invariant if it induces the same graph in both \(G\) and \(H\). \(S\) is an invariant path(cycle,etc.) if \(S\) is invariant and induces a path(cycle,etc.).

leaf: a leaf of a tree is any vertex except the root which has degree one.

line graph: let \(G\) be a graph. The line graph of \(G\), \(L(G)\), has \(|E(G)|\) vertices. Each vertex of \(L(G)\) corresponds to an edge of \(G\). Two vertices of \(L(G)\) are adjacent if the corresponding edges of \(G\) have a vertex in common.

line graph of bipartite graphs(LGBG): see above.

maximal: a stable set (clique) is maximal if it is not contained in a larger stable set (clique).

maximum: a stable set, in a graph \(G\), is maximum if it contains \(\omega(G)\) vertices. A clique, in a graph \(G\), is maximum if it contains \(\omega(G)\) vertices.

Meyniel closure: let \(B\) be a class of graphs. A graph \(G\) is in the Meyniel closure of \(B\), \(B^M\), if for every subgraph \(H\) of \(G\) either:

(i) \(H\) has an even pair,

(ii) \(H\) has an even pair, or

(iii) \(H\) is in \(B\).

Meyniel graph: a graph \(G\) is Meyniel if every odd cycle of \(G\) with at least five vertices has at least two chords.

minimal imperfect: a graph is minimal imperfect if it is not perfect but all of its induced subgraphs are.

misses: in a graph \(G\), \(x\) misses \(y\) if \(xy\) is not an edge of \(G\).

mixed: let \(G\) be a graph and let \(D\) be a disk in \(G\). A vertex \(x\) of \(G-D\) is mixed with respect to \(D\) if it sees some but not all of the vertices of \(D\).

neighbor: in a graph \(G\), \(y\) is a neighbor of \(x\) if \(xy\) is an edge of \(G\). The neighborhood of \(x\) is the set of neighbors of \(x\).
non-adjacent: two vertices $x$ and $y$ in a graph $G$ are non-adjacent if $xy$ is not an edge of $G$.

odd pair: a pair of vertices, $x$ and $y$, in a graph $G$, is odd if every induced path from $x$ to $y$ in $G$ has an odd number of edges.

orientation: we orient a graph $G$ by assigning a direction to each of the edges of $G$. Thus for an edge $xy$, we can choose either $xy$ or $yx$ to be in the orientation. An orientation, $U$, is transitive if $\{xy$ and $yz$ in $U\}$ $\Rightarrow$ $xz$ in $U$. The orientation is semi-transitive if $\{xy$ and $yz$ in $U\}$ $\Rightarrow$ $xz$ or $zx$ in $U$.

ornamental edge: an edge is ornamental if it is in no maximum clique and if its deletion does not create a maximum stable set.

$P_4$: an induced path with three edges and four vertices.

$P_4$-free graphs: a graph is $P_4$-free if it does not contain a $P_4$.

$P_4$-isomorphic: two graphs $G$ and $H$, with common vertex set $V$, are $P_4$-isomorphic if each set of four vertices of $V$ induces a $P_4$ in $G$ if and only if it induces a $P_4$ in $H$.

$P_4$-structure: let $G$ be a graph. The $P_4$-structure of $G$, $P_4(G)$, is the set of quadruples of vertices of $G$ which induce $P_4$s in $G$.

partially ordered set: a partially ordered set is a set $P$ of elements and an binary relation $\prec$ on $P$ such that:

(i) for $a,b,c$ elements of $P$ if $a \prec b$ and $b \prec c$ then $a \prec c$ ($\prec$ is transitive).

(ii) there do not exist $a,b$ elements of $P$ such that $a \prec b$ and $b \prec a$

($\prec$ is anti-symmetric).

path: a path is a set of vertices $v_0, v_1, \ldots, v_k$ such that $v_i$ sees $v_{i+1}$. The endpoints of a path are $v_0$ and $v_k$.

perfect graphs: a graph $G$ is perfect if each of its induced subgraphs is good.

perfection preserving operation: a perfection preserving operation is an algorithm which takes as input a set of graphs $\{G_1, \ldots, G_k\}$ and constructs a new graph $G'$ such that if each of the input graphs is perfect then $G'$ is perfect.

planar graph: a graph is planar if it can be drawn in the plane so that no edges cross and the only vertices touched by an edge $xy$ are $x$ and $y$.

proper endomorphism: see endomorphism.

quasi-parity graphs (QPG): a graph $G$ is a quasi-parity graph if for every induced subgraph $H$ of $G$, with more than one vertex, either $H$ or $\overline{H}$ has an even pair.

remote: let $G$ be a graph and let $D$ be a disk in $G$. A vertex $x$ in $G$-$D$ is remote with respect to $D$ if it sees no vertex of $D$. 
root: given a tree, we can specify any vertex to be the root.

sees: a vertex x sees a vertex y in a graph G if xy is an edge of G.

semi-transitive orientation: see orientation.

separator: let x and y be non-adjacent vertices in a graph G. A cutset S of G is a separator of x and y if x and y are in different components of G-S.

simplicial vertex: a vertex is simplicial if its neighborhood induces a clique.

singleton: a graph is a singleton if it consists of a single vertex.

skew partition: a skew partition of a graph G is a partition of the vertices of G into four disjoint non-empty sets, A,B,C, and D, such that:

(1) If x is in A and y is in D then xy is not an edge of G. (Equivalently B ∪ C is a cutset of G).

(2) If x is in B and y is in C then xy is an edge of G. (Equivalently A ∪ D is a cutset of G).

small hitting set: see hitting set.

spanning tree: let G be a connected graph. A spanning tree of G is a tree which contains all the vertices of G.

stability number: the stability number of G, α(G), is the size of the largest stable set in G.

stable set: a stable set in a graph is a set of pairwise non-adjacent vertices. A k-stable set is a stable set of size k.

star cutset: see cutset.

star closure: let B be a class of graphs. We define the star-closure of B, B*, as follows. A graph G is in B* if every unbreakable subgraph of G is in B.

strongly perfect graph: a graph is strongly perfect if each of its induced subgraphs has a good stable set (that is a stable set which meets all the maximal cliques).

substitution: let G and H be graphs and let x be a specified vertex of G. A new graph, G', arises from substituting H for x in G in the following manner.

To obtain G' take the disjoint union of H and G-x and for every pair of vertices y and z with y in G-x and z in H add the edge zy if and only if xy is an edge of G.

subgraph: H is a subgraph of G if V(H) ⊆ V(G) and E(H) ⊆ E(G). H is an induced subgraph if E(H) contains all the edges of G which have both of their endpoints in V(H).
transitive orientation: see orientation.

tree: a graph is a tree if it contains no cycle.

triangle: a triangle is a clique with three vertices.

triangulated graph: a graph is triangulated if it contains no hole with more than three vertices.

true twins: see twins

twins: two vertices \( x \) and \( y \) of a graph \( G \) are twins if \( N_{G-x-y}(x) = N_{G-x-y}(y) \). False twins are adjacent, true twins are not.

undecorated: a graph is undecorated if the removal of any of its edges either creates a maximum stable set or destroys a maximum clique.

universal: let \( G \) be a graph and let \( D \) be a disk in \( G \). A vertex \( x \) of \( G-D \) is universal with respect to \( D \) if \( x \) sees all the vertices of \( D \).

variant pair: let \( G \) and \( H \) be \( P \)-isomorphic graphs on a common vertex set \( V \). A pair of vertices \( x \) and \( y \) in \( V \) is variant if \( x \) misses \( y \) in one graph and sees \( y \) in the other.

weakly bipartite graph: a graph is weakly bipartite if it contains no odd cycle of length greater than 3.

weakly triangulated graph: a graph is weakly triangulated if it contains no disc.

wreath checkerboard graph: a graph is a wreath checkerboard graph if it is a decoration of \( C_{\alpha \omega+1} \) for some \( \alpha \) and \( \omega \).
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