FINITE-ELEMENT MODELLING OF MIDDLE-EAR PROSTHESES IN CAT

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements of the degree of Master of Engineering

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Montréal, Canada
November 1993

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To my parents, Amina and Badrudin, for their support and encouragement.
ABSTRACT

Discontinuity of the middle-ear ossicular chain results in conductive hearing loss. Two prostheses commonly used to surgically repair the ossicular chain are the MSA and the MFA. In the MSA, a strut is connected between the malleus and the head of the stapes, while in the MFA a strut is connected between the malleus and the footplate of the stapes. In this work, finite-element models of the MSA, MFA and normal cat middle ear are developed. The models are valid for low frequencies (below 300 Hz) and for physiological sound levels. The mechanical behaviour of the normal middle-ear model is compared with that of the MFA and MSA models. Several parameters are then varied in order to study their relative importance in the models. The effects of strut placement on the behaviour of the MSA and MFA models are also investigated.
RÉSUMÉ

Une discontinuité dans la chaîne des osselets de l'oreille moyenne occasionne la perte d'ouïe conductive. Deux prothèses couramment utilisées pour réparer la chaîne des osselets sont la prothèse entre le marteau et l'étier (PME) et la prothèse entre le marteau et la base d'étier (PMB). Dans ce travail, la méthode des éléments finis est utilisée pour développer des modèles de la PME, de la PMB et de l'oreille moyenne normale du chat. Les modèles sont valides pour les basses fréquences (moins de 300 Hz) et pour les niveaux de son physiologiques. Le fonctionnement mécanique du modèle de l'oreille moyenne normale est comparé avec celui de la PME et celui de la PMB. Plusieurs paramètres sont variés pour étudier leur importance pour les modèles. L'effet du placement de la greffe sur le fonctionnement de la PME et de la PMB est examiné.
ACKNOWLEDGEMENTS

I am indebted to my supervisor, Dr. W. Robert J. Funnell, for the guidance and support he has provided throughout the course of this work. His patience, despite my many, many questions, is greatly appreciated. (Remember, you just have to look at the thesis one more time!)

I am also grateful to Dr. M. D. Sc'loss, Chairperson of the Department of Otolaryngology, for sharing with me his wealth of knowledge in the area of middle-ear surgery.

I would like to thank several of my friends for helping with the preparation of this thesis. Jennifer Day, Mark Meehan and Luckshman (Lucky) Parameswaran made heroic efforts to proofread various chapters of this thesis. Their honest yet considerate criticisms of this work have helped much in improving its quality. I would also like to thank Thomas Ranger and Baidar Chaudhry, the first for helping with the translation of the abstract, and the second for providing me with some yummy meals and assistance in putting together the manuscript.

This work was supported by the Natural Sciences and Engineering Research Council of Canada and the Medical Research Council of Canada.
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NOTATION

LATIN SYMBOLS

\( A_f \)  
Area of footplate.

\([B]\)  
Strain-displacement matrix.

\([C]\)  
Matrix of material stiffnesses.

\( \{d\}, \{D\} \)  
Element and structure nodal d.o.f. vectors, respectively.

\( D \)  
Average displacement amplitude due to deformation.

\( E \)  
Young's modulus.

\( \{F\} \)  
Vector of body forces.

\([k], [K]\)  
Element and structure stiffness matrices, respectively.

\( F_i \)  
Complex force acting on stapedial footplate.

\([N]\)  
Matrix of interpolation functions.

\( \{P\} \)  
Vector of concentrated nodal loads.

\( P_i \)  
Pressure acting on stapes, a complex quantity.

\( R \)  
Average displacement amplitude due to rigid-body motion.

\( \{r\}, \{R\} \)  
Element and structure load vectors, respectively.

\( r, r' \)  
Position vectors.

\( \{s\} \)  
Intra-element displacements.

\( s \)  
Displacement vector.

\( U \)  
Strain energy of structure.

\( U_i \)  
Volume velocity of stapes, a complex quantity.

\( u, v \)  
Displacements in x and y directions.

\( V_i \)  
Linear velocity of the stapes, a complex quantity.

\( x, y \)  
Cartesian coordinates.

\( x', y' \)  
Cartesian coordinates attached to footplate.

\( Z_{ac} \)  
Acoustic impedance of stapediocochlear complex.

\( Z_{me} \)  
Mechanical impedance of stapediocochlear complex.

GREEK SYMBOLS

\( \alpha, \beta \)  
Generalized coordinates.

\( \{\epsilon\} \)  
Strain vector.

\( \xi_1, \xi_2, \xi_3 \)  
Area coordinates.

\( \{\sigma\} \)  
Stress vector.

\( \nu \)  
Poisson's ratio.

\( \Pi \)  
Potential energy of a structure.

\( \Omega \)  
Potential energy of applied loads.
\( \Delta \theta \) Angle of rotation.
\( \{ \Phi \} \) Vector of surface tractions.
CHAPTER 1
INTRODUCTION

Over the past three decades, several factors have led to a resurgence of interest in the surgical treatment of hearing loss caused by middle-ear disease (Smyth, 1987). In that time, a plethora of surgical techniques and prostheses have emerged that claim to restore middle-ear function; even as recently as the past one or two years, several new prostheses have been proposed (e.g., Fleischer, 1991; Ledoux et al., 1991; Beoni, 1993). None of these prostheses result in the complete reduction of hearing loss. Two of the issues involved in the design of such prostheses are the biocompatibility of the materials used to make the prostheses and the mechanical behaviour of these prostheses. Although much attention has focused on the biocompatibility of middle-ear prostheses, little theoretical or experimental work has been done to assess the degree to which the mechanics of the normal middle ear are replicated by such prostheses. The overall objective of this research is to develop finite-element models of the surgically corrected middle ear that will aid in the design and evaluation of middle-ear prostheses. The specific objective of this thesis is to model the mechanics of the surgically corrected cat middle ear. Two prostheses are modelled in the course of this work: the malleus-stapes assembly (MSA) and the malleus-footplate assembly (MFA). Both prostheses are used to correct a discontinuous middle-ear ossicular chain. A model of the normal cat middle ear is also developed in order to permit comparison with models of the surgically corrected middle ear. The models are restricted to physiological sound levels and to low frequencies. The middle-ear cavities and the middle-ear muscles are not modelled in this
work; this is consistent with experimental situations in which the cavities are widely opened, and the middle-ear muscles are relaxed.

In order to appreciate the task that middle-ear prostheses must perform, it is necessary to review the anatomy and mechanics of the normal middle ear as will be done in Chapters 2 and 3, respectively. The two common types of ossicular discontinuities and the prostheses used to correct them, the MSA and the MFA, will be discussed in Chapter 4. Fundamentals of the finite-element method will be reviewed in Chapter 5. Finite-element models of the MSA and MFA will be presented in Chapter 6, and simulation results in Chapter 7. Conclusions and areas for future research are discussed in Chapter 8.
CHAPTER 2

ANATOMY OF THE MIDDLE EAR

2.1 INTRODUCTION

The anatomy of the human middle ear is reviewed in Section 2.2. As this thesis is concerned with modelling the mechanics of prostheses in the cat middle ear, the anatomy of this species is compared with that of the human in Section 2.3. The surgically corrected cat middle ear was modelled instead of the human middle ear for two reasons. First, there are more experimental data available for the normal cat middle ear than for the human. Second, experiments on cats can be done on live animals, whereas most experiments on humans must be done in cadavers and are thus susceptible to degenerative effects (Zwislocki and Feldman, 1963; Rosowski et al., 1990).

2.2 THE HUMAN MIDDLE EAR

2.2.1 Introduction

Figure 2.1 is a schematic diagram of the human middle ear. The middle ear includes several interconnected air-filled cavities that are separated from the ear canal by the eardrum, or tympanic membrane. Traversing these cavities is a chain of three bones, or ossicles, which serve to link the eardrum to the cochlea; this so-called ossicular chain is suspended by various ligaments and is acted upon by two muscles. The following sections further discuss the details of the middle-ear cavities, the eardrum and the ossicular chain.
Figure 2.1 The human middle ear. After Palmer (1993).

AC = Air Cells, HM = Head of Malleus, EC = Ear Canal, TTT = Tendon of Tensor Tympani muscle, MM = Manubrium of Malleus, U = Umbo, TS = Tendon of Stapedius muscle, ED = Eardrum, TY = Tympanum, TSu = Tympanic Sulcus, TPT = Tympanic Portion of Temporal bone, AN = Antrum, AD = Aditus, ER = Epitympanic Recess, TT = Tegmen Tympani, BI = Body of Incus, PPT = Petrous Portion of Temporal bone, LOI = Long process of Incus, LEI = Lenticular process of Incus, NS = Neck of Stapes, VE = Vestibule, SF = Stapedial Footplate, CS = Crura of Stapes, OW = Oval Window, BC = Basal turn of cochlea, RW = Round Window, PE = Pyramidal Eminence, TTM = Tensor Tympanic Muscle, ET = Eustachian Tube, CET = Cartilaginous portion of Eustachian Tube
2.2.2 The Middle-Ear Cavities

As illustrated in Figure 2.1, the middle-ear cavities are quite complex in form. The chief cavity is the tympanum proper which is bounded laterally by the eardrum. Medially, it is bounded by the labyrinthine wall formed in part by the cochlea; two windows, the round one and the oval one, open into this wall. The small region above the tympanum and superior to the eardrum is the epitympanic recess, or attic. Posterior to the epitympanic recess and connected to it by an aperture known as the aditus is a small space called the mastoid antrum. Numerous irregularly shaped air cells in the mastoid portion of the temporal bone are connected to the antrum. All of these cavities are lined with mucous membrane and communicate with the external environment via the Eustachian tube. As mentioned previously, the middle-ear cavities are not modelled in this work; the effects of neglecting the cavities are explained in Chapter 6.

2.2.3 The Eardrum

Funnell and Laszlo (1982) provide a detailed review of eardrum structure and function; some aspects of the gross anatomy and microscopic structure of the eardrum are discussed below.

Gross Anatomy

As illustrated in Figure 2.2a, the eardrum has a conical shape. The cone points medially; its deepest point is called the umbo. The curvature of the sides of the cone formed by the eardrum is convex outward. The major diameter of the eardrum has a
Figure 2.2 The human eardrum. (a) Schematic diagram showing conical shape. From Rabbitt (1985, p.25). (b) Outline of the eardrum showing gross anatomical regions. From Funnell (1975, p. 4), after Kojo (1954).
length of approximately 9.0 mm to 10.2 mm; the minor diameter has a length of 8.5 mm to 9.0 mm (Donaldson et al., 1992). Figure 2.2b shows an outline of the tympanic ring, or annulus, of the eardrum. The major parts of the eardrum (the pars tensa and the pars flaccida) and the manubrium of the malleus can also be seen. The pars tensa is a very thin sheet of connective tissue, about 30 μm to 90 μm in thickness (Lim, 1970); it represents most of the surface area of the eardrum. The manubrium of the malleus is firmly attached to the pars tensa at the level of the umbo, but the attachment becomes progressively looser towards the top of the manubrium (Graham et al., 1978). An annular ligament firmly attaches the pars tensa to bone around most of its circumference and separates it from the pars flaccida, a very small and very elastic portion of the drum located superior to the pars tensa. The pars flaccida has a thickness of 30 μm to 230 μm (Lim, 1970).

**Microscopic Structure**

The pars tensa consists of three layers as shown in Figure 2.3: an outer epidermal layer that is continuous with the epidermal lining of the ear-canal, an inner mucosal layer that is continuous with the mucosal lining of the middle-ear cavities, and a fibrous layer (the lamina propria) that is sandwiched between these two layers. The lamina propria in turn consists of 4 layers as shown in the figure. The radial and circular layers of the lamina propria form the main structural components of the eardrum. The radial layer consists of collagen fibrils and finer fibrils that radiate outward from the manubrium to the annular ligament around the eardrum. The circular layer consists of fibres arranged
Figure 2.3 Pars tensa. Schematic diagram of a cross section through the pars tensa of the squirrel monkey; the organization of the pars tensa in humans is similar to this. From Funnell (1972, p. 8), after Lim (1968).
more or less concentrically around the manubrium. The pars flaccida is similar to the
pars tensa but lacks the highly organized radial and circular layers of the lamina propria.

2.2.4 The Ossicular Chain

Three bones, the malleus, the incus and the stapes, form the ossicular chain that
serves to link the eardrum to the cochlea. As shown in Figure 2.4, the malleus possesses
a head, a neck, a handle (manubrium), and two processes, the lateral one and the anterior
one. The head of the malleus is connected to the incus at the incudomalleal joint. The
incus has a body and two processes, the long one and the short one. The extremity of
the long process, which is called the lenticular process, articulates with the head of the
stapes at the incudostapedial joint. The stapes consists of a more or less oval footplate,
or base, surmounted by two crura that merge laterally into a neck and a head; the crura,
the neck and the head are collectively referred to as the stapes superstructure. The
footplate of the stapes fills the oval window of the cochlea.

Several ligaments suspend the ossicular chain in the middle-ear space. The
malleus is supported by the anterior malleal and the lateral malleal ligaments; the
superior malleal ligament is merely a fold of mucous membrane (Kobayashi, 1954). The
incus is held in place by the posterior incudal ligament which connects the short process
of the incus to the posterior wall of the epitympanic recess; a superior incudal ligament
has been described but is absent in general (Kobayashi, 1954). As shown in Figure 2.5,
an annular ligament surrounds the footplate and holds it in place in the oval window.
Figure 2.4 The middle-ear ossicular chain. The malleus, incus and stapes are shown with their various processes. From Donaldson et al. (1992, p. 224).

Figure 2.5 Footplate and annular ligament. A typical adult human footplate (FP) and stapediovestibular joint (SVJ) are shown. The stapediovestibular joint contains the annular ligament. An articular cavity (as shown by the arrow) is found in most humans near the posterior (POST) end of the footplate. ANT indicates the anterior end. From Bolz and Lim (1972).
The two muscles acting on the ossicular chain are the tensor tympani and the stapedius; both are penniform muscles. The fibres of the tensor tympani are encased in a canal alongside the Eustachian tube; the tendon of this muscle is attached to the medial aspect of the malleal neck. The fibres of the stapedius lie in a canal posterior to the tympanic cavity; its tendon is attached to the posterior aspect of the neck of the stapes. As mentioned previously, the middle-ear muscles are not modelled in this work; the effects of neglecting the muscles are explained in Chapter 6.

2.3 THE CAT MIDDLE EAR

The overall configuration of the cat middle ear is similar to that of the human middle ear. However, several differences in detail exist (Funnell, 1989); the significance of these differences is not very well understood. Figure 2.6 shows schematic diagrams of the cat and human middle ears. The middle-ear cavities of the cat are somewhat different from those of humans: the tympanum is larger and divided into two cavities, the ectotympanum (also called the bulla cavity) and the entotympanum. The cavities are separated by a bony septum which contains an aperture that permits communication between the two cavities. Furthermore, the attic in cats is smaller than in humans, and there are no mastoid air cells.

The eardrum of the cat has a conical shape as in humans. The curvature of the sides of the cone formed by the eardrum is convex outward. As shown in Figure 2.7, the outline of the cat eardrum is somewhat elongated, whereas as that of humans is more or less circular. The length of the cat eardrum parallel to the manubrium is approximately
Figure 2.6 Schematic representation of the cat and human middle ears. From Funnell (1989, p. 12), after Funnell (1972).
Figure 2.7 Outlines of (a) human and (b) cat eardrums. From Funnell (1975, p. 6), (a) after Fumigalli (1949), (b) after Khanna (1970).
9.2 mm; the length perpendicular to the manubrium is approximately 6.3 mm.

As in the human middle ear, there are three ossicles in the cat middle ear: the malleus, the incus and the stapes. The manubrium of the malleus is firmly attached along its entire length to the pars tensa, and the stapes is attached to the oval window by an annular ligament. The angle formed between the manubrium and the ossicular axis of rotation is approximately 30° (Khanna, 1970); on the other hand, in humans the manubrium is almost perpendicular to this axis. (The axis of rotation is shown in Figure 3.1.) The geometrical relationship between the manubrium and the footplate is also different for the two species: in cats, the footplate is located behind the upper end of the manubrium; in humans, it is located at the level of the umbo as shown in Figure 2.8. Also, in cats the annular ligament does not contain articular spaces; on the other hand, articular spaces have been found in most human temporal bones (Botz and Lim, 1972). The only true ligament in the cat besides the annular ligament of the footplate is the posterior incudal ligament.
Figure 2.8 Ossicles of the (a) cat and (b) human (right ear, lateral view). From Funnell (1972, p. 10), (a) after Jayne (1898), (b) after Nager and Nager (1953).
CHAPTER 3
MECHANICS OF THE NORMAL MIDDLE EAR

3.1 INTRODUCTION

In the typical mammalian ear, sound waves are funnelled towards the eardrum by the ear canal; these waves cause the eardrum and the ossicular chain to vibrate. Vibrations of the stapes, the most medial ossicle in the middle ear, set up waves of pressure in the cochlear fluids; these waves are converted by sensory receptors in the cochlea to neural activity that is ultimately processed by the brain. In an effort to better understand the mechanical function of the middle ear, measurements of eardrum and ossicular vibrations have been made by numerous investigators; some of these results are presented in Section 3.2. In Section 3.3, theories and models of middle-ear mechanics that have emerged from these observations are discussed.

3.2 EXPERIMENTAL OBSERVATIONS

3.2.1 Vibrations of the Eardrum

Various attempts have been made to measure the vibration pattern of the eardrum; an historical review of these investigations is presented by Funnell and Laszlo (1982). In general, it is found that at low frequencies the displacements of the manubrium are smaller than those of the surrounding eardrum. In humans and cats, the maximum displacement amplitude occurs in the area of the pars tensa posterior to the manubrium; a smaller local displacement maximum also occurs in the anterior region. Figure 3.1
**Figure 3.1** Hologram of low-frequency eardrum vibration pattern in cat. The vibration amplitude ($\times 10^{-7}$ m) is marked for each iso-amplitude contour. From Khanna and Tonndorf (1972, p. 1915).
shows an example of such a low-frequency vibration pattern measured by Khanna and Tonndorf (1972) in anesthetized cats using time-averaged holography; the pattern can be interpreted like the equal-amplitude contours on a topographic map. Similar vibration patterns were also measured in human cadaver middle ears (Tonndorf and Khanna, 1972). In the models developed in this work, inertial and damping effects are ignored, thus the models are only valid for low frequencies. It is therefore permissible to compare modelling results to low-frequency experimental results.

Although the models developed in this work are limited to low frequencies, it is desirable to discuss the experimental data that are available for high frequencies; such data will be useful when extending the frequency range for which the present models are valid. The few measurements that have been made at high frequencies indicate that the vibration pattern of the eardrum breaks up and becomes quite complex with increasing frequency. For instance, Khanna and Tonndorf (1972) found that the simple mode of vibration measured at low frequencies (see Figure 3.1) remained essentially unchanged up to about 2.5 kHz; beyond this, it became quite complex, breaking up into sectional vibrations. More recently, Decraemer et al. (1989) made point-by-point measurements of the amplitude and phase of eardrum and manubrium vibrations in cats up to 20 kHz using a homodyne interferometer. Their results also indicated break-up of the vibration pattern with increasing frequency.

3.2.2 Vibrations of the Ossicular Chain

Measurements of ossicular vibrations in cats and humans are discussed separately.
Measurements in Cat

Wever and Lawrence (1954) reviewed work on the mechanics of the ossicular chain. They also measured the displacements of the ossicles in cats. They found that the malleus and incus rotate as a single rigid body around an axis running from a point on the posterior incudal ligament to a point on the anterior malleal ligament.

Møller (1963) determined the amplitude and phase of vibrations of the malleus, incus and round-window membrane in anesthetized cats using a capacitive probe. The malleus and incus were found to be rigidly coupled below 2500 Hz. The incudostapedial joint was inferred to be rigid in this frequency range based on measurements of incus and round-window membrane displacements; the joint was not directly observed.

Guinan and Peake (1967) measured ossicular motion visually under stroboscopic illumination. They found that, for sound pressures less than 140 to 150 dB SPL and for frequencies less than 3 kHz, the malleus and incus rotated as a single rigid body around an axis running in the antero-posterior direction. (The notation dB SPL means that the sound pressure level is measured in decibels relative to the reference pressure of $2 \times 10^4$ dyn cm$^2$.) Although this axis could not be determined precisely by them, its apparent orientation was in agreement with the results of Wever and Lawrence. Above 3 kHz, the incus was not rigidly coupled to the malleus, possibly due to flexing of the incudomalleal joint. No appreciable flexing of the incudostapedial joint was found at moderate sound levels (below about 150 dB SPL); above this level, the incudostapedial joint was found to stretch during rarefaction of the sound wave and compress while slipping sideways during condensation. The work of Guinan and Peake is perhaps the
only detailed study of the mode of vibration of the stapes in cats. They concluded that, to a first approximation, the stapes was rigid and moved in pure translation; the motion was further described as being piston-like (i.e., into and out of the cochlea along a line running from the centre of the stapedial head to the centre of the footplate and perpendicular to it).

Vlaming (1987) measured ossicular vibrations in anesthetized cats using a laser Doppler interferometer. Displacements of the stapedial head and of the lenticular process of the incus were found to be similar over a frequency range of about 10 kHz, indicating that the incudostapedial joint was nearly slipless. Displacements of the malleus and incus measured near the incudomalleal joint suggested that this joint was slipless for most frequencies except possibly in the band of frequencies from 3 kHz to 8 kHz.

Recent observations by Decraemer et al. (1991a) showed that the malleus mode of vibration in cats changes with frequency: at some frequencies the malleus vibration is purely rotational, at others it is purely translational and at most frequencies it consists of both rotation and translation. Even at frequencies where the motion is purely rotational, the axis shifts throughout the cycle of motion. However, the classical concept of the malleus rotating around a fixed axis was found to fit their data at low frequencies. A significant finding of their experiments was that the manubrium bends near the umbo at high frequencies.

Measurements in Human

In humans, experimental results indicate that the incudomalleal joint is rigid, whereas the incudostapedial joint is flexible (Gyo et al., 1987). The malleus and incus
rotate as a single rigid body around an axis running from the anterior malleal ligament to the posterior incudal ligament (Kirikae, 1960; Gundersen, 1972; Gundersen and Høgmoen, 1976). The axis of rotation shifts with frequency (Gundersen, 1972; Gundersen and Høgmoen; 1976; Gyo et al., 1987). Recently, Donahue et al. (1991) have suggested that, for frequencies above 600 Hz, malleal motion may be more complicated than pure rotation; it may consist of both rotational and translational motions, where the magnitudes of the two motions vary in a frequency-dependent manner.

The mode of vibration of the stapedial footplate remains controversial in man. A few researchers feel that it moves in a piston-like manner, whereas others feel that it also tilts. Békésy (1960, pp. 112-113) determined that, for physiological sound levels, the posterior end of the footplate remained stationary whereas other points on the footplate tilted about a vertical axis passing through this end; for very high sound levels, he found that the footplate tilted about a longitudinal axis running between the crura. The results of Kobrak (1959) essentially agreed with those of Békésy. Kirikae (1960), however, felt that the mode of vibration was more complicated, consisting of a combination of piston-like motion and tilting about the longitudinal and vertical axes; he did not find that the posterior end of the footplate remained stationary. Recent investigations by Gyo et al. (1987) showed that, for frequencies below 1 kHz, the motion of the footplate consists of rotation about an axis near the posterior end superimposed on a piston-like motion; above 1.2 kHz, the footplate was found to rotate around its longitudinal axis.
In contrast to the above, Dankbaar (1970), Gundersen (1972), and Vlaming and Feenstra (1986a) found that the footplate moves in a piston-like manner. Høgmoen and Gundersen (1977) found that the footplate moves predominantly in a piston-like manner, but possibly with some tilting. There does not seem to be any consensus on the mode of vibration of the stapes in humans; differences in experimental techniques may explain some of these discrepancies. For example, Gyo et al. (1987) point out that removal of the cochlear fluids may affect the mechanics of the footplate.

3.2.3 Input Impedance of the Stapediocochlear System

The cochlea loads the mechanical system of the middle ear. If the stapes motion is one dimensional and if the stapediocochlear system is linear then this mechanical load can be characterized by the input impedance of the system (Lynch et al., 1982). The mechanical impedance of the stapediocochlear complex, \( Z_m \), is defined as the complex ratio of the force acting on the stapes, \( F_s \), to the resulting linear velocity of the stapes, \( V_s \):

\[
Z_m = \frac{F_s}{V_s}. \tag{3.1}
\]

In auditory physiology the acoustic impedance is usually used rather than the mechanical impedance. The acoustic impedance of the stapediocochlear system, \( Z_a \), is defined as the complex ratio of the sound pressure acting on the stapes, \( P_s \), to the resulting volume velocity of the stapes, \( U_s \):
The mechanical impedance of this system can be converted to acoustic impedance through use of the relation:

$$Z_{sc}^e = \frac{P_s}{U_s}.$$  \hspace{1cm} (3.2)

where $A_{fp}$ is the area of the footplate.

In cats, stapes motion has been shown to be one dimensional (Guinan and Peake, 1967), and the stapediocochlear system has been shown to be linear up to 140 dB SPL (Lynch et al., 1982); therefore, input impedance is a useful way of characterizing the stapediocochlear load. The earliest attempts to measure the magnitude of this impedance in anesthetized cats were made by Tonndorf et al. (1966). Tones were delivered to the stapedial footplate, and the peak linear displacement of the round-window membrane was measured and converted to volume velocity by making certain assumptions; furthermore, the volume velocity of the round window was assumed to be equal to that of the footplate. Not satisfied with some of these assumptions, Khanna and Tonndorf (1971) used time-averaged holography to directly measure the volume displacement and hence the volume velocity of the round-window membrane, which was once again assumed to be equal to the volume velocity of the footplate. Only the magnitude of the impedance could be measured, not the phase.

The most extensive measurements of stapediocochlear input impedance are those of Lynch et al. (1982). Tones were applied to the stapes, and the linear velocity of the
stapes was measured using the Mössbauer effect. By assuming that the stapes moves in a piston-like manner, they were able to convert its linear velocity to volume velocity. Both the phase and magnitude of the impedance were determined. Their results indicate that the input impedance of the stapediocochlear system is stiffness-dominated for frequencies below 300 Hz; the stiffness is primarily determined by the annular ligament of the footplate. Between 500 Hz and 5000 Hz, the impedance is primarily resistive and is determined by the cochlear fluids and by the basilar membrane, a structure within the cochlea.

Although the mode of vibration of the human stapes may or may not be one dimensional (see previous section), attempts have been made to characterize the load of the human stapediocochlear system by measuring its input impedance. For example, Békésy (1960, pp. 435-436) and Onchi (1961) have measured the input impedance of the stapes and cochlea in human cadaver middle ears. Recently, Merchant et al. (1992) have measured the stapediocochlear impedance in human cadaver middle ears over a broad range of frequencies; their results indicate that it is stiffness-dominated for low frequencies, resistive for medium frequencies, and variable for higher frequencies.

3.3 THEORIES AND MODELS OF MIDDLE-EAR FUNCTION

3.3.1 Introduction

The middle ear is thought to aid the transmission of sound energy from the ear canal to the cochlea by partially matching the impedance of the air in the canal to the impedance of the fluids in the cochlea; the mechanisms involved in this process are not
well understood and are only briefly discussed in the next section. In an effort to better quantify the function of the middle ear, lumped-parameter models have been proposed by numerous investigators; these models will be discussed in Section 3.3.3. Analytical and finite-element models will be reviewed in sections 3.3.4 and 3.3.5, respectively.

3.3.2 Impedance-Matching Function

The transmission of sound from one medium to another is reduced if the acoustic impedances of the two media are different; full transmission is only possible if the impedances are matched (Kinsler et al., 1982). Since the impedance of air is much lower than that of the cochlear fluids, direct stimulation of the cochlea by sound would not be very effective in transmitting energy to the sensory receptors within it. The middle ear improves sound transmission by partially matching the impedances of these two media; such matching results in an increase in the pressure at the stapedial footplate relative to the pressure at the eardrum.

Three mechanisms are thought to be involved in this pressure transformation process. The primary mechanism involves the ratio of the area of the eardrum to that of the footplate: sound pressure acts over the large surface of the eardrum, and the resultant force (after some amplification due to the ossicular chain and the curvature of the drum as discussed below) is applied to the smaller area of the footplate; since this force is now applied over a much smaller area, the pressure at the footplate will be larger than that at the eardrum. The ratio of the area enclosed by the tympanic ring of the eardrum to the area of the footplate is 30 in cats and 15 in humans (Kirikae, 1960;
The functionally significant area of the eardrum is probably larger than that enclosed by the tympanic ring, but it is likely to be smaller than the surface area of the eardrum since not all of the eardrum is tightly coupled to the manubrium, and hence to the ossicular chain.

Secondary contributions are thought to be made by the ossicular lever and the curvature of the eardrum. The malleus and incus form a lever with the fulcrum being the axis of rotation mentioned in Section 3.2.2; the lever serves to amplify the force acting on the manubrium before it is applied to the stapes. Also, as discussed by Tonndorf and Khanna (1970), the curvature of the eardrum amplifies the force acting on its surface before it is applied to the manubrium; the curved membrane effect, as it is called, was originally proposed by Helmholtz (1869). It is not clear how much each of the three mechanisms contributes to the total value of the middle-ear transformer ratio.

3.3.3 Lumped-Parameter Models

Numerous researchers have proposed lumped-parameter models of the middle ear (e.g., Möller, 1961; Onchi, 1961; Zwislocki, 1962; Shaw and Stinson, 1986; Vlaming, 1987; Kringlebotn, 1988; Peake et al., 1992). In these models, a distributed system such as the eardrum is represented by discrete elements such as point masses, dashpots and springs; hence, response variables depend only on time, not space. The models can be drawn in terms of acoustical, electrical or mechanical elements since these are all mathematically equivalent (Beranek, 1954, pp. 47-90). As an example, consider the model of Zwislocki (1962) shown in Figure 3.2. The block diagram shown in part (a)
Figure 3.2 A lumped-parameter model of the human middle ear. (a) Block diagram. (b) Corresponding circuit diagram. Refer to the text for a description of the elements. From Zwislocki (1962).
of the figure is based on the functional anatomy of the middle ear. Block 1 represents the middle-ear cavities. Block 2 represents the portion of the drum that is loosely coupled to the ossicles; the motion of this part of the drum differs from that of the ossicles. Block 3 represents the malleus, the incus and the portion of the drum closely coupled to the ossicles. Block 4 represents the incudostapedial joint which Zwislocki thought was flexible, and therefore shunted energy past the cochlea. Finally, block 5 represents the stapediocochlear complex. The complete model is shown in part (b) of the figure; all elements have been cast in terms of electrical components. Elements denoted with subscripts $a, p, m,$ and $t$ belong to block 1. Those with subscripts $d, o, s,$ and $c$ belong to blocks 2, 3, 4 and 5, respectively. The values of some elements (e.g., compliances representing the cavities) were determined from anatomical considerations; others were determined by fitting predictions of the model to impedance measurements at the eardrum in normal and pathological ears. The model was capable of matching impedance curves for frequencies between 100 Hz and 2000 Hz.

One problem with such models is that the parameter values depend on the mode of vibration of the eardrum and the ossicles. Since the mode of vibration of the middle-ear structures depends on frequency, the parameter values also depend on frequency. The dependence of parameters on frequency makes the mathematical analyses of such systems difficult.
3.3.4 Analytical Models

As discussed by Funnell (1975), analytical models of the eardrum have been proposed by Esser (1947) and Gran (1968). Esser modelled the drum as a circular membrane under tension, clamped along its circumference and with a force applied to a point at its centre representing the malleus. Esser’s analysis resulted in several multiple integrals which, in some cases, could only be solved if the curvature of the eardrum was ignored; however, the curvature of the drum has been shown to be important to its function (Funnell and Laszlo, 1978).

Based on Békésy’s (1960) observations of low eardrum tension but significant bending stiffness, Gran (1968) modelled the drum as a plane circular plate with a radially positioned manubrium. Even with these oversimplifications, analysis of the model was hampered by the mathematical complexity of the system.

Recently, Rabbitt and Holmes (1986) have proposed a fibre-composite shell model of the eardrum that accounts for its geometry and ultrastructure. Equations were derived to describe structural damping, transverse inertia and membrane restoring forces. A closed-form asymptotic solution was found by making several geometric and material assumptions. As pointed out by Funnell (1989), each assumption affects subsequent problem formulation, making it difficult to change them.

3.3.5 Finite-element Models

The finite-element method, fundamentals of which will be discussed in Chapter 5, overcomes some of the difficulties of analytical approaches. The method handles
irregular geometries and complex material properties quite readily; furthermore, changes in the geometric and material assumptions of a finite-element model are quite easily accommodated.

Funnell and Laszlo (1978) modelled the cat eardrum as a three-dimensional thin shell using this method. Their model was valid for low frequencies (below 1-2 kHz) and for physiological sound levels. Vibration patterns and amplitudes calculated with the model were similar to experimentally observed patterns (Khanna, 1970). The most important factors determining the low-frequency behaviour of the eardrum model were found to be the stiffness and thickness of the pars tensa, the curvature and conical shape of the drum, and the anisotropic material properties of the drum. Boundary conditions, Poisson's ratio, ossicular loading and air loading were found to be less important. In a later model (Funnell, 1983), inertial effects were included and natural frequencies and mode shapes were computed. Damped frequency responses have also been calculated (Funnell et al., 1987). Recently, a more realistic representation of the manubrium has been incorporated into the eardrum model; calculations have indicated that the manubrium bends and twists at frequencies above a few kHz and possibly even for frequencies as low as 1-2 kHz (Funnell et al., 1992).

Models of the human eardrum have also been proposed (Lesser and Williams, 1988; Williams and Lesser, 1990; Stuhmiller, 1989), and recently a model of the human eardrum and ossicles has been developed (Wada et al., 1992). The finite-element method has also been used to study the mechanical effects of various surgical procedures on the human middle ear, including the effects of the Fisch II Spandrel prosthesis (Williams and
Lesser, 1992) and of the MSA (Lesser et al., 1991) and the effects of grommets (Lesser et al., 1988). Note that the models developed by Lesser et al. (1988, 1991) were two-dimensional, neglecting the three-dimensional shape of the eardrum which has been shown to be very important to its function (Funnell and Laszlo, 1978).

3.4 CONCLUSION

Experimental studies on the vibrations of the eardrum and ossicles have been discussed in this chapter. The mechanical behaviour of the eardrum and ossicles is quite complex and not very well understood; however, the behaviour at low frequencies is reasonably simple.

Lumped-parameter and analytical models have been developed by previous investigators to explain the mechanics of the middle ear. Such models have difficulty dealing with the complex geometry and material properties of the middle ear. However, some finite-element models have been developed that show promise in dealing with these complexities.
CHAPTER 4

OSSICULAR DISCONTINUITY

4.1 INTRODUCTION

As discussed in Chapter 3, the middle ear serves to transmit sound energy from the ear canal to the cochlea. In order to facilitate the efficient transmission of this energy, the eardrum and ossicles act as an impedance-matching transformer, partially matching the high impedance of the cochlear fluids to the low impedance of air in the ear canal. Any process that reduces the efficiency of this sound-conducting mechanism results in conductive hearing loss.

There are many causes of conductive hearing loss (Harris and Cueva, 1992); however, only hearing loss caused by discontinuity of the ossicular chain will be discussed in this work. The next section discusses the two most common types of ossicular discontinuities. Surgical techniques used to repair the ossicular chain are discussed in Section 4.3. In Section 4.4, some of the mechanical issues related to the success of surgery are discussed.

4.2 COMMON DEFECTS

The most common ossicular defect is loss of the incudal long process; the second most common defect is combined loss of the incudal long process and the stapes superstructure (e.g., Austin, 1971; Sade and Luntz, 1991). These situations are illustrated in Figure 4.1. Austin (1971) has reported that loss of the incudal long process
Figure 4.1 Two types of ossicular discontinuities. a) Missing incudal long process. b) Missing long process and stapes superstructure. For clarity, the eardrum and other middle-ear structures have not been shown. From Donaldson et al. (1992, p.250).
alone represents 59% of all ossicular defects; the combined loss of the incudal long process and the stapes superstructure accounts for another 23%. Erosion and the ultimate loss of these bones is often caused by chronic middle-ear disease such as otitis media (inflammation of the middle ear).

Discontinuities of the ossicular chain result in conductive hearing loss since vibrations of the eardrum can no longer be transmitted to the cochlea; moreover, the eardrum shields the cochlea from direct stimulation by sound. The extent of hearing loss in humans is measured in terms of the air-bone gap. By definition, this is the number of decibels (dB) by which the air-conduction threshold exceeds the bone-conduction threshold (Katz, 1985; p. 1060). The air-conduction threshold is measured by applying an air-conducted stimulus to the ear canal; it is affected by both the conductive and the sensorineural components of the auditory system. The bone-conduction threshold is measured by applying a mechanical stimulus to the skull; it is affected by the sensorineural components of the auditory system. It is not possible to tell from the air-conduction threshold alone whether hearing loss is conductive or sensorineural; however, the difference between the two thresholds, the air-bone gap, can be used as an indicator of conductive hearing loss. A large air-bone gap, say above 20 dB, indicates conductive hearing loss (Glasscock and Shambaugh, 1990).

Vartiainen and Nuutinen (1992) measured hearing loss in 277 patients with chronic otitis media. Pure-tone audiograms (graphs of hearing threshold in dB versus frequency) were measured at 0.5, 1 and 2 kHz. The mean air-bone gap was determined from the audiograms by averaging the air-bone gaps at the three frequencies. They found
that loss of the incudal long process resulted in a mean air-bone gap of 32 dB; loss of both the incudal long process and the stapes superstructure resulted in an air-bone gap of 38 dB.

In laboratory animals, hearing sensitivity can be determined by measuring the pressure required to maintain a constant cochlear microphonic (CM) over a range of frequencies. The CM is a stimulus-related electric potential generated by sensory receptors in the cochlea (Durrant and Lovrinic, 1984, pp. 152-156). It can be measured by placing an electrode on the round-window membrane. Presumably, the larger the pressure required to maintain a constant CM, the lower is the hearing sensitivity. Weyer et al. (1948) determined the hearing sensitivity up to 10 kHz in cats before and after interruption of the incudostapedial joint. They found that after such an encroachment, larger sound pressure was required to maintain the same CM as before; in fact, the sound pressure required was 60 dB larger than in the normal ear, indicating a hearing loss of 60 dB relative to the normal ear.

4.3 SURGICAL REPAIR

Methods of repair

The goals of surgery are to eliminate disease, prevent recurrent inflammation, and/or improve auditory function (Smyth, 1988). Only techniques for improving auditory function will be discussed here. These techniques have been described by a number of authors (e.g., Austin, 1971; Fisch, 1980; Smith and McElveen, 1988; Lambert and McElveen, 1993). In the case where only the long process of the incus is
missing, sound transmission is re-established by fitting a prosthesis between the manubrium and the head of the stapes; often a homograft or autograft incus or malleus head is used to make the prosthesis (Emmett, 1989). (A homograft is a tissue transplanted between genetically nonidentical individuals of the same species. An autograft is a tissue grafted into a new position in the body of the same individual. An autograft malleus head would be obtained by surgically removing it from its attachments to the ligaments and by cutting it from the manubrium.) Essentially, the graft is shaped to form a slim strut, a hole is drilled into the medial end of the strut to accept the head of the stapes, and a groove is carved into the lateral end to articulate with the manubrium. Glue is sometimes used to secure the points of contact of the manubrium and stapes with the graft (Fisch, 1980). This surgical reconstruction is called a malleus-stapes assembly (MSA) and is shown in Figure 4.2a.

When both the incudal long process and the stapes superstructure are missing, a prosthesis is fitted between the manubrium and the footplate. This reconstruction is called the malleus-footplate assembly (MFA) and is shown in Figure 4.2b. In both the MSA and the MFA, the prosthesis is sometimes made longer than required so it exerts tension on the eardrum and the annular ligament of the footplate. This tension stabilizes the strut.

Many variations of the basic MFA and MSA exist (e.g., Fisch, 1980; Wehrs, 1982; Frootko, 1987; Mills, 1991). Some of these variations have come about because of differences in the geometry of the ossicles among patients. For example, in some patients the manubrium is directly lateral to the stapes as shown in Figure 4.3a, but in
Figure 4.2 Ossicular prostheses. a) MSA. From Fisch (1980, p. 21) b) MFA. From Fisch (1980, p. 22).
Figure 4.3 Ossicular geometry. a) Manubrium directly lateral to stapes. b) Manubrium lateral and anterior to stapes. c) Horizontal MSA used in situation (b). Parts (a) and (b) are from Frootko (1987, p.249) and part (c) is from Mills (1991, p.477).
others it is located anterior to the stapes as shown in Figure 4.3b. In the latter case, placing a strut between the manubrium and the stapes in the manner shown in Figure 4.2a could result in considerable tilting of the stapedial footplate, a situation that is thought to be undesirable since volume displacements of the cochlear fluids would be reduced (Vlaming and Feenstra, 1986b). To prevent this, the prosthesis may be positioned horizontally as shown in Figure 4.3c.

Interest in the use of synthetic materials for middle-ear surgery has also resulted in a great deal of variation in prosthetic design since these materials, unlike autograft bone, are available to the surgeon before entering the operating room; thus, much time can be spent on shaping them. Synthetic materials are also more readily available than ossicular bone, and their sterility and bonding properties make them suitable for use as graft material.

A variety of graft materials have emerged in the past few years. Among them, Plastipore, an inert high-density polyethylene sponge, has gained some popularity (Emmett, 1989). Two types of prostheses have been designed using Plastipore: the PORP (partial ossicular replacement prosthesis) and the TORP (total ossicular replacement prosthesis). Both prostheses are shown in Figure 4.4. The PORP is similar to the MSA except that sometimes its platform makes contact with the eardrum, while the TORP is similar to the MFA. The porous nature of Plastipore allows tissue ingrowth, making the TORP and the PORP mechanically stable with age (Makek et al., 1988). Recently, a modified version of the TORP, called the Fisch II Spandrel, has been developed (Williams and Lesser, 1992). It is shown in Figure 4.4c.
Figure 4.4 Synthetic ossicular prostheses. a) PORP (Partial Ossicular Replacement Prosthesis). b) TORP (Total Ossicular Replacement Prosthesis). c) Fisch II Spandrel. Parts (a) and (b) are from Frootko (1987, p.257) and part (c) is from Williams and Lesser (1992, p.263).
Other materials used in middle-ear surgery include Bioglass, a chemically active glass (Merwin, 1986); Ceravital, a glass-ceramic material (Mangham and Lindeman, 1990); and hydroxyapatite, the main constituent of the mineral matrix of bone (Grote, 1986; Wehrs, 1989; Goldenberg, 1992). Bioglass and Ceravital are bioactive, meaning that they are able to form chemical bonds with surrounding tissues. Hydroxyapatite comes in both bioactive and porous forms. These materials have been used to produce a variety of prostheses; however, the basic MFA and MSA designed from ossicular bone remain popular (Emmett, 1989) and will be the only prostheses to be modelled in this work.

Hearing Results

The success of surgery in humans is assessed in terms of the reduction of conductive hearing loss, which is often measured by the degree of closure of the air-bone gap. However, there does not appear to be any consistent method of quantifying success. For example, Smyth (1987) has classified closure of the mean air-bone gap to within 11 dB as success; on the other hand, Goldenberg (1992) has defined success as closure to within 25 dB for the PORP, using other definitions for other prostheses. Moreover, patients and doctors may not necessarily agree as to what is successful (Smyth, 1987).

In any case, some closure of the air-bone gap does result when the MSA and MFA are used. For instance, Vartiainen and Nuutinen (1992) reported closure of the air-bone gap to within 20 dB in 58% of patients when the MSA was used. The situation was worse for the MFA: only 30% of the cases resulted in closure to within 20 dB. Results
obtained by other authors are shown in Table 4.1. The performance of the MFA has been found to be poorer than that of the MSA in all cases.

Table 4.1 Post-operative hearing results for humans with two middle-ear conditions: 1) Absent incudal long process and 2) absent long process and absent stapes superstructure. Shelton and Sheehy used FORP's and TORP's; MSA's and MFA's made from ossicular bone were used in the other studies. *N* indicates the number of patients with the particular condition. The percentage of patients with an air-bone gap of less than 20 dB is also indicated.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Follow-up (years)</th>
<th>Absent incudal long process</th>
<th>Absent long process &amp; superstructure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wehrs (1982)</td>
<td>1</td>
<td>207</td>
<td>119</td>
</tr>
<tr>
<td></td>
<td></td>
<td>89%</td>
<td>73%</td>
</tr>
<tr>
<td>Shelton &amp; Sheehy (1990)</td>
<td>≤3</td>
<td>86</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td></td>
<td>79%</td>
<td>67%</td>
</tr>
<tr>
<td>Vartiainen &amp; Nuutinen (1992)</td>
<td>≤10</td>
<td>210</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td></td>
<td>58%</td>
<td>30%</td>
</tr>
</tbody>
</table>

The effectiveness of prostheses has also been determined in cats. Tonndorf and Pastaci (1986) measured hearing sensitivity over a frequency range of 10 Hz to 15 kHz.
in anesthetized cats with normal middle ears. They then broke the incudostapedial joint and removed the crura. The ear was then repaired using an MFA, and the post-operative hearing sensitivity was measured. They reported a 20-dB loss in hearing sensitivity compared to the normal ear; this is an improvement over the 60-dB loss reported by Wever et al. (1948) in cats after interruption of the incudostapedial joint.

Benitez et al. (1974) determined audiometric thresholds at 0.5, 1 and 2 kHz in three normal cats. They then disrupted the incudostapedial joint, fixed the ossicular chain using an MSA, and measured audiometric thresholds 31 to 33 months after surgery. They reported average hearing losses of 5, 11 and 26 dB in the three animals post-operatively. Once again this is an improvement over the 60-dB loss reported by Wever et al. after interruption of the ossicular chain.

4.4 MECHANICAL CONSIDERATIONS FOR THE MFA AND MSA

4.4.1 Introduction

Relatively little work has been done to assess the mechanics of the surgically corrected middle ear or the importance of mechanical variables. Section 4.4.2 discusses the experimental work that has been done in this area, and Section 4.4.3 discusses the theoretical work that has been done.

4.4.2 Experimental Work

Several mechanical variables may be considered to be of interest in the application of the MSA and MFA: the position of the prosthesis, the tension exerted on the
surrounding structures, and the types of connections formed between the prosthesis and the surrounding tissues. Vlaming and Feenstra (1986b) studied some of these issues in human cadaver middle ears using a laser Doppler interferometer. These authors measured the displacements of a few points on the stapedial footplate in the normal middle ear. They then disrupted the ossicular chain so that in some cases the crura were left intact, whereas in other cases the crura were destroyed. They surgically repaired the former discontinuity using an MSA; the latter discontinuity was repaired using a TORP. The displacements of the footplate were then measured after surgery. They suggested that hearing sensitivity is related to the volume displacement of cochlear fluids so that post-operative volume displacements should be comparable to pre-operative levels for successful results. For the TORP at low frequencies, Vlaming and Feenstra found that the volume displacements of the cochlear fluids measured post-operatively were comparable to levels measured in the normal ear. On the other hand, volume displacements were lower than normal when the MSA was used, suggesting that the TORP should give better hearing results than the MSA. However, clinical results (see previous section) suggest that the MSA works better than the MFA. Since the MFA is similar to the TORP if the platform of the latter does not contact the eardrum, then, based on clinical results, it would be expected that the MSA also works better than the TORP. Vlaming and Feenstra did not report whether the TORP contacted the eardrum or not in all of their preparations, but from their discussion it is likely that it contacted the drum in some cases; nevertheless, in all cases, they found the TORP to work better than the MSA. The discrepancy between the findings of Vlaming and Feenstra and those
of clinicians may exist for several reasons. First, Vlaming and Feenstra used ears that were disease-free; however, when surgery is performed in living patients, the middle-ear is usually not free of disease. The extent of middle-ear disease necessitating the use of the TORP or the MFA is probably more advanced than that requiring the MSA or PORP. Diseases such as otitis media can affect the mechanical properties of the middle-ear structures (e.g., von Unge et al., 1992) and would thus account for differences between the results determined by Vlaming and Feenstra and those by clinicians. Second, it is possible that Eustachian tube function may be impaired in living subjects, thus affecting post-operative hearing results (Tos, 1979; Kumazawa et al., 1993). Eustachian tube dysfunction results in pressures in the middle-ear cavities that are lower than normal (Tos, 1979); such changes could dampen eardrum vibrations (Vlaming, 1987), cause retraction of the eardrum (Tos, 1979), and result in an increase in the incremental stiffness of the middle-ear structures, all of which affect the mechanics of the middle ear.

Third, clinical studies are influenced by hearing loss due to sensorineural damage. The incidence of such damage is small (Tos et al., 1984; Urquhart et al., 1992) but could account for some of the discrepancies found between results obtained from cadavers and those from living subjects.

Vlaming and Feenstra also studied how the position of a prosthesis affects the displacements of the footplate. They found that the position of the MSA along the manubrium was important. They suggested that the optimal orientation for the MSA is such that it lies along an imaginary line passing through the head of the stapes and the
centre of the footplate. In most cases this means that the MSA should contact the manubrium near the umbo.

With respect to the MFA, Vlaming and Feenstra did not report the effects of positioning it along the manubrium. However, the position of the MFA on the footplate was varied, and it was found not to significantly affect the displacements of the footplate. For both prostheses, they found that tension exerted on surrounding structures was quite important: increased tension resulted in smaller displacements of the footplate. Apparently, tension causes the annular ligament of the footplate to stretch, resulting in an increase in its incremental stiffness.

Other authors have studied the effects of the mechanical variables just described in human cadaver middle ears by determining the amount of sound transmission that occurs pre-operatively and post-operatively. Andersen et al. (1962) applied sound to the oval window and measured the resulting pressure in the ear canal; presumably, the higher the degree of sound transmission, the better is the hearing sensitivity. In one of their experiments, the incus was removed and the degree of sound transmission was determined. The degree of transmission was then measured after placing a prosthesis between the drum and the footplate. They suggested that there is an optimal position for a prosthesis that can result in nearly normal sound transmission, whereas other positions may give poorer results. No detailed information was given as to how it should be placed.

Elbrond et al. (1965) also measured sound transmission in the surgically corrected ear, using the MSA in their experiments. They applied sound to the ear canal and
measured pressure variations in the vicinity of the round window caused by vibrations of its membrane. Like Vlaming and Feenstra, their results also indicate that tension in the MSA is important: increased tension results in lower transmission, especially at low frequencies.

Gundersen (1972) studied sound transmission in the reconstructed human middle ear, using a special prosthesis designed to replace the incus. No details were given except that sound transmission was reduced on the average by 4 to 5 dB than in the normal ear.

Very few studies have been done in cats. Tonndorf and Pastaci (1986) found that the position of a prosthesis along the manubrium is relatively unimportant for the MFA as long as the connection between the manubrium and the prosthesis is ‘inflexible’. No results were given for a flexible connection.

Benitez et al. (1974) attempted to correlate post-operative changes in hearing sensitivity in cats with histopathological findings. Their results indicate that the connections between the prosthesis and surrounding tissues are stabilized by the proliferation of fibres. Hearing results were found to be the best in one cat in which the connections between the prosthesis and bones were deemed to be ‘excellent’. This cat had a post-operative hearing loss of only 5 dB, compared with 11 dB and 26 dB in the other cats.
4.4.3 Theoretical Work

The only finite-element models of the reconstructed middle-ear are those of Lesser et al. (1991) and Williams and Lesser (1992). Lesser et al. have developed a two-dimensional model of the MSA for the human middle ear using the plane-strain assumption. Results of their model for various placements of the prosthesis along the manubrium indicate that the stress levels at the prosthesis-stapes interface increase as the prosthesis is moved down the manubrium. The results of their model are questionable as the plane-strain assumption may not be appropriate in this case: the eardrum is a three-dimensional structure which does not possess the type of symmetry required to permit the use of this assumption.

Williams and Lesser (1992) have developed a three-dimensional model for the Fisch II Spandrel prosthesis in the human middle ear. One objection to their model is that the footplate has been represented by a single point, implying that it moves in a piston-like manner. It is possible for the footplate to tilt in the surgically corrected middle ear depending on the placement of the prosthesis, and tilting of the footplate may affect volume displacements of the cochlear fluids and hence hearing sensitivity (Vlaming and Feenstra, 1986b).

Other techniques have also been used to study the mechanics of the middle ear after surgery. For example, Wada et al. (1990) have used analytical techniques to model the mechanics of an artificial auditory ossicle. Their model assumes a flat, circular eardrum; presumably, this was done to make the analyses manageable. However, the
results of Funnell and Laszlo (1978) indicate that the three-dimensional shape of the eardrum is very important to its function.

Peake et al. (1992) developed a lumped-parameter model of type IV tympanoplasty, a surgical technique used to shield the round window from acoustic stimulation when the eardrum, malleus and incus are missing; however, this technique is not in any way similar to the MSA or MFA.

4.5 CONCLUSIONS

Existing middle-ear prostheses do not result in complete closure of the air-bone gap. The types of connections formed between a prosthesis and the surrounding tissues may determine if a location exists that results in improved closure of the air-bone gap. For rigid connections the position of a prosthesis would not be expected to have any effect on closure of the air-bone gap since the prosthesis and the surrounding bones would form a single rigid body, however the position of a prosthesis may be very important when the connections are flexible. It is difficult to study experimentally how the rigidity of prosthesis-tissue connections determines the optimal location of a prosthesis since it is difficult in practice to control or even measure the rigidity of these connections. Finite-element models of the surgically corrected middle ear would permit investigation of how the nature of prosthesis-tissue connections affects prosthesis behaviour.
5.1 INTRODUCTION

The finite-element method is a numerical technique that has been applied to solve problems in many fields; the discussion in this chapter will focus on its application to the static (or low-frequency) analysis of structures. In structural analysis, one may wish to ascertain the stresses, strains and displacements throughout a body. These three quantities must satisfy several governing equations: stresses must satisfy the differential equations of equilibrium and force boundary conditions; displacements and strains must satisfy both compatibility and boundary conditions; furthermore, constitutive relations must hold between stresses and strains. Analytical solutions for stresses, strains and displacements can only be found for structures with simple geometries, material properties and boundary conditions; for complex structures, it is necessary to employ numerical techniques.

Another approach to finding the displacements and hence the strains and stresses throughout a complex structure involves assuming a displacement field (e.g., a polynomial of high enough order) over the entire region of interest; the coefficients of the polynomial can be found by using techniques such as the variational method. However, the assumed displacement field is usually quite complex, and such an approach is not very practical (Bathe, 1982, pp. 171-172).
The finite-element method follows a slightly different approach that allows the use of simpler displacement fields. In this method, the region to be analyzed is first divided up into subregions called *elements* which are connected at points called *nodes*; the process of dividing a region into elements is known as *mesh generation*. The desired field is then represented by functions defined over each element; these functions are much simpler than those required for the entire region. Next, the mechanical response of each element is analyzed; this yields a stiffness matrix and a load vector for each element in the structure. Once all of the element stiffness matrices and load vectors have been obtained, they are combined into a structure matrix equation which relates nodal displacements for the entire structure to nodal loads. After applying boundary conditions, the structure matrix equation can be solved to obtain unknown nodal displacements; intra-element displacements can be interpolated from nodal values using the functions defined previously over each element.

The next section describes the process of mesh generation. In Section 5.3, the mechanical behaviour of each element, and hence the entire structure, will be analyzed using the variational method; this will yield general forms for the element stiffness matrix, the element load vector, and the structure matrix equation. As mentioned above, the element stiffness matrices and the element load vectors are computed first and then assembled to form the structure matrix equation; the process of assembling the structure matrix equation will be demonstrated in Section 5.4. Finally, in Section 5.5, the application of boundary conditions and the solution of unknown displacements will be briefly discussed.
5.2 MESH GENERATION

The first step in finite-element analysis is the idealization of the structure to be analyzed as an assembly of simpler parts called elements. For example, an irregularly shaped structure such as the eardrum may be idealized as an assembly of regularly shaped triangular plates as shown in Figure 6.2. This process of idealization, or mesh generation, involves the selection of the numbers, types and orders of elements to be used to model the structure. The number of elements used in an analysis affects the quality of the solution. In general, as the number of elements increases, displacements converge to true values. However, a finer mesh with more elements places greater demands on the computing resources than does a coarse one. A good compromise is to select the coarsest mesh for which displacements are close enough to the true values. Such a mesh can be found by using finer and finer meshes to model the structure until the results converge; the coarsest mesh for which results are adequate can then be selected and used in future simulations.

The type of element used in any analysis depends on the mechanical nature of the structure to be analyzed. Theoretically, a three-dimensional solid element could be used to model any structure; however, by taking advantage of certain characteristics of a structure, one can use elements that are less general but also less complicated. For example, one may model the footplate using thin-plate elements instead of solid elements; the thin-plate assumption leads to the simplification that the stress through the thickness of the structure is zero.
The order of the functions used to interpolate intra-element displacements from nodal ones also affects the accuracy of the solution. For example, Silvester (1969) has shown that the accuracy of a finite-element solution is superior for a problem modelled using one fourth-order element than it is for the same problem modelled using sixteen first-order elements; moreover, the computational requirements are similar in the two cases.

5.3 GENERAL FORMULATION

5.3.1 Overview

General forms for the element stiffness matrix, element load vector and structure matrix equation are derived in this section; the derivations are based on those by Cook et al. (1989, pp. 109-112). In the derivation, the unknowns will be taken to be the nodal degrees of freedom (d.o.f.'s) which can include nodal displacements or nodal rotations or a combination of the two. Next, interpolation functions will be defined over each element which express intra-element displacements in terms of the nodal d.o.f.'s. Then intra-element stresses and strains will also be expressed in terms of the nodal d.o.f.'s. The variational method will be used to derive the element stiffness matrix, the element load vector and the structure matrix equation. In this method, the potential energy of the system is expressed in terms of the nodal d.o.f.'s and then minimized with respect to variations in them.

Some things need to be said about notation at this point. Scalars will be denoted by letters, either uppercase or lowercase, that are not bold. Matrices will be denoted by
bold uppercase letters enclosed in brackets. Vectors will be denoted by bold letters enclosed in braces.

5.3.2 Some Preliminaries

1. The Unknowns

The unknowns for a typical element and hence for the entire structure are taken to be the nodal d.o.f.'s. For the triangular element shown in Figure 5.1 which has two translational d.o.f.'s at each node, the element nodal d.o.f. vector, \( \{d\} \), may be defined as

\[
\{d\} = [u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3]^T
\]

(5.1)

where \( u_i \) and \( v_i \) are the components of displacement in the x- and y-directions, respectively, for node \( i \). For a mesh with \( n \) nodes, the structure nodal d.o.f. vector is

\[
\{D\} = [u_1 \ v_1 \ ... \ u_n \ v_n]^T.
\]

(5.2)

As mentioned previously, displacements within an element are interpolated from the nodal d.o.f.'s. For element \( i \) with two translational d.o.f.'s at each node, the displacements within the element are denoted by the vector \( \{s\}_i = [u(x,y) \ v(x,y)]^T \) and are interpolated using the formula

\[
\{s\}_i = [N]\{d\}_i,
\]

(5.3)

where \([N]\) is a matrix of displacement interpolation functions; the components of this matrix are functions of the coordinates. As discussed in Section 5.2, the choice of interpolation functions affects the quality of the solution.
Figure 5.1 A triangular element. The element has two translational degrees of freedom, $u$ and $v$, at each node.
2. Strains

The strains within any element are related to the derivatives of displacements. For two-dimensional problems, the strain vector contains three components: 
\[ \{\varepsilon\} = [\varepsilon_x \varepsilon_y \gamma_{xy}]^T \]  where \( \varepsilon_x \) and \( \varepsilon_y \) are normal strains in the \( x \) and \( y \) directions, respectively, and \( \gamma_{xy} \) is the shear strain. (It should be noted that strains and stresses are actually tensors of order two; however, vectorial notation has been used here for simplicity.) The strains for element \( i \) may be found from the displacement vector using the relation

\[ \{\varepsilon\}_i = [\partial] \{u\}_i \]  \hspace{1cm} (5.4)

where \([\partial]\) is a differential operator; for problems in two-dimensional elasticity, it is defined as

\[ [\partial] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \]  \hspace{1cm} (5.5)

Combining Equations (5.3) and (5.4) yields

\[ \{\varepsilon\}_i = [B] \{d\}_i \]  \hspace{1cm} (5.6)

where

\[ [B] = ([\partial])[N] \]  \hspace{1cm} (5.7)

is the so-called strain-displacement matrix.
3. Stresses

Stresses may be found from strains using known material properties. For problems in two-dimensional elasticity, the stress vector is denoted by \( \{ \sigma \} = [\sigma_x, \sigma_y, \tau_{xy}]^T \) where \( \sigma_x \) and \( \sigma_y \) are normal stresses in the \( x \)- and \( y \)-directions, respectively, and \( \tau_{xy} \) is the shear stress; the stresses can be found from the strains using the constitutive relation

\[
\{ \sigma \} = [C] \{ \varepsilon \}
\]

(5.8)

where \([C]\) is a matrix of material stiffnesses. For linearly elastic, isotropic, homogeneous materials in plane stress, the material stiffness matrix is

\[
[C] = \frac{E}{1-v^2} \begin{bmatrix}
1 & v & 0 \\
v & 1 & 0 \\
0 & 0 & \frac{1-v}{2}
\end{bmatrix}
\]

(5.9)

where \( E \) and \( v \) are the Young's modulus and Poisson's ratio of the material, respectively.

5.3.3 The Variational Method

For conservative systems, the variational method is the most common approach for formulating element stiffness matrices, element load vectors and the structure matrix equation; discussions of other methods such as the weighted residual method may be found elsewhere (e.g., Cook et al., 1989, pp. 455-473). In the variational method, the principle of stationary potential energy is used. This principle states that admissible configurations of a system that satisfy the equations of equilibrium make the potential energy stationary.
energy of the system stationary with respect to small variations in displacement (e.g., Cook et al., 1989, pp. 70-73). An admissible configuration is one that satisfies both internal compatibility and essential boundary conditions (e.g., prescribed values of nodal d.o.f.'s).

In order to apply the above principle to the analysis of finite elements, it is first necessary to write the potential energy of the structure, \( \Pi \), in terms of its nodal d.o.f.'s; symbolically, it is possible to write \( \Pi = \Pi(D_1, D_2, \ldots, D_n) \) where the \( D_i \) represent the d.o.f.'s of the \( i \)th node in the structure. Application of the principle of stationary potential energy yields

\[
d\Pi = \sum_{i=1}^{n} \frac{\partial \Pi}{\partial D_i} dD_i = 0. \tag{5.10}
\]

The variations \( dD_i \) are independent and arbitrary, thus the potential energy can be made stationary only if the coefficients of each \( dD_i \) vanish separately. Mathematically, this means

\[
\frac{\partial \Pi}{\partial D_i} = 0; \quad i = 1, 2, 3, \ldots, n. \tag{5.11}
\]

This yields \( n \) algebraic equations in the \( n \) values of the structure's d.o.f. 's that define its equilibrium configuration.
For a linearly elastic body in static equilibrium, the potential energy can be written in the form

\[ \Pi = U + \Omega \]  

(5.12)

where \( U \) is the strain energy of the system and \( \Omega \) is the potential of loads. The strain energy is

\[ U = \int \frac{1}{2}(\epsilon)^T \sigma dV \]  

(5.13)

where the integral is evaluated over the entire volume of the structure. Substituting Equation (5.8) into the above yields

\[ U = \int \frac{1}{2}(\epsilon)^T [C][\epsilon] dV. \]  

(5.14)

The potential of the loads is

\[ \Omega = -\int_v (s)^T [F] dV - \int_s (s)^T [\Phi] dS - [D]^T [P] \]  

(5.15)

in which \( \{F\} = [F, F, F]^T \) is a vector of body forces,

\( \{\Phi\} = [\Phi, \Phi, \Phi]^T \) is a vector of surface tractions,

\( \{P\} \) is a vector of concentrated nodal loads,

and the two integrals in the formula are evaluated over the volume and surface of the body, respectively.
5.3.4 Element Stiffness Matrix and Structure Matrix Equation

The variational method will now be applied to obtain the element stiffness matrix, the element load vector and the structure matrix equation. In order to do this, the potential energy of the finite-element model must be expressed in terms of its nodal d.o.f.'s. The potential energy of the model is the sum of the strain energy of the model and the potential of loads. The strain energy is in turn the sum of the strain energies of the $M$ individual elements:

$$U = \sum_{i=1}^{M} U_i$$  \hspace{1cm} (5.16)

where $U_i$ is the strain energy of element $i$:

$$U_i = \int_{V_i} \frac{1}{2} \{\varepsilon\}_i^T [C]_i \{\varepsilon\}_i dV.$$  \hspace{1cm} (5.17)

The integral in the above equation is evaluated over the volume of element $i$. The strain energy of each element can now be expressed in terms of the element's nodal d.o.f.'s by substituting Equation (5.6) into Equation (5.17):

$$U_i = \{d\}_i^T \int_{V_i} \frac{1}{2} [B]^T [C]_i [B] \{d\}_i dV.$$  \hspace{1cm} (5.18)

Similarly, the potential of loads may be expressed as

$$\Omega = \sum_{i=1}^{M} \Omega_i - \{D\}^T \{P\}$$  \hspace{1cm} (5.19)

where $\Omega_i$ is the potential of surface tractions and body forces acting on element $i$; it can
be written as

$$\Omega_i = -\int_{V_i} \{s\}_i^T [F] dV - \int_{S_i} \{s\}_i^T [\Phi] dS \quad (5.20)$$

where the first integral is evaluated over the volume of the element and the second one over the surface of the element. By using Equation (5.3), it is possible to write Equation (5.20) in terms of the nodal d.o.f.'s of element $i$:

$$\Omega_i = -\{d\}_i^T \int_{V_i} [N]^T [F] dV - \{d\}_i^T \int_{S_i} [N]^T [\Phi] dS . \quad (5.21)$$

The potential energy of the structure is now

$$\Pi = \sum_{i=1}^{M} (U_i + \Omega_i) - \{D\}_i^T [P] . \quad (5.22)$$

Substitution of Equations (5.18) and (5.21) into the above equation yields

$$\Pi = \frac{1}{2} \sum_{i=1}^{M} \{d\}_i^T [k]_i \{d\}_i - \sum_{i=1}^{M} \{d\}_i^T \{r\}_i - \{D\}_i^T [P] \quad (5.23)$$

where

$$[k]_i = \int_{V_i} [B]^T [C] [B] dV \quad (5.24)$$

is the element stiffness matrix for element $i$, and

$$\{r\}_i = \int_{V_i} [N]^T [F] dV + \int_{S_i} [N]^T [\Phi] dS \quad (5.25)$$

is the element load vector. Note that in Equation (5.24) the element stiffness matrix may be different from element to element, thus allowing one to model inhomogeneous
materials. Furthermore, to handle anisotropies, an appropriate material stiffness matrix, \([C]\), may be used.

Since each nodal d.o.f. in \(\{d\}\) appears in the structure d.o.f. vector, \(\{D\}\), it is possible to replace \(\{d\}\) by \(\{D\}\) in Equation (5.23) provided that the element stiffness matrices and the element load vectors are expanded to structure size; the concept of expanding matrices to structure size will be clarified in Section 5.4. Equation (5.23) can now be rewritten as

\[
\Pi = \frac{1}{2} (D)^T [K] (D) - (D)^T [R]
\]

(5.26)

where

\[
[K] = \sum_{i=1}^{M} [k]_i \quad \text{and} \quad [R] = \{P\} + \sum_{i=1}^{M} \{r\}_i.
\]

(5.27)

\([K]\) is the structure stiffness matrix, and \([R]\) is the structure load vector.

Now that the potential energy of the system has been expressed in terms of the nodal d.o.f.'s, it is possible to apply the principle of stationary potential energy as embodied in Equation (5.11); this yields the following structure matrix equation, or equation of static equilibrium

\[
[K] (D) = [R].
\]

(5.28)

After taking into account the boundary conditions, the above equation can be solved for the nodal d.o.f.'s; this will be illustrated in Section 5.5. Intra-element displacements
may then be interpolated from nodal d.o.f.'s using Equation (5.3), and intra-element stresses and strains may be computed using Equations (5.6) and (5.8), respectively.

5.3.5 The Constant-Strain Triangle

To demonstrate the formulation of an element stiffness matrix, let us consider the constant-strain triangle (CST). This element can be used to solve problems in plane stress and plane strain. Each of the three nodes in the element has two translational d.o.f.'s, \( u \) and \( v \). Linear interpolation functions are used to represent intra-element displacements in terms of these nodal d.o.f.'s; the resulting strains within the element will be constant as illustrated in the following discussion.

Consider the triangular element shown in Figure 5.1. The element nodal d.o.f. vector is given by Equation (5.1). Displacements, \( u \) and \( v \), within the element can be interpolated using the formulae

\[
\begin{align*}
  u(x,y) &= \alpha_1 + \alpha_2 x + \alpha_3 y \\
  v(x,y) &= \beta_1 + \beta_2 x + \beta_3 y \\
\end{align*}
\]

where \( \alpha_i \) and \( \beta_i \) are referred to as generalized coordinates; these may be expressed in terms of the nodal d.o.f.'s by substituting the coordinates of each point into Equation (5.29) and solving a set of algebraic equations. For example, substituting the coordinates of nodes 1, 2, and 3 of the element in Figure 5.1 into Equation (5.29) will yield the following six equations in the six unknown generalized coordinates:
\[ u(x, y) = \alpha_1 + \alpha_2 x_i + \alpha_3 y_i \quad (i = 1, 2, 3) \]
\[ v(x, y) = \beta_1 + \beta_2 x_i + \beta_3 y_i . \]

The above equations may be solved to express the generalized coordinates in terms of the nodal d.o.f.'s; the generalized coordinates can then be inserted into Equation (5.29) to express the intra-element displacements in terms of the nodal d.o.f.'s.

Another approach is to use the so-called area coordinates: this coordinate system allows one to write the intra-element displacements in terms of the nodal d.o.f.'s directly. To define this coordinate system, consider triangle 1-2-3 shown in Figure 5.2; the point \( P \) divides it into three subareas \( A_1, A_2 \) and \( A_3 \). The ratios of each of these areas to the total area, \( A \), locate point \( P \) and represent the area coordinates:

\[ \xi_1 = \frac{A_1}{A}, \quad \xi_2 = \frac{A_2}{A}, \quad \xi_3 = \frac{A_3}{A} . \]

The area coordinates are not independent since

\[ \xi_1 + \xi_2 + \xi_3 = 1 . \]

Note that at node \( i \)

\[ \xi_i = 1 \quad (i = 1, 2, 3) \]

and on side \( i \)

\[ \xi_i = 0 . \]

The relationship between area coordinates and Cartesian coordinates is given by the equation

\[ \xi_i = 1 \quad (i = 1, 2, 3) \]
Figure 5.2 Area coordinates. The triangular element is divided into three subregions, $A_1$, $A_2$, and $A_3$, by the point $P$. 
\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{bmatrix} = \frac{1}{2A} \begin{bmatrix}
x_2y_3 - x_3y_2 & y_{23} & x_{32} \\
x_3y_1 - x_1y_3 & y_{31} & x_{13} \\
x_1y_2 - x_2y_1 & y_{12} & x_{21}
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\] (5.35)

where \(x_y = x_i - x_j\) and \(y_y = y_i - y_j\).

The displacements within element \(i\) may be interpolated as:

\[
\{s\}_i = \begin{bmatrix} u \\ v \end{bmatrix}_i = [N]\{d\}_i
\] (5.36)

where

\[
[N] = \begin{bmatrix}
\xi_1 & 0 & \xi_2 & 0 & \xi_3 & 0 \\
0 & \xi_1 & 0 & \xi_2 & 0 & \xi_3
\end{bmatrix}
\] (5.37)

Note that at node \(j\)

\[
\{s\}_i = \begin{bmatrix} u_j \\ v_j \end{bmatrix}_i
\] (5.38)

Equation (5.37) indicates that the displacements within an element are a linear combination of the spatial coordinates. The strains are given by Equation (5.6) where

\[
[B] = \frac{1}{2A} \begin{bmatrix}
y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\
0 & x_{32} & 0 & x_{13} & 0 & x_{21}
\end{bmatrix}
\] (5.39)

For an element of constant thickness \(t\) and with constant material properties, the element stiffness matrix is:

66
For a state of plane stress, the material stiffness matrix, $[C]$, is given by Equation (5.9).

From Equation (5.40), it is apparent that the strains within any element are constant, hence the name of the element. The CST performs poorly in bending since the linearly-varying strain fields that are present in problems such as the bending of a cantilever beam (e.g., Ugrual and Fenster, 1987) are not represented by the assumed fields. At best, it is possible to construct a piecewise-constant approximation to the actual strains. The approximation improves as more elements are used to model the structure. Improvements can also be made by using higher-order interpolation functions as are used in the linear-strain triangle (Cook et al., 1989, pp.157-159). The improvements brought about by using more elements or higher-order interpolation functions hold in general for most elements.

5.4 ASSEMBLING THE STRUCTURE MATRIX EQUATION

Element stiffness matrices and element load vectors are assembled to obtain the structure stiffness matrix and structure load vector, respectively. The process of assembling the structure stiffness matrix and the structure load vector is symbolized by the summations in Equation (5.27). The summations in that equation can only be performed if each element stiffness matrix is made to operate on $\{D\}$, the structure nodal d.o.f. vector. This means that, for a structure with $n$ nodal d.o.f.'s, each element stiffness matrix must be expanded to the size $n \times n$ so that the multiplication $[k]\{D\}$ is defined.
To demonstrate this process, consider, for simplicity, the two-element structure shown in Figure 5.3; each node has one d.o.f., \( d \). The structure nodal d.o.f. vector is

\[
\{D\} = [d_1 \ d_2 \ d_3 \ d_4]^T
\]  

(5.41)

The stiffness matrix and the load vector for element 1 (i.e., the element with nodes 1-2-3) are

\[
[k]_1 = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}, \quad \{r\}_1 = \begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3
\end{bmatrix}.
\]  

(5.42)

For element 2 with nodes 2-3-4, the element stiffness matrix and load vector are

\[
[k]_2 = \begin{bmatrix}
    b_{11} & b_{12} & b_{13} \\
    b_{21} & b_{22} & b_{23} \\
    b_{31} & b_{32} & b_{33}
\end{bmatrix}, \quad \{r\}_2 = \begin{bmatrix}
    g_1 \\
    g_2 \\
    g_3
\end{bmatrix}.
\]  

(5.43)

Element 1 does not contain node 4. However, the d.o.f. for this node can be included in Equation (5.42) if a row of zeros is added to the bottom of the matrix; a column of zeros must also be included at the end of the matrix. A zero must be added to the bottom of the load vector:

\[
[k]_1 = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & 0 \\
    a_{21} & a_{22} & a_{23} & 0 \\
    a_{31} & a_{32} & a_{33} & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}, \quad \{r\}_1 = \begin{bmatrix}
    f_1 \\
    f_2 \\
    f_3 \\
    0
\end{bmatrix}.
\]  

(5.44)

Similarly, element 2 does not contain the d.o.f. for node 1; to include this d.o.f., a row of zeros must be added to the top of the stiffness matrix in Equation (5.43), and a column
Figure 5.3 Assembling the system stiffness matrix. A simple two-element system is shown. The large numbers correspond to elements, whereas the small ones correspond to the nodes.
of zeros must be added to the front of this matrix. A zero must be added to the top of
the element load vector:

\[
[k]_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & b_{11} & b_{12} & b_{13} \\
0 & b_{21} & b_{22} & b_{23} \\
0 & b_{31} & b_{32} & b_{33}
\end{bmatrix}, \quad \{r\}_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}.
\] (5.45)

The addition of element stiffness matrices and the addition of element load vectors can
now be performed to yield the structure stiffness matrix and the structure load vector:

\[
[K] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 \\
a_{21} & a_{22} + b_{11} & a_{23} + b_{12} & b_{13} \\
a_{31} & a_{32} + b_{21} & a_{33} + b_{22} & b_{23} \\
0 & b_{31} & b_{32} & b_{33}
\end{bmatrix}, \quad \{R\} = \begin{bmatrix}
f_1 \\
f_2 + g_2 \\
f_3 + g_3 \\
g_4
\end{bmatrix}.
\] (5.46)

5.5 SOLUTION OF STRUCTURE MATRIX EQUATION

Before solving for the nodal d.o.f.'s, it is necessary to take into account any
displacement boundary conditions that have been specified. It is possible to partition the
structure matrix equation in the form

\[
[K] = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & 0 \\
a_{21} & a_{22} + b_{11} & a_{23} + b_{12} & b_{13} \\
a_{31} & a_{32} + b_{21} & a_{33} + b_{22} & b_{23} \\
0 & b_{31} & b_{32} & b_{33}
\end{bmatrix}, \quad \{R\} = \begin{bmatrix}
f_1 \\
f_2 + g_2 \\
f_3 + g_3 \\
g_4
\end{bmatrix}.
\] (5.46)
where \( \{D_x\} \) and \( \{D_c\} \) are vectors of unknown and constrained displacements, respectively, and \( \{R_x\} \) and \( \{R_c\} \) are vectors of unknown reactions and applied loads, respectively. The unknown displacements can be calculated using the equation

\[
\{D_x\} = [K_{11}]^{-1}(\{R_x\} - [K_{12}][D_c]).
\]  

(5.48)

Once unknown displacements have been calculated, reaction forces may be calculated using the equation

\[
\{R_x\} = [K_{21}][D_x] + [K_{22}][D_c] + [K_{22}][D_c].
\]  

(5.49)

Practical methods of imposing boundary conditions and solving for unknown d.o.f.'s are discussed by Bathe (1982).

### 5.6 SUMMARY

The finite-element method has been described in this chapter. The method is capable of handling complex geometries, material properties, boundary conditions and loading patterns, thus making it suitable for modelling complex biological structures.
CHAPTER 6
DESCRIPTION OF MODELS

6.1 INTRODUCTION

Finite-element models of the MFA and MSA are described in this chapter. A model of the normal cat middle ear was also developed for comparison with the MFA and MSA models; this model is described in Section 6.4. All models in this work have been implemented using the SAP IV finite-element program (Bathe et al., 1974). The range of validity of these models, the type of stimulus used, and the adequacy of the meshes are discussed in the final three sections.

6.2 MFA MODEL

6.2.1 Introduction

Figure 6.1 shows a medial view of the finite-element model for the MFA. The MFA model includes the eardrum and manubrium, the footplate with the cochlear load acting upon it, and the prosthesis; models of each of these structures are described below.

6.2.2 Eardrum Model

It is assumed that the eardrum being modelled has not been damaged by middle-ear disease or by the surgical procedure; this permits use of an existing eardrum model (Funnell, 1983; Funnell et al., 1987).
Figure 6.1 MFA model with ossicular axis of rotation. View showing the medial aspect of the eardrum. The cochlear load is omitted for clarity. The two rigid plate elements representing the head and neck of the malleus and the body of the incus (labelled simply as malleus and incus in the figure) are in front of the tympanic ring, but should dip behind it; this is an artifact due to the simplified geometry assumed for these structures. This simplification has no effects on the mechanical behaviour of the eardrum and the footplate.
**Geometry of the Model**

Figure 6.2a shows a lateral view of the eardrum model in the plane of the tympanic ring. The eardrum mesh has a nominal resolution of 12 elements/diameter; the mesh generator used to create the mesh is described elsewhere (Funnell, 1983). Each triangle of the mesh represents a thin-shell element corresponding to element type 6 of the SAP IV finite-element program; the formulation of this element is described elsewhere (Clough and Felippa, 1969).

Figure 6.2b is a section of the drum through the manubrium, showing how the drum points medially. As with previous models (Funnell, 1983; Funnell et al., 1987, 1992), the curvature of the sides of the cone formed by the eardrum is represented by circular arcs lying in planes perpendicular to the plane of the tympanic ring; each arc has a ‘normalized radius of curvature’ of 1.19. As explained by Funnell (1983), the radius of curvature is normalized with respect to the chord length of the arc.

The ossicular axis of rotation is assumed to be fixed, an assumption that appears to be valid for low frequencies (Guinan and Peake, 1967; Decraemer et al., 1991a). The axis lies in the plane of the tympanic ring; its position is shown in Figure 6.2a. All degrees of freedom on the axis are constrained to be zero except for rotation about it. Two rigid triangular plate elements representing the neck and head of the malleus and the body of the incus couple the manubrium to this axis; the shapes of these elements are not important since they are thick enough to be effectively rigid (Funnell et al., 1992). In some simulations, the axis of rotation and the two elements representing portions of the malleus and incus were not included in order to simulate the effects of surgery; the
Figure 6.2 Eardrum model. a) Lateral view showing the outlines of the triangular elements. b) Section through the manubrium showing the curvature of the drum.
head of the malleus and the body of the incus are sometimes removed during surgery, thus destroying the axis of rotation.

Mechanical Properties

Although most biological tissues, including the eardrum, can be characterized as being viscoelastic (e.g., Decraemer et al., 1980; Özkaya and Nordin, 1991, pp. 333-354), only linearly elastic material behaviour is considered in this model. It is also assumed that the eardrum is isotropic and uniform across its thickness, thus ignoring the possible mechanical implications of its highly organized and layered structure (Funnell and Laszlo, 1982). Inhomogeneities of the eardrum and the supporting structures are also ignored; quantitative data on the inhomogeneities of the middle-ear structures are not available. With these assumptions, it is possible to characterize the mechanical properties of the eardrum by its Young's modulus, Poisson's ratio and thickness; these quantities are given below and are rough estimates based on an extensive review by Funnell and Laszlo (1982). The same parameter values have been used in previous models of the eardrum (Funnell and Laszlo, 1978; Funnell, 1983; Funnell et al., 1987).

The Young's modulus of the pars tensa is $2 \times 10^9$ dyn cm$^2$; this is the effective stiffness of the combined epidermal, fibrous and mucosal layers. The pars tensa has an overall thickness of 40 $\mu$m; this value is much smaller than the diameter of the eardrum, and therefore permits the use of thin-shell finite elements. The pars tensa is fully clamped at its attachments to the annular ligament and the manubrium; in other words, the displacements and rotations of the pars tensa are zero with respect to the supporting
structures. The pars flaccida is modelled as being much less stiff than the pars tensa (Young's modulus of $10^8$ dyn cm$^{-2}$); the pars flaccida is also fully clamped around its periphery. The portion of the annular ligament separating the pars tensa from the pars flaccida has a Young's modulus identical to that of the pars tensa but a thickness of 300 $\mu$m. The Poisson's ratio of all materials is assumed to be 0.3.

The manubrium in this model is essentially rigid. It has a Young's modulus of $2 \times 10^{11}$ dyn cm$^2$, a Poisson's ratio of 0.3 and a thickness of 0.1 cm. The manubrium is a thick structure, yet thin-shell elements may be used to model it. Thin-shell elements have also been used to represent the manubrium in previous models of the eardrum (e.g., Funnell et al., 1992). In the formulation of the thin-shell element, assumptions about its thickness relate to stresses and strains induced within the element. However, for a thick structure that is effectively rigid, no stresses and strains are induced within it during its motion, thus it is permissible to use thin-shell elements to model a thick structure as long as it undergoes rigid-body motion only.

The load exerted on the eardrum and the footplate by the air in the middle-ear cavities is not represented; this is equivalent to an experimental situation in which the middle-ear cavities and septum are open. At low frequencies, inclusion of the middle-ear cavities can be expected to reduce manubrial displacements by 5 dB (Guinan and Peake, 1967); however, the form of eardrum displacement patterns should not be greatly affected by the cavities at these frequencies (Funnell and Laszlo, 1978). The tensor tympani and stapedius muscles are not modelled, approximately simulating the relaxed state of the middle-ear muscles in temporal bone preparations and in anesthetized
animals. (Presumably, the passive mechanical effects of the middle-ear muscles are negligible.) The load due to the ossicular chain is represented by an effective angular stiffness of $10^4$ dyn cm about the axis of rotation; the angular stiffness is implemented using a torsional spring element (element type 7 of the SAP library). The value of the angular stiffness was obtained by subtracting the value for the angular stiffness of the cochlea ($18 \times 10^3$ dyn cm) given by Funnell and Laszlo (1978) from the total angular stiffness about the axis in their model ($28 \times 10^3$ dyn cm); the resultant stiffness represents contributions due to the posterior incudal ligament and anterior process of the malleus. The stiffness of the cochlea in the present model is represented in a different manner as discussed below.

6.2.3 Footplate and Cochlear Load

Figure 6.3 shows the model for the footplate; each triangle represents a thin-plate element of SAP (i.e., the same element used to model the eardrum). The shape of the footplate was obtained from a picture of a histological section (Guinan and Peake, 1967). The location and orientation of the footplate were determined from a computer reconstruction of the cat middle ear; the software and methodology used for graphically reconstructing the middle ear from serial histological sections have been described elsewhere (Funnell, 1989; Funnell et al., 1992). The mesh for the footplate has a nominal resolution of 6 elements/diameter. It was generated independently of the eardrum mesh and then combined with it, using software developed in this laboratory. A square region of area 0.04 mm$^2$ is defined at the centre of the footplate mesh; the
Figure 6.3 Model of footplate. The thin lines attached to the perimeter of the footplate represent the in-plane springs. The out-of-plane springs are not shown since they are perpendicular to the plane of the footplate; these springs are attached to the external nodes. The $x'-y'$ axes are shown along with the origin $O$. These axes will be referred to in Chapter 7.
medial end of the prosthesis is connected to this square.

The material properties for the footplate are the same as those for the manubrium: it has a Young's modulus of $2 \times 10^{11}$ dyn cm$^{-2}$ and a Poisson's ratio of 0.3. The rim of the footplate has a thickness of 200 $\mu$m; the central portion has a thickness of 20 $\mu$m; these dimensions were measured from a histological section (Guinan and Peake, 1967).

As discussed in Chapter 3, the cochlear load acting upon the footplate is stiffness-dominated for low frequencies. The stiffness is mainly due to the annular ligament (Lynch et al., 1982) and is represented in the model by axial springs distributed along the periphery of the footplate. These springs, corresponding to element type 7 of SAP, constrain both the in-plane and out-of-plane displacements of the footplate. (Element type 7 of SAP can be used to represent both torsional and axial springs. As the name implies, a torsional spring resists torsion. An axial spring, on the other hand, resists tensile and compressive forces.)

The axial springs constraining the footplate's out-of-plane motion are perpendicular to the plane of the footplate; one spring is used for each of the evenly spaced external nodes on the footplate. The stiffness of each spring is determined by dividing the total stiffness of the stapedio cochlear complex by the number of external nodes, $n_{ex}$. Lynch et al. (1982) have given a value of $0.36 \times 10^{-9}$ cm$^4$ dyn$^{-1}$ for the acoustic compliance of the stapedio cochlear complex. This can be converted to mechanical compliance by dividing by the square of the area of the footplate. The area of the footplate is 1.26 mm$^2$ (Guinan and Peake, 1967), giving a mechanical compliance...
of $2.3 \times 10^4$ cm $\text{dyn}^{-1}$; inverting this value gives a mechanical stiffness of $4.4 \times 10^5$ dyn $\text{cm}^{-1}$. Therefore, each spring has a stiffness of

$$\frac{4.4 \times 10^5}{n_{\text{cm}}} \text{ dyn cm}^{-1}. \quad (6.1)$$

For the model shown in Figure 6.1, there are 16 external nodes on the footplate, giving an individual spring stiffness of $2.76 \times 10^4$ dyn $\text{cm}^{-1}$.

In-plane displacements of the footplate cause portions of the annular ligament to be compressed or stretched; shearing effects are ignored. A portion of the annular ligament of length $l_i$ between external nodes $n_1$ and $n_2$ as shown in Figure 6.4 can be considered to be a bar of uniform cross-sectional area; its stiffness is given by:

$$k_i = \frac{E t_i}{w} \quad (6.2)$$

where $E$ is the Young's modulus of the annular ligament ($10^5$ dyn $\text{cm}^2$ according to Lynch et al.), and $t$ (200 $\mu$m) and $w$ (20 $\mu$m) are its thickness and width, respectively.

The dimensions of the annular ligament are assumed to be constant around its perimeter. They were measured from a histological section (Guinan and Peake, 1967). The stiffness of each segment of the annular ligament was computed and distributed equally to the springs at the two nodes; these springs are perpendicular to the perimeter of the footplate at the nodes in question and are in the plane of the footplate.
Figure 6.4 Schematic diagram of annular ligament around the footplate. The perimeter of the footplate has been represented by straight line segments corresponding to the sides of the triangular finite elements representing it. The top part of the figure shows a portion of the ligament of length $l$, between nodes $n_1$ and $n_2$. The bottom part is a cross-sectional view illustrating the width, $w$, and thickness, $t$, of the ligament.
6.2.4 Prosthesis

The prosthesis is considered to be rigid and can thus be modelled using a single brick element. In the present model, an 8-node brick element with 3 degrees of freedom at each node has been used (element type 5 of SAP). The formulation of this element is described elsewhere (Bathe, 1982, Chapter 5). The Young’s modulus and Poisson’s ratio of the prosthesis are the same as those of the other bones in this model (manubrium and footplate); in fact, since the prosthesis is effectively rigid, it could equally well model rigid synthetic materials such as ceramic. The prosthesis is connected between a quadrilateral region on the manubrium and a square region on the footplate; the quadrilateral region on the manubrium is defined by the manubrial mesh. In all simulations the prosthesis was connected to the uppermost quadrilateral region of the manubrial mesh except in some simulations where it was positioned further down the manubrium (see Section 7.6).

When using the brick element, the joints between the prosthesis and the manubrium and between the prosthesis and the footplate are assumed to be inflexible. In some simulations (see Section 7.6), a truss element (element type 1 of SAP) was used for the prosthesis in order to model pin joints. The truss element has a Young’s modulus that is equal to that of the manubrium and footplate. The cross-sectional area of the truss, 10 cm², is high enough to make it effectively rigid.
6.3 MSA MODEL

For the MSA, it is necessary to model the crura as shown in Figure 6.5; the stapedius muscle is not modelled. The crura are modelled using thin-plate elements. The material properties, including the thickness, are exactly the same as those for the manubrium; therefore, the crura are effectively rigid. The perpendicular distance from the incudostapedial joint to the footplate has been estimated from a histological section (Guinan and Peake, 1967) to be 1.25 mm. The medial end of the prosthesis is attached to the tops of the crura.

6.4 MODEL OF THE NORMAL CAT MIDDLE EAR

The finite-element model of the normal cat middle ear is similar to that of the MSA with two exceptions: (1) the prosthesis is no longer present and (2) the incus is modelled in its entirety. The incus is modelled by four rigid plates having the same mechanical properties as the manubrium; the shapes of these elements are not important as they are effectively rigid. The plates are connected at one end to the axis of rotation and at the other end to the tops of the crura; the incudostapedial and incudomalleal joints are thus modelled as being effectively rigid. This model is shown in Figure 6.6.

6.5 RANGE OF VALIDITY

The range of displacements for which the models are valid is limited by the assumption of linearity. The cat middle ear has been shown to be linear for sound pressure levels up to 130 dB SPL for frequencies below 1500 Hz and up to 140-150 dB...
Figure 6.5 MSA model without an ossicular axis. View showing the lateral aspect of the eardrum. The springs representing the cochlear load are not shown for clarity.
Figure 6.6 Normal middle-ear model. The springs representing the cochlear load are not shown for clarity.
SPL for higher frequencies (Guinan and Peake, 1967). It is reasonable to assume that the reconstructed middle ear also behaves linearly for these sound levels.

One other possible limit on the displacements of the models is due to a constraint common to thin-plate and thin-shell elements: the displacements must be small compared to the thickness of the structure in order to permit the assumption that in-plane and out-of-plane displacements are not coupled. This criterion has been met in all simulations; for instance, in the MFA model without an ossicular axis of rotation, the peak drum displacement is 742 nm which is less than 2% of the thickness of the drum (4C μm).

The upper frequency for which the models are valid is limited to 300 Hz by the omission of damping and inertial effects at the cochlear level (Lynch, et al., 1982). As has been shown by Funnell et al. (1987), the inertial and damping effects of the eardrum and ossicles can be neglected up to 1-2 kHz, thus the drum and the ossicles do not contribute such effects below 300 Hz in the normal middle-ear model. For the MSA and MFA models, it seems reasonable to assume that the inertial and damping effects of the prosthesis are not as large as that of the ossicles.

6.6 STIMULUS

In all simulations, a uniformly distributed pressure of 100 dB SPL is applied to the lateral surface of the eardrum. At the frequencies for which the above-mentioned models are valid, the wavelength of sound is much larger than the dimensions of the middle-ear structures; thus, it is reasonable to assume that sound pressure is uniformly distributed over the eardrum at these frequencies.
6.7 ADEQUACY OF MESH RESOLUTION

As stated previously, meshes for the eardrum and footplate were generated separately and then combined to form the MFA, MSA and normal middle-ear models. The resolution of each mesh is specified in terms of the number of elements across the diameter of the structure (Funnell, 1983). The diameters of the footplate and eardrum are different, thus the effects of mesh resolution on displacements were tested separately.

Figure 6.7 shows the maximum drum displacement as a function of mesh resolution; the effects of the prosthesis and of the cochlear load present in the MFA and MSA models are represented by four springs attached to the manubrium. The springs are attached to the nodes where the prosthesis would be located. Each spring has stiffness of $1.1 \times 10^3$ dyn cm$^{-1}$ (i.e., the stiffness of the cochlea divided by four). As the mesh resolution increases up to 12 elements/diameter, the displacement amplitudes also increase, agreeing with the fact that displacements are usually underestimated when using the displacement-based finite-element method (Bathe, 1982). For mesh resolutions of 12 elements/diameter and greater, the drum displacements are approximately constant at 612 nm, indicating convergence of the behaviour of the eardrum model. The minimum mesh resolution that can be used for the drum is therefore 12 elements/diameter; this resolution is used for all models in this work.

To determine an adequate mesh resolution for the footplate model, concentrated loads of 1 dyn were applied to the four corners of the 'square' at the centre of the footplate. (The square serves as the attachment site for the prosthesis.) Springs representing the cochlear load were distributed around the footplate as discussed in
Figure 6.7 Convergence of the behaviour of the eardrum model. The maximum displacement of eardrum is plotted against mesh resolution. The unfilled triangle indicates a mesh resolution of 12 elements/diameter.
Section 6.2.3. The maximum displacement was computed for mesh resolutions from 6 to 15 elements/diameter. The displacements were found to be approximately constant at 97.6 nm for all mesh resolutions, indicating that a resolution of 6 elements/diameter is sufficient.

It should be noted that the above convergence test for the footplate takes into account displacements due to both deformation and rigid-body motion. To investigate the effect of mesh resolution on displacements due to deformation only, the footplate was fully clamped around its periphery to eliminate rigid-body motion. Concentrated loads were then applied as outlined before. Figure 6.8 shows the displacement of the centre of the footplate as a function of mesh resolution. For resolutions of 12 elements/diameter and greater, the displacements are approximately constant with an average value of 6.76 nm. A mesh resolution of 6 elements/diameter gives a displacement amplitude of 4.64 nm; this is 31% smaller than the value computed using mesh resolutions greater than or equal to 12 elements/diameter. In assessing the adequacy of this mesh, it should be kept in mind that displacements due to the deformation of the footplate are much smaller than those due to rigid-body motion. For example, displacements due to deformation of the footplate in the MFA model are 17 times smaller than those due to rigid-body motion (see Section 7.4). Practically no deformation of the footplate occurs in the MSA and normal middle-ear models. Therefore, for the purposes of this work a mesh resolution of 6 elements/diameter was deemed adequate.
Figure 6.8 Convergence of the behaviour of the footplate model. The displacement of centre of footplate is plotted against mesh resolution. For the present convergence test, the footplate is fully clamped around its perimeter. The unfilled triangle corresponds to a mesh resolution of 6 elements/diameter.
7.1 INTRODUCTION

Displacement patterns for the normal middle-ear, MSA and MFA models are presented in Sections 7.2 to 7.4, respectively. Eardrum displacements are quantified using two measures: (1) the maximal displacement of the drum and (2) the displacement of the umbo. The first quantity is a measure of overall drum displacement; the second is a measure of the input to the middle ear. Displacements of the footplate are reported in terms of components within its plane and components normal to its plane.

Since the mechanical properties of the middle-ear structures are only roughly known, and since they show a great deal of inter-animal variability and may even change after surgery, the sensitivity of the footplate's displacement pattern to variations in these properties is explored; results may be found in Section 7.5. The effects of varying the position of the prosthesis in both the MFA and MSA models are explored in Section 7.6.

7.2 DISPLACEMENT PATTERNS FOR THE NORMAL MIDDLE-EAR MODEL

Displacements of the Eardrum

Figure 7.1 shows the displacement contours calculated for the eardrum. The contours are lines of constant vibration amplitude, equally spaced on an amplitude scale. Only displacements in a direction normal to the tympanic ring are considered; these
Figure 7.1 Eardrum displacement contours. The low-frequency displacement amplitudes were calculated using the normal middle-ear model. The contours represent amplitudes evenly spaced between 0 and the maximum displacement. The position of the maximum is indicated by a filled triangle.
correspond to the component of displacement measured in the experiments of Tonndorf and Khanna (1971). The contours indicate that the displacements of the manubrium are smaller than those of the surrounding eardrum; these contours are qualitatively similar to those determined experimentally. The small triangle in the figure represents the point of maximal drum displacement, having a magnitude of 484 nm for a low-frequency pure-tone input of 100 dB SPL; this maximum occurs in the posterior region of the pars tensa. By comparison, Tonndorf and Khanna reported a maximal drum displacement of 1500 nm at 600 Hz and 111 dB SPL; this is equivalent to 420 nm at 100 dB SPL. (The frequency of 600 Hz is low enough that the displacement is practically the same as at 0 Hz.) Funnell et al. (1987) calculated a maximal drum displacement of 662 nm using their model of the normal cat eardrum. This value is higher than that calculated using the present model and may reflect the different way in which the cochlear load was modelled by them.

The umbo displacement calculated using the present model is 161 nm. Tonndorf and Khanna (1971) reported the ratio of peak drum displacement to umbo displacement to be 2.2; this corresponds to an umbo displacement of 191 nm for a peak drum displacement of 420 nm. Funnell et al. (1987) computed a ratio of 2.7 using their model, which corresponds to an umbo displacement of 245 nm. The umbo displacement measured by Tonndorf and Khanna is 19% larger than that calculated using the present model, whereas the value computed by Funnell et al. is 52% larger.
Displacements of the Footplate

It is convenient to describe the three-dimensional motion of the footplate by decomposing this motion into components within the plane of the footplate and components normal to its plane. The description of footplate motion in terms of these components is arbitrary but has some physiological significance: normal components of footplate motion cause displacement of cochlear fluids, whereas in-plane components are not effective in displacing these fluids.

Figure 7.2a shows contours for the normal component of footplate displacement. The displacements are 99.0 nm for the anterior end (point A in Figure 7.2a), 91.6 nm for the centre (point C), and 82.7 nm for the posterior end (point P). The displacement of the central point was used by Vlaming and Feenstra (1986b) to compare the mechanics of the footplate in various cadaver preparations. If the footplate is perfectly rigid, then the normal component of displacement of this point, when multiplied by the area of the footplate, gives the volume displacement of the cochlear fluids; on the other hand, if the footplate deforms or bulges at the centre then this product will overestimate the volume displacement of the cochlear fluids. Any bulging of the footplate that might occur at its centre can be quantified using the parameter $\beta$ (henceforth called the bulge ratio):

$$\beta = \frac{n_c - \frac{1}{2}(n_A + n_P)}{n_c} \quad (7.1)$$
Figure 7.2 Footplate displacement contours. The displacements were calculated using the normal middle-ear model. (a) Component of displacement normal to plane of footplate. Displacement amplitudes are labelled in nm. The anterior (A), central (C), inferior (I), posterior (P) and superior (S) points of the footplate are indicated on the diagram. (b) $x'$-component of displacement. (c) $y'$-component of displacement. The footplate in parts (b) and (c) has been scaled relative to that in part (a).
where \( n_A, n_C \) and \( n_P \) are the normal components of displacement at the anterior, central and posterior points of the footplate, respectively. If the footplate is rigid, then the displacement of the central point will be equal to the average of the displacements of the anterior and posterior points and the bulge ratio will be zero; this is more or less the case for the present model since the bulge ratio has a very small value of 0.8%. The bulge ratio should be zero in this model since no forces are applied to the centre of the footplate; however, it is nonzero since the central point is not exactly half-way between the anterior and posterior ends but is 0.8% closer to the anterior end.

The difference in displacement amplitudes between the anterior and posterior ends indicates that the footplate tilts. The amount of antero-posterior tilting relative to the displacement of the centre can be characterized by the parameter \( \Delta_{AP} \) (henceforth called the antero-posterior tilt ratio):

\[
\Delta_{AP} = \frac{n_A - n_P}{n_C}.
\]  

(7.2)

The antero-posterior tilt ratio is 0.18 for this model, indicating that the difference in displacements of the two ends is 18% as large as the displacement of the centre of the footplate; a positive ratio indicates that the anterior end displaces more than the posterior end. Tilting of the footplate also results in differences in the displacements of the inferior and superior ends (points I and S, respectively, in Figure 7.2a). An infero-superior tilt ratio, \( \Delta_{IS} \), may be calculated using the formula
where \( n_i \) and \( n_s \) are the normal components of displacement of the inferior and superior ends, respectively. The displacement of the superior end is 77.0 nm and that of the inferior end is 106.2 nm, giving an infero-superior tilt ratio of 0.32; a positive result indicates that the inferior end displaces more than the superior end.

Figures 7.2b and 7.2c show the contours for the in-plane components of footplate motion. The in-plane components have been further decomposed into \( x' \) and \( y' \) components; the \( x' \)- and \( y' \)-axes are attached to the footplate as shown in Figure 6.3. Since the contours for the \( x' \) component consist of evenly spaced horizontal lines and those for the \( y' \) component consist of evenly spaced vertical lines, it was hypothesized that the in-plane motion of the footplate was that of a rigid body. (See Appendix A for a justification of this interpretation.) In order to test this hypothesis, an attempt was made to fit a translation and a rotation to the finite-element results as described below and in Appendix A; nodal displacements caused by this rigid-body motion were then compared to the nodal displacements calculated using the finite-element software. The finite-element results in general will include displacements due to both deformation and rigid-body motion. In order to compare the magnitudes of displacements caused by deformation to those caused by rigid-body motion, the ratio \( D/R \) was computed where

\[
D = \frac{1}{n} \sum_{i=1}^{n} \sqrt{(u'_i - u)^2 + (v'_i - v)^2}
\]  

(7.4)
and

\[ R = \frac{1}{n} \sum_{i=1}^{n} \sqrt{u_i^2 + v_i^2}. \]  

(7.5)

Primed quantities represent nodal displacements calculated using the finite-element software, and unprimed quantities represent nodal displacements calculated using the hypothesized translation and rotation; the summations in Equations (7.4) and (7.5) take into account all of the nodes of the footplate. The quantity \( D \) represents the average displacement amplitude due to deformation, whereas \( R \) represents the average displacement amplitude due to rigid-body motion. For the footplate in the normal middle-ear model, the ratio \( D/R \) was found to be \( 2.5 \times 10^{-4} \), indicating that the average displacement due to deformation is almost four orders of magnitude smaller than that due to rigid-body motion; it is thus possible to approximate the in-plane motion of the footplate as that of a rigid body.

To estimate the translation and rotation, a point was arbitrarily chosen about which the angle of rotation was estimated. The point \( O \) shown in Figure 6.3 was chosen. The displacement of this point, calculated using the finite-element software, gives the translation of the footplate which, in this case, has an \( x' \) component of 66.7 nm and a \( y' \) component of 87.4 nm.

The angle of rotation can then be estimated using the formula

\[ \theta = \frac{\Delta u}{\Delta y}; \]  

(7.6)

or
The angle of rotation was found to be 8.9 μrad in the counterclockwise direction.

It should be noted that the maximum in-plane displacement amplitude of the footplate is 116 nm. This value is much smaller than the width of the annular ligament (20 μm), thus justifying the assumption that the ligament behaves linearly.

Comparison to Experimental Results

The only detailed measurements of footplate motion in cats with which these results can be compared are those of Guinan and Peake (1967). As discussed in Chapter 3, these authors found that, to a first approximation, the footplate moves in a piston-like manner. Figure 14 (p. 1248) of their paper indicates that the magnitude of footplate displacement per unit sound pressure at the eardrum is approximately $3.0 \times 10^{-7}$ cm$^3$ dyn$^{-1}$, giving a stapes displacement of 85 nm for a pressure of 100 dB SPL (i.e., 28.28 dyn cm$^{-2}$) applied to the eardrum; this agrees well with the results calculated using the present model. The results of Guinan and Peake suggest that all points on the footplate move by this amount since no tilting or bulging of the footplate occurred in their preparation. The results of the present model indicate that the footplate does not bulge significantly, but it does tilt.

The model also predicts in-plane motion of the footplate; the magnitude of this motion is comparable to that of the medio-lateral, or normal, component of motion. The results of Guinan and Peake indicate that in-plane motion of the footplate, if any, is small.
compared to the normal component of motion. Wada et al. (1992) have also reported large in-plane displacements of the footplate in their middle-ear model. They suggested that one reason for this type of motion may be that the incudostapedial joint in their model was assumed to be rigid. In the present model, the malleus and the incus are assumed to rotate as a single rigid body around the ossicular axis. Since the incudostapedial joint has also been modelled as being rigid, the stapes is forced to rotate about the ossicular axis, thus resulting in in-plane, as well as medio-lateral, motion of the footplate. A flexible incudostapedial joint could, however, permit rotation of the incus and malleus about the ossicular axis without forcing the stapes to rotate about the same axis. More work needs to be done to estimate the stiffness of the incudostapedial joint and its effects on footplate motion.

7.3 DISPLACEMENT PATTERNS FOR THE MSA MODEL

7.3.1 Introduction

Recall that for the MSA a prosthesis is placed between the manubrium and the stapedial head. The ossicular axis may or may not be intact; results are reported for both cases.
7.3.2 Intact Ossicular Axis

Displacement patterns and amplitudes for the eardrum and for the footplate calculated using the MSA model with an intact ossicular axis are virtually identical to those calculated using the normal middle-ear model; all displacement amplitudes are within 2% of those calculated for the normal model. This similarity is to be expected since the ossicular chain in the normal model acts as a single rigid body that is, in effect, equivalent to the manubrium-prosthesis-stapes system in the MSA model.

7.3.3 Absent Ossicular Axis

Displacements of the Eardrum

The shapes of the displacement contours for the eardrum calculated using the MSA model without an ossicular axis are similar to those calculated using the previous models. These patterns remain unchanged since they are controlled primarily by the properties of the eardrum, which do not change from model to model. However, the magnitudes of the displacements do change when the axis is removed. The maximum drum displacement calculated using the present model is 736 nm which is approximately 1.5 times larger than that calculated using the previous models. Removing the ossicular axis of rotation removes the constraints it places on the eardrum, thus permitting larger drum displacements.

The umbo displacement is 311 nm in the present model, which is almost twice as large as that calculated using the previous models. The larger umbo displacement
indicates that the input to the middle ear is larger in the present model than in the previous models.

**Displacements of the Footplate**

The normal component of displacement amplitude at the centre of the footplate is 107 nm in this model, a value that is 15.4 nm larger than that calculated using the previous models. The anterior end has a larger displacement amplitude than the posterior end, giving an antero-posterior tilt ratio of 0.54; this value is approximately three times larger than that calculated for the MSA model with an intact ossicular axis and for the normal middle-ear model. The inferior end has a slightly larger displacement amplitude than the superior end; the infero-superior tilt ratio is 0.16, a value that is about half as large as that calculated using the previous models. The footplate in this model does not bulge significantly, having a bulge ratio of 0.5%.

Once again, the in-plane motion of the footplate can be described by a rigid-body motion. The $x'$ component of translation is 65.0 nm; the $y'$ component is -55.5 nm. The $x'$ component of translation is similar to that calculated for the previous models; the $y'$ component of translation has changed sign. The angle of rotation is 10 µrad but in the clockwise direction.

**Comparison to Experimental Results**

No measurements of footplate displacements have been made in cats to assess the mechanics of the surgically corrected middle ear; however, such measurements have been
made in human cadaver middle ears (Vlaming and Feenstra, 1986b). In comparing the mechanics of the reconstructed cat and human middle ears, it is necessary to keep in mind the anatomical differences between these two species that were outlined in Section 2.3. In addition, it is necessary to keep in mind that there are differences in surgical technique. Vlaming and Feenstra did not give a detailed description of their surgical technique. For instance, no mention was made of what was done with the ossicular remnants. It is reasonable to assume, however, that a conservative approach was used in which the remnants, and hence the ossicular axis, were left intact. The results of the MSA model with an intact ossicular axis can therefore be compared with their results. At low frequencies, they found that displacements of the footplate's centre were 10-20 dB lower (i.e., about 3-10 times lower) than those for the ear before surgery. Modelling results indicate that the displacement of the footplate's centre is virtually identical to that calculated for the normal case. The discrepancy between experimental results and modelling results indicates that the flexibility of the manubrium-prosthesis-stapes system needs more study.

7.4 DISPLACEMENT PATTERNS FOR THE MFA MODEL

7.4.1 Introduction

In this model, the prosthesis contacts the footplate directly, rather than contacting the stapedial head as in the MSA model. The ossicular axis may or may not be intact; results are reported for both cases.
7.4.2 Intact ossicular axis

Displacements of the Eardrum

The contour patterns for eardrum displacements are qualitatively similar to the previous ones. The maximum drum displacement is 480 nm which is 1% smaller than the value calculated for the normal middle-ear model. The displacement of the umbo is 155 nm which is approximately 4% smaller than that calculated using normal model.

Displacements of the Footplate

Contours for the normal component of displacement amplitude are shown in Figure 7.3. The displacement amplitude at the centre of the footplate is 92.5 nm, a value which is comparable to that calculated using the normal middle-ear model. The anteroposterior tilt ratio is 0.10, which is slightly smaller than for the normal case; the inferosuperior tilt ratio is 0.30, which is similar to that calculated using the normal middle-ear model. The bulge ratio is an order of magnitude larger than for the normal middle-ear model, having a value of 6%; it is larger since the prosthesis directly loads the footplate. Notice that the contours in Figure 7.3 no longer consist of straight lines since the footplate bulges.

The in-plane motion of the footplate in this model is similar to that of the footplate in the normal middle-ear model. The $D/R$ ratio is $6.2 \times 10^{-4}$. The $x'$ component of translation is 67.9 nm, and the $y'$ component is 87.7 nm. The angle of rotation is $9.0 \mu$rad.
Figure 7.3 Displacement contours for the footplate in the MFA model. The ossicular axis is intact. The normal component of displacement is shown. Displacement amplitudes are labelled in nm.
7.4.3 **Absent Ossicular Axis**

**Displacements of the Eardrum**

Once again, the contours for eardrum displacement are similar to those computed for the previous models. The maximum drum displacement is 742 nm, and the displacement of the umbo is 337 nm; both values are larger than for the normal case and for the MFA model with an ossicular axis.

**Displacements of the Footplate**

The normal component of displacement is 112 nm at the centre of the footplate. The antero-posterior tilt ratio is 0.38, and the infero-superior tilt ratio is 0.14. The bulge ratio has a value of 6%, which is equal to that computed for the MFA model with an ossicular axis. The bulge ratio indicates that the displacement of the centre of the footplate due to deformation is almost 17 times smaller than the total displacement of the centre.

The $D/R$ ratio is $3.1 \times 10^{-4}$, thus the in-plane motion can be described by a rigid-body motion. The $x'$ component of translation is 63.5 nm; the $y'$ component is -55.0 nm. The $y'$ component of translation is negative just as in the case of the MSA with no ossicular axis. Also as in that case, the angle of rotation is clockwise, having a value of 10 μrad.
Comparison to Experimental Results

Vlaming and Feenstra (1986b) found that, for the TORP at low frequencies, displacements of the footplate's centre were comparable to what they had measured pre-operatively. (Recall that the TORP is similar to the MFA.) Results calculated using the MFA model with an intact ossicular axis also indicate that the normal component of footplate displacement at its centre is comparable to that calculated using the normal middle ear model.

7.5 SENSITIVITY ANALYSES

7.5.1 Introduction

For simplicity and brevity, the effects of parameter variations on footplate displacements were determined only for the MFA model with no ossicular axis. It is assumed that the other models in this work will show similar sensitivity to these parameters.

The effects of varying the stiffness, thickness and Poisson's ratio of the footplate and the stiffness of the annular ligament were explored. Variations in the mechanical properties of the eardrum and prosthesis were not explored. Previous results of modelling indicate that the low-frequency mechanical behaviour of the normal cat eardrum is primarily determined by its stiffness and thickness, curvature and conical shape, and anisotropy (Funnell and Laszlo, 1978). Boundary conditions, Poisson's ratio, ossicular loading and air loading were found to be less important. It is also expected that
the present models will show similar sensitivity to these parameters. Since the prosthesis was assumed to be rigid, it was not necessary to vary its material properties.

7.5.2 Variation of Young's Modulus

The Young's modulus of the footplate was varied from 100 Mdyn cm\(^{-2}\) to 500 Mdyn cm\(^{-2}\); the standard value used in the model is 200 Mdyn cm\(^{-2}\). Figure 7.4 shows the normal component of displacement for the anterior, posterior and central points of the footplate as a function of the Young's modulus. For values of 200 Mdyn cm\(^{-2}\) or more, the displacements of these points remain approximately constant; the footplate continues to tilt so that displacements of the anterior end are larger than those of the posterior end. When the Young's modulus is decreased below the standard value, the footplate becomes less rigid, and the bulge ratio increases by 83\%. (Bulging of the footplate can be seen from Figure 7.4 since the displacement of the central point is not close to the average of the displacements of the two ends.) The amount of tilting for this value of the Young's modulus is less pronounced than that for the higher values of the Young's modulus; the antero-posterior tilt ratio has decreased by 28%.

The in-plane motion of the footplate does not change significantly as the Young's modulus is varied. The \(x'\) component of translation decreases by only 2.8\% when the Young's modulus is reduced to 100 Mdyn cm\(^{-2}\); the magnitude of the \(y'\) component decreases by even less (1.0\%). As the Young's modulus is increased to 500 Mdyn cm\(^{-2}\),
Figure 7.4 Variation of Young's modulus of the footplate. The normal components of displacement amplitudes for the anterior, central and posterior points of the footplate are plotted against the Young's modulus.
the $x'$ component of translation increases by 1.7%, whereas the magnitude of the $y'$ component increases by 0.6%. The angle of rotation does not vary at all.

7.5.3 Variation of Poisson's Ratio

Varying the Poisson's ratio from 0.0 to 0.5 has little effect on the normal component of footplate displacement. The displacement of the central point increases by only 0.4% when the Poisson's ratio is reduced to zero, and it decreases by 0.3% when the Poisson's ratio is increased to 0.5. Note that for problems in three-dimensional elasticity and for problems in plane strain, a Poisson's ratio of 0.5 cannot be handled using the displacement-based finite-element formulation; a mixed formulation must be used (Gallagher, 1975). However, this does not pose a problem in the analysis of thin plates and shells.

The in-plane displacements of the footplate do not change significantly either as the Poisson's ratio is varied. The $x'$ component of translation decreases by 0.3% when the Poisson's ratio is reduced to zero; the $y'$ component decreases by 0.1%. Increasing the Poisson's ratio to 0.5 increases the $x'$ component by 0.2%; the $y'$ component increases by 0.1%. The angle of rotation remains constant for all values of the Poisson's ratio.

7.5.4 Variation of Footplate Thickness

The thicknesses of the central portion of the footplate and of the rim were varied separately.
Thickness of the Central Portion

The thickness of the central portion was varied from 10 μm to 50 μm; the standard value used in the model is 20 μm. Figure 7.5 shows the normal component of footplate displacement as a function of thickness. For thicknesses of 20 μm or more, the displacements remain approximately constant with the anterior end moving more than the posterior end. However, for a 10-μm thick footplate the displacements of the anterior and posterior ends are approximately equal, indicating that tilting is negligible.

Note that a 20-μm thick footplate is already quite rigid so that increases in the thickness above this value do not produce significant changes in the amount of bulging; however, decreasing the thickness below 20 μm makes the footplate less rigid and more prone to bulging. Changes in the thickness of the footplate have a more pronounced effect on its rigidity than do changes in its Young's modulus alone; for example, cutting the thickness in half from 20 μm to 10 μm increases the bulge ratio by a factor of 5.9, whereas cutting the Young's modulus in half increases it by only a factor of 1.8; this is to be expected since the bending stiffness of a plate is proportional to the cube of its thickness, whereas it is only directly proportional to its Young's modulus.

Just as for the normal component of displacement, increasing the thickness of the footplate above 20 μm has little effect on its in-plane displacements. When the thickness is decreased from 20 μm to 10 μm, the $D/R$ ratio increases from $3.1 \times 10^4$ to $1.3 \times 10^3$; the in-plane motion of the footplate can still be described by a rigid-body motion. However, the $x'$ component of translation decreases by 18%. The magnitude of the $y'$ component increases by only 6.4%. The angle of rotation increases by 8.7%.
Figure 7.5 Variation of the thickness of the central portion of the footplate. The normal components of displacement amplitudes for the anterior, central and posterior points of the footplate are plotted against the thickness.
**Thickness of the Rim**

The effects of varying the thickness of the rim from 20 μm (i.e., the same thickness as the central portion) to 400 μm are shown in Figure 7.6; the standard value of thickness used in the model is 200 μm. Increasing the thickness of the rim beyond 200 μm or decreasing it to 100 μm has little effect on the normal component of displacement; however, decreasing the thickness of the rim to 20 μm results in greater bulging of the footplate and decreased displacements of the anterior and posterior ends. When the thickness is reduced to 20 μm, the bulge ratio increases by a factor of 9.2; the displacements of the anterior and posterior ends decrease by 36% and 39%, respectively, and the centre displaces 2-3 times as much as the ends.

The in-plane motion of the footplate similarly does not change much as the thickness of the rim is increased beyond 200 μm or decreased to 100 μm. When the thickness is decreased from 200 μm to 20 μm, the \(D/R\) ratio increases from \(3.1 \times 10^4\) to \(7.5 \times 10^4\). The \(x'\) component of translation decreases by 25%, while the magnitude of the \(y'\) component decreases by only 8.4%, and the angle of rotation increases by 8.7%.

**7.5.5 Variation of Annular Ligament Stiffness**

The effects of the annular ligament on the displacements of the footplate are represented by in-plane and out-of-plane springs attached along the periphery of the footplate. The effects of varying the stiffness of the annular ligament were investigated by varying the in-plane and out-of-plane stiffnesses separately and then simultaneously; it is permissible to vary the in-plane and out-of-plane stiffnesses separately since the
Figure 7.6 Variation of footplate rim thickness. The normal components of displacement amplitudes for the anterior, central and posterior points of the footplate are plotted against the rim thickness.
annular ligament may be anisotropic. These stiffnesses were reduced to one-quarter and one-half of their values, and they were also increased to twice and four times their values.

Variation of Out-of-plane Stiffness

Figure 7.7a shows the effects of varying the out-of-plane stiffness on the normal component of footplate displacement. Increasing the stiffness above the standard value results in a decrease in the displacement amplitudes, whereas decreasing it below the standard value results in an increase in the displacement amplitudes. This is expected, of course, since the displacements of a spring are inversely proportional to its stiffness.

As can be seen from Figures 7.8a to 7.8c, the in-plane motion of the footplate is also affected by variations in the out-of-plane stiffness of the annular ligament. As the stiffness is decreased below the standard value, both the $x'$ and $y'$ components of translation decrease; presumably, when the out-of-plane stiffness is decreased, it is easier for the footplate to respond to the force exerted by the prosthesis by moving into and out of the cochlea than by moving sideways. When the stiffness is increased, the two components of translation also increase since it is now easier for the footplate to move in its own plane than to move out of its plane. Notice that the sign of the $x'$ component changes as the stiffness is reduced to one-quarter of its value.
Figure 7.7 Variation of out-of-plane annular-ligament stiffness. The normal components of displacement amplitude for the anterior, central and posterior points of the footplate are plotted against the stiffness.
Figure 7.8 Variation of out-of-plane annular-ligament stiffness. The effects on the in-plane motion of the footplate are shown. (a) $x'$ component of translation. (b) $y'$ component. (c) Angle of rotation.
Variation of In-plane Stiffness

Figures 7.9a to 7.9c show how the in-plane motion of the footplate depends on the in-plane stiffness of the annular ligament. As expected, the $x'$ and $y'$ components of translation decrease as the in-plane stiffness is increased above the standard value; the angle of rotation also decreases. The two components of translation and the angle of rotation all increase as the in-plane stiffness is decreased below the standard value. Changes in the $x'$ component are not symmetric for symmetric changes in the stiffness around the standard value. For instance, the $x'$ component increases by only 3% when the stiffness is reduced to one-quarter of the standard value, whereas it decreases by 37% when the stiffness is increased by a factor of four.

As shown in Figure 7.10, increasing the in-plane stiffness of the annular ligament has some effect on the normal component of displacement. The displacement at the centre of the footplate increases by 16% when the in-plane stiffness is increased by a factor of four, whereas it decreases by 12% when the in-plane stiffness is reduced to one-quarter of its value. As the in-plane stiffness is increased, there is a greater tendency for the footplate to move into and out of the cochlea than to move sideways.

Simultaneous Variation of In-plane and Out-of-plane Stiffnesses

Figure 7.11 shows the normal component of footplate displacement as a function of annular ligament stiffness; both the in-plane and out-of-plane stiffnesses were varied simultaneously. These curves are visually indistinguishable from those obtained when the out-of-plane stiffness was varied by itself (cf. Figure 7.7), suggesting, as one might
Figure 7.9 Variation of in-plane annular-ligament stiffness. The effects on the in-plane motion of the footplate are shown. (a) $x'$ component of translation. (b) $y'$ component of translation. (c) Angle of rotation.
Figure 7.10 Variation of in-plane annular-ligament stiffness. The normal components of displacement amplitude for the anterior, central and posterior points of the footplate are plotted against the in-plane stiffness.
Figure 7.11 Variation of total annular-ligament stiffness. Both the in-plane and out-of-plane stiffnesses are varied. The normal components of displacement amplitude for the anterior, central and posterior points of the footplate are plotted against the stiffness.
expect, that the normal component of displacement is primarily controlled by the out-of-plane stiffness.

As shown in Figure 7.12, the $x'$ component of translation varies in a manner similar to that seen when the out-of-plane stiffness was varied alone. On the other hand, the $y'$ component and the angle of rotation vary in a manner similar to that seen when the in-plane component was varied by itself.

7.6 PROSTHESIS POSITION

It is possible to position the prosthesis in the MFA and MSA at various points along the manubrium. In the simulations presented above, the prosthesis was positioned near the upper end of the manubrium; in this section, the effects of positioning the prosthesis further down on the manubrium are explored. For example, Figure 7.13 shows a case where the prosthesis is positioned half-way down the manubrium. Moving the prosthesis down the manubrium does not significantly alter the mechanical behaviour of the footplate. As an example of the changes, consider the MFA model with no ossicular axis; moving the prosthesis down to the umbo results in a decrease of 0.3% in the normal component of displacement of the centre of the footplate. The in-plane displacements do not change significantly either: the $x'$ component of translation decreases by 0.4%, the $y'$ component by 0.7%; and the angle of rotation remains constant. These results are consistent with those of Tonndorf and Pastaci (1986). This behaviour is expected since the manubrium, prosthesis and footplate in the model effectively act as a single rigid body.
Figure 7.12 Variation of total annular-ligament stiffness. The effects on the in-plane motion of the footplate are shown. (a) $x'$ component of translation. (b) $y'$ component of translation. (c) Angle of rotation.
Figure 7.13 MFA model with prosthesis positioned half-way down the manubrium. The position of the prosthesis on the footplate has not been changed.
If the joints between the prosthesis and the bones are made flexible, the footplate’s mechanical behaviour will be affected by prosthesis position. For example, consider an extreme case where pin joints are assumed, but the prosthesis is still rigid; this can be modelled using a rigid truss element for the prosthesis. Figure 7.14 shows the out-of-plane displacements at the centre of the footplate for the MFA model without an ossicular axis as the truss element is positioned at various locations along the manubrium; displacement amplitudes for the case of inflexible joints (i.e., using a brick element) are also shown. Clearly, the position of the prosthesis is important if the joints are flexible. The out-of-plane displacements of the footplate increase in magnitude as the prosthesis is positioned closer to the upper end of the manubrium; the in-plane displacements also increase.

For the MFA, it is also possible to position the prosthesis at various points on the footplate. An off-centre position affects the mechanical behaviour of the footplate. For example, positioning the prosthesis at the anterior end of the footplate results in greater displacement of this end and greater tilting compared to that seen when the prosthesis is positioned at the centre; the out-of-plane displacement of the anterior end is 149 nm, whereas that of the posterior end is 59.3 nm. In-plane displacements are not significantly affected.
Figure 7.14 Effect of prosthesis position in the MFA model. The ossicular axis is not intact. The normal component of footplate displacement for the central point is plotted against the distance along the manubrium from the umbo. The top curve shows the displacements for rigid joints (a brick element was used for the prosthesis), while the bottom one for pin joints (a truss element was used).
CHAPTER 8
CONCLUSIONS AND FUTURE DEVELOPMENTS

8.1 CONCLUSIONS

Finite-element models of the normal and surgically corrected cat middle ear were developed in this work. Two surgical techniques were modelled, the malleus-stapes assembly (MSA) and the malleus-footplate assembly (MFA). The models are valid for frequencies below 300 Hz and for physiological sound levels.

Displacement contours were calculated for the normal middle-ear, MSA and MFA models. Contours for the eardrum were found to be qualitatively similar for all models. Moreover, the eardrum contours for the normal middle-ear model were found to be similar to experimentally measured ones; no experimental data on drum displacements are available for the surgically corrected middle ear.

Components of footplate displacement normal to its plane constitute the mechanical input to the cochlea and, for a rigid footplate, the displacement of the centre reflects the volume displacement of the cochlear fluids. The normal component of displacement at the centre of the footplate in the normal middle-ear model was found to be similar to that determined experimentally. For the MSA model with an intact ossicular axis, the displacement was virtually identical to that calculated for the normal case; this result is to be expected since in the normal model the ossicular chain was modelled as being rigid, while in the MSA the manubrium and prosthesis were also modelled as being rigid. Removal of the axis resulted in displacement amplitudes which
were larger but not significantly different from normal levels. For the MFA model with an intact ossicular axis, the displacement of the footplate was similar to that for the normal middle-ear model; removal of the axis in the MFA model resulted in larger displacements. Significant in-plane displacements of the footplate were also predicted by all of the models.

The effects of parameter variations on the displacements of the footplate were investigated for the MFA model with no ossicular axis. Displacements of the footplate were found to be sensitive to its Young’s modulus and thickness and to the stiffness of the annular ligament. The mechanical behaviour of the footplate was not sensitive to the Poisson’s ratio. The MSA and normal middle-ear models are expected to show similar sensitivity to the above parameters.

The position of the prosthesis along the manubrium in both the MFA and MSA models, with or without an ossicular axis, was not found to affect the mechanical behaviour of the footplate as long as the bones, prosthesis and joints were all rigid; however, if the joints were made to be completely flexible, the largest out-of-plane displacements of the footplate occurred when the prosthesis was positioned closer to the upper end of the manubrium. The mechanical behaviour of the footplate was also affected by the position of the prosthesis on the footplate in the MFA model.
8.2 FUTURE DEVELOPMENTS

8.2.1 Introduction

The objective of this thesis was to develop models of the mechanics of the surgically corrected cat middle ear. The finite-element method was used to develop the models. Although the finite-element method is a powerful technique for developing such models, it is not possible at this point to take complete advantage of the capabilities of the method due to our limited knowledge about the mechanical properties of the middle-ear structures, and how these properties change after surgery. As discussed in the next section, further work needs to be done to obtain this information. Before using new experimental data to refine the existing models, it would be desirable to enhance the mesh-generation software as outlined in Section 8.2.3. Possible ways in which the existing models can be enhanced are discussed in Section 8.2.4. Potential clinical applications of finite-element models are discussed in the last section.

8.2.2 Experimental Work

Data are required for both the normal and the surgically corrected middle ear; these two situations are discussed separately.

Normal Middle Ear

Since finite-element models of the middle ear can only predict the mechanical input to the cochlea, it is necessary to understand the relationship between this input (i.e., displacements of the footplate) and hearing sensitivity. In cats, it would be possible to
measure displacements of the footplate and simultaneously assess changes in hearing sensitivity by measuring the cochlear microphonic, auditory nerve activity, etc. Such experiments would not, however, be possible in humans.

The success of surgery may be evaluated by comparing the mechanical behaviour of the footplate post-operatively with its behaviour in the normal ear. In order to make this comparison, it is first necessary to fully describe the three-dimensional motion of the footplate in the normal ear. With the exception of the work by Guinan and Peake (1967), experimental studies have only measured one component of footplate displacement. Interferometric methods have already been used to measure the three-dimensional motion of the malleus (Decraemer and Khanna, 1992, 1993); such methods could also be adapted to measure the three-dimensional motion of the footplate.

More quantitative information is needed on the geometry of the middle-ear structures. The shape and thickness of the eardrum are thought to be important to its function (Funnell and Laszlo, 1978). Phase-shift moiré topography has been used to measure the shape of the eardrum (Decraemer et al., 1991a); however, only a few such measurements have been made. It would be desirable to make more measurements of eardrum shape in order to understand the variation in eardrum geometry among individuals. The variation in thickness of the eardrum over its entire surface also needs to be determined. In addition, the thickness and width of the annular ligament must be measured around the entire circumference of the footplate since these dimensions are thought to affect the motion of the footplate (Guinan and Peake, 1967).
In modelling the middle ear, it was assumed that the material properties of the eardrum and annular ligament are isotropic and homogeneous. However, the fibrous composition of these structures suggests that their properties could depend on direction. Moreover, the variable distribution of fibres in these tissues suggests that these materials may be inhomogeneous. The dependence of the material properties of these tissues on position and direction could potentially be measured using techniques such as electronic speckle pattern interferometry (Charette et al., 1993).

The Surgically Corrected Middle Ear

Just as for the normal middle ear, the three-dimensional motion of the footplate and the geometry of the eardrum need to be measured in the surgically corrected middle ear. Changes in the shape of the eardrum can be caused by both middle-ear disease and by the forces exerted on the eardrum by the prosthesis. It would be useful to evaluate the contributions that each of these factors makes in changing the shape of the eardrum. The effects of middle-ear disease on eardrum shape can be evaluated by measuring the shape of the eardrum pre-operatively. The combined contributions due to both factors can be evaluated post-operatively. The changes caused by the prosthesis itself can be determined by ‘subtracting’ the pre-operative geometry from the post-operative one.

The material properties of the middle-ear structures may also be affected by disease and need to be measured. For example, scarring of the eardrum could result in a change in its Young’s modulus; the Young’s modulus is a very important factor in determining the function of the eardrum (Funnell and Laszlo, 1978).
Changes in loading patterns due to surgery affect the mechanical properties of the middle-ear structures. For instance, the stiffness of the annular ligament is increased when a prosthesis loads the footplate (Vlaming and Feenstra, 1986b); as demonstrated in this work, the motion of the footplate is very sensitive to changes in annular ligament stiffness. Measurements of such changes would greatly aid the enhancement of the MFA and MSA models.

The joints formed between the prosthesis and the bones strongly affect the mechanical behaviour of the footplate. The joints are stabilized somewhat by the growth of fibres into the joint space, but the articulations may still remain somewhat flexible. Estimates of the stiffnesses of these joints would certainly help the modelling process.

8.2.3 Enhancements to the Mesh-Generation Software

The mesh-generation software used in this thesis is capable of generating triangular meshes over surfaces. The software has several shortcomings that make it difficult to model the normal and surgically corrected middle ears. First, it cannot handle multiple objects with different orientations, such as the eardrum and footplate. To overcome this difficulty, meshes for the footplate and eardrum were generated separately and then combined; the orientation and position of the footplate relative to the eardrum were adjusted using auxiliary software. It would be desirable to modify the existing mesh-generation software to be able to generate meshes for multiple structures with different orientations; this would eliminate the need to orient and position the structures using auxiliary software.
The second shortcoming of the mesh generator is that it does not allow the specification of different resolutions for the various components of a structure. The mesh generator could be modified to accept input in which each component is labelled as having a different resolution. After accepting the input, the software could generate meshes for each component of the structure individually.

Third, the software at this time cannot generate meshes for solid objects such as the prosthesis. Meshes for such objects must be generated manually and then added to the eardrum and footplate meshes. This procedure is tolerable if only a few elements need to be added; however, it becomes cumbersome when many elements need to be used, which would be the case if a flexible prosthesis were being modelled. Three-dimensional mesh-generation software does exist in our laboratory (Boubez, 1986) and could be combined with the two-dimensional software to handle structures with both two-dimensional and three-dimensional components.

8.2.4 Enhancements to the Models

Apart from refining the parameters of the models using the data derived from experimental studies, more effort must focus on modelling the incudostapedial joint and the ligaments, and on extending the frequency range of the models. Once three-dimensional mesh-generation capabilities are incorporated into the mesh-generation software, it will be possible to model the effects of a compliant incudostapedial joint on the motion of the footplate. If a compliant joint is found to affect the motion of the footplate, then it would be desirable to measure the stiffness of this joint experimentally.
The axis of rotation in the present models was assumed to be fixed. However, it is known that the axis not only shifts with frequency but may also shift throughout the cycle of motion. Furthermore, one might expect the axis to be different post-operatively than in the normal ear. The position and orientation of the axis are determined in part by the ligaments, which, with the exception of the annular ligament, were not explicitly modelled in this work. Since the presence of the axis was found to have an effect on the mechanical behaviour of the footplate, it would be useful to model the three-dimensional structure of the ligaments and hence their effects on the position and orientation of the ossicular axis.

In order to extend the frequency range of the models developed here, it is necessary to include damping and inertial effects. It is relatively easy to take into account inertial effects. However, some work may need to be done in order to take into account damping effects. Rayleigh (or proportional) damping is used by the SAP finite-element software (Funnell et al., 1987). One set of damping coefficients can be specified for any one model; these coefficients apply to all of the components making up the structure being modelled. This could be a potential problem since, for a structure such as the middle ear, the damping at the cochlear level is quite different than that at the level of the eardrum.

8.2.5 Clinical Applications

Good models of the normal and surgically corrected human middle ear could potentially guide the design of prostheses. The models developed here could be adapted
to the human ear by modifying the geometry, material properties and mechanical constraints. Such models could, for example, be used to assess the effects of different prosthesis shapes and stiffnesses on the mechanical behaviour of the footplate.

Presently, there is much controversy concerning the use of glues in middle-ear surgery. Glue can be used to produce secure contacts between a prosthesis and the middle-ear tissues. On the other hand, for some prosthesis materials, such as Plastipore, no glue may be required since the contact points are made more or less stable by the proliferation of fibres into the pores of the prosthesis. However, the resulting contacts may be more compliant than those produced by the use of glues. The mechanical advantages and disadvantages associated with rigid or flexible contacts can be evaluated using finite-element models. Mechanical considerations, in conjunction with other criteria such as the biocompatibility of prosthesis materials, may be used to select one material or approach rather than another.

Ultimately, it might be possible to use high-resolution magnetic resonance imaging (MRI) and computed tomography (CT) to measure the anatomy of the middle ear of an individual patient. Such data could be used to develop a finite-element model of the middle ear for that patient. The surgeon could then use this model to predesign and prefabricate a prosthesis for that individual.
APPENDIX A

DISPLACEMENT CONTOURS FOR RIGID-BODY MOTION

A.1 INTRODUCTION

As mentioned in Chapter 7, the displacement-amplitude contours calculated for the eardrum and the footplate are lines of constant vibration amplitude, equally spaced on an amplitude scale. These lines are drawn on the original body. The nature of the contours that result when a body undergoes rigid-body motion will be discussed here.

When rigid-body motion occurs, the distance between any two points on the body remains fixed. This type of motion can consist of either pure translation or pure rotation or a combination of the two. Translation does not produce any displacement contours since all points on the body will have the same displacement amplitude. Rotation will produce contours since not all points will have the same displacement amplitude: points farther away from the axis of rotation will have larger displacement amplitudes than those that are closer to the axis of rotation.

A.2 THE NATURE OF CONTOURS FOR ROTATION

Consider the body shown in Figure A.1. Let \( A(x', y') \) be any point on it. Suppose that the body rotates about the point \( O(x'_o, y'_o) \) through an angle of \( \Delta \theta \) radians so that point \( A \) is mapped into point \( A_1 \); it is assumed here that the angle of rotation is small.
Figure A.1 Rigid-body undergoing pure rotation. The angle of rotation, $\Delta \theta$, has been greatly exaggerated. See text for an explanation of the symbols.
\[ \Delta \theta < 1 \text{ rad.} \quad (A.1) \]

\( r \) and \( r' \) are position vectors from point \( O \) to point \( A \) and from point \( O \) to point \( A' \), respectively; \( \theta \) is the angle between the vector \( r \) and the horizontal (all angles are in radians and are considered to be positive when measured in the counterclockwise direction). \( s \) is the displacement vector from point \( A \) to point \( A' \). (The notation used here is slightly different from that in Chapter 5 since bold letters representing vectors are not enclosed in braces.)

For pure rotation

\[ |r| = |r'|, \quad (A.2) \]

and

\[ \phi = \theta. \quad (A.3) \]

Since the angle of rotation is small, the amplitude of the displacement vector is

\[ |s| = |r| \Delta \theta = |r'| \Delta \theta. \quad (A.4) \]

The \( x' \) component of displacement, \( u \), is

\[ u = -|s| \sin \phi = -|r| \Delta \theta \sin \theta, \quad (A.5) \]

and the \( y' \) component of displacement, \( v \), is

\[ v = |s| \cos \phi = |r| \Delta \theta \cos \theta. \quad (A.6) \]
From Figure A.1, it is apparent that

\[ |r| \cos \theta = x' - x_0', \quad |r| \sin \theta = y' - y_0'. \tag{A.7} \]

Therefore, Equation (A.5) becomes

\[ u = -(y' - y_0') \Delta \theta, \tag{A.8} \]

and Equation (A.6) becomes

\[ v = (x' - x_0') \Delta \theta. \tag{A.9} \]

From Equation (A.8), it is apparent that, for small angles of rotation, the \( x' \) component of displacement \((u')\) will result in evenly spaced horizontal lines with spacing

\[ \Delta y' = \frac{-\Delta u}{\Delta \theta}. \tag{A.10} \]

Similarly, the \( y' \) component of displacement \((v')\) will result in evenly spaced vertical lines with spacing

\[ \Delta x' = \frac{\Delta v}{\Delta \theta}. \tag{A.11} \]

A.3 DESCRIBING RIGID-BODY MOTION

Considering the results of Chapter 7, one might be tempted to think that the in-plane behaviour of the footplate is like that of a rigid body since the contours for the \( x' \) component of displacement consist of evenly spaced horizontal lines and those for the \( y' \) component consist of evenly spaced vertical lines; furthermore, the spacing of the contours for the \( x' \) component is equal to that for the \( y' \) component. To test this
hypothesis, one can estimate a suitable translation and rotation from the results produced by the finite-element method (FEM) and apply these to the nodes of the original body. The nodal displacements computed in this fashion may then be compared to those computed using the FEM alone. (The displacements predicted by the FEM may in general be due to deformation as well as rigid-body motion.) If the results agree closely then it implies that the body undergoes mainly rigid-body motion.

As mentioned above, it is necessary to estimate a suitable translation and rotation from the finite-element results. Figure A.2 shows a plane body undergoing translation followed by rotation. To describe the rotational component of motion, it is first necessary to choose a centre of rotation such as point $O(x_0', y_0')$. This point will have zero displacement due to rotation; its displacement will be due to translation alone. Each point on the body will undergo this translation; the $x'$ and $y'$ components of this translation, denoted $u'$ and $v'$ respectively, can be equated to the $x'$ and $y'$ components of displacement computed by the FEM for this point:

$$u' = u'_0; \quad v' = v'_0.$$  \hfill (A.12)

The primed components are calculated using the FEM. The $x'$ and $y'$ components of displacement due to rotation can be computed for any point $P(x', y')$ on the body by using the formula

$$u' = -(y' - y'_0)\Delta \theta$$  \hfill (A.13)

and
Figure A.2 Rigid-body motion. The triangular body undergoes translation by the amount $u'$ in the $x'$ direction and by the amount $v'$ in the $y'$ direction. It then rotates about the point $O$ by the amount $\Delta \theta$ radians; the angle of rotation has been greatly exaggerated.
\begin{equation}
\nu' = (x' - x_0')\Delta \theta \tag{A.14}
\end{equation}

where \(\Delta \theta\) can be computed from Equation (A.10) or (A.11).

The total components of displacement of point \(P(x', y')\) will then be

\begin{equation}
u = u' + u'' = u_0' - (y' - y_0')\Delta \theta \tag{A.15}\end{equation}

and

\begin{equation}
v = v' + v'' = v_0' + (x' - x_0')\Delta \theta. \tag{A.16}\end{equation}

\(u\) and \(v\) can be computed for all points on the footplate using Equations (A.15) and (A.16). To compare these displacements to those predicted by the FEM, \(u'\) and \(v'\), the ratio \(D/R\) can be computed where

\begin{equation}
D = \frac{1}{n} \sum_{i=1}^{n} \sqrt{(u_i' - u) + (v_i' - v)^2} \tag{A.17}
\end{equation}

and

\begin{equation}
R = \frac{1}{n} \sum_{i=1}^{n} \sqrt{u_i^2 + v_i^2}, \tag{A.18}
\end{equation}

and nodes on the footplate are assumed to be labelled from 1 to \(n\). \(D\) represents the average displacement amplitude due to deformation; \(R\) represents the average displacement amplitude due to rigid-body motion. If the ratio \(D/R\) is very small, then the in-plane behaviour of the footplate can be described as a rigid-body motion.
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