
This thesis deals primarily with matrix norms. The preliminary general treatment is largely confined to non-Archimedean norms, i.e., norms which satisfy the strong triangle inequality. It includes the establishment of a correspondence between a class of such norms and certain submodules of a vector space over a non-Archimedean field as well as a discussion of the properties of bounds (special norms on spaces of linear transformations) and of unit spheres in normed spaces and algebras. Finite dimensional vector spaces over an arbitrary valuated field are considered next. Duality is discussed and a new necessary and sufficient condition for a matrix norm to be a bound established as well as a relation between non-Archimedean matrix norms and the "natural" bound. There follow results concerning "\( \mathfrak{U} \) -unitary" matrix groups, spectral radii, convergence, and methods of successive approximation over the field of p-adic numbers.
NON-ARCHIMEDEAN NORMS AND BOUNDS

by

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A thesis submitted to the Faculty of Graduate Studies and Research of McGill University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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July 1967

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The author wishes to thank Professor H. Schwerdtfeger for suggesting the topic for this thesis and for his helpful advice and direction.
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§ 0. Introduction

Non-Archimedian normed spaces have been studied by I. S. Cohen, I. Fleischer, A. F. Monna, J.-P. Serre, T. A. Springer and others. The concept of K-convexity which leads to the investigation of R-submodules of a vector space over a non-Archimedian valued field is due to A. F. Monna ([18]). This concept is employed by J. van Tiel for the study of certain topological spaces in a dissertation which also contains a summary of previous work ([29]).

The study of matrix norms has a fairly long history but non-Archimedian matrix norms do not appear to have been considered. The association of norms with convex bodies has been systematically exploited by A. S. Householder (see [9]). Ju. I. Ljubič in [13] established a necessary and sufficient condition for a matrix norm to be a bound. The first systematic treatment of the use of norms in convergence proofs of numerical analysis was given by V. N. Faddeeva in [7].

Chapter I of the present thesis contains a general treatment of norms, mainly non-Archimedian. In §1 we prove the existence of a canonical pseudonorm on a quotient module of a pseudonormed module over a valued ring. In §2 with the help of an algebraic definition of boundedness we establish an association between a class of non-Archimedian norms on a vector space $E$ over a non-Archimedian field and certain $R$-submodules of $E$. In §3 we prove that the open unit sphere of a complete non-Archimedian algebra $A$ is an ideal contained in the Jacobson radical of the closed unit sphere of $A$ with resultant effects on convergence and invertibility.

Chapter 2 deals with finite dimensional vector spaces over a valued field $K$ and particularly with $K^n$ and $K_m$. In §4 we show that, although
duality in such a space need not be an involutory relation, the dual norm has useful properties. In particular, we obtain a new proof of Ljubic's theorem. We investigate the properties of the natural norm and bound for a non-Archimedean $K$. We prove an inequality satisfied by non-Archimedean matrix norms and establish a necessary and sufficient condition for such a norm being a natural bound. In § 5 we consider duality for norms determined by $R$-submodules of $K^m$ and prove a new necessary and sufficient condition for a matrix norm over a valued field to be a bound. We also investigate the relation between non-Archimedean bounds and spectral radii and use the concept of an $S$-unitary matrix to establish the non-singularity of a class of matrices over a non-Archimedean field as well as certain properties of their inverses and spectra. § 6 contains estimates for the rate of convergence of certain iterative processes over a non-Archimedean field and examples of the use of methods of successive approximation for the solution of systems of linear equations and for the inversion of matrices over the field of $p$-adic numbers.

We shall use the following notation:

Implication will be denoted by $\Rightarrow$. $\emptyset$ will stand for the empty set. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ will denote the natural numbers, the integers, non-negative integers, the reals, non-negative reals, and the complex numbers respectively. For sets $A$ and $B$, $A \subseteq B$ will mean $x \in A \Rightarrow x \in B$.

All rings and algebras will contain the identity usually denoted by 1 and all modules will be unital. If $R$ is a ring

$\bigoplus R = \{(a_i) \mid a_i \in R\}$ will stand for the restricted external direct sum.
of a number of copies of $R$, when this number is countable, $\oplus R$ will be the $R$-module of sequences in $R$ with a finite number of non-zero terms.

$K^m$ and $K_m$ will denote respectively the space of $m$-dimensional column vectors and the algebra of $m \times m$ matrices over the field $K$. If $x, y \in K^m$, $A \in K_m$ then

$$
\begin{align*}
\mathbf{x} &= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \\
\mathbf{y} &= \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_m \end{pmatrix} \quad \text{and} \quad A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}
\end{align*}
$$

The transpose of a vector $\mathbf{x}$ or a matrix $A$ will be denoted by $\mathbf{x}'$ and $A'$ respectively. Thus $x'y$ will be the scalar product of $x$ and $y$ while $xy'$ will be a matrix of rank $\leq 1$. $I$ will stand for the identity matrix and $e_i$ for the $i$'th column of $I$. 
§ 1. Normed and Pseudonormed Modules.

A valuation on a ring $D$ is a function $a \rightarrow |a|$ from $D$ to $\mathbb{R}^+$ which satisfies the following conditions:

1. $|a| = 0 \iff a = 0$,
2. $|ab| = |a||b|$ for all $a, b \in D$,
3. $|a + b| < |a| + |b|$ for all $a, b \in D$.

A valuation defines a topology on $D$ under which $D$ is a topological ring ([11], p.329). The couple $(D, | |)$ is called a valued ring. Since $\mathbb{R}$ is a field, a valued ring cannot have any divisors of zero (ibid, p.294).

A valuation is said to be non-Archimedean if, instead of the triangle inequality (3), it satisfies the stronger condition:

$$|a + b| < \max (|a|, |b|)$$

for all $a, b \in D$.

If $D$ is a non-Archimedean valued ring, then

$$R = \{a \in D \mid |a| \leq 1\}$$

is a subring of $D$ called the valuation ring of $D$. The set

$$P = \{a \in D \mid |a| < 1\}$$

is a prime ideal of $R$ called the valuation ideal.

If $D$ is a valued ring and $M$ a torsion-free $D$-module, a norm may be defined on $D$ entirely analogous to a norm on a vector space over a valued field (see [11], p.65). The resultant definition of a module norm must be distinguished from the one used by Monna and Springer (see [22], p.613).
Definition

Let $D$ be a valuated ring and $M$ a torsion-free $D$-module. A norm on $M$ is a function $x \mapsto \|x\|$ from $M$ to $R^+$ which satisfies the following conditions:

1. $\|x\| = 0 \iff x = 0$,
2. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in D$, $x \in M$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in M$.

The couple $(M, \|\|)$ is a normed module which is said so be non-Archimedian if $\|\|$ satisfies the strong triangle inequality:

$$\|x + y\| \leq \max(\|x\|, \|y\|)$$ for all $x, y \in M$.

A valuated ring $D$ may be regarded as a normed $D$-module. If $M$ is a normed $D$-module and $C$ is a subring of $D$ then every submodule of $M$ is a normed $C$-module ([2], p.49) under the norm defined on $M$.

Since for $\lambda, \mu \in D$, $0 \neq x \in M$, $|\lambda| < |\mu| \iff \|\lambda x\| < \|\mu x\|$, a non-Archimedian norm on $M$ implies a non-Archimedian valuation on $D$ (cf [15], p.1046), but the converse statement is false. In this connection Monna considered the spaces of sequences $(\lambda_1)$, $\lambda \in K$, $K$ a non-Archimedian valuated field, for which the series $\sum_{i=1}^{\infty} |\lambda_i|^p$ is convergent (see [16], p.480). We prove a general result:

Theorem 1.1

If $D$ is a valuated ring and $M$ a free $D$-module, then each monotonic norm on $@R$ determines a norm on $M$.

Proof:

Note first that $M$ is torsion-free ([12], p.134).

Establish a partial order on $@R$ by setting $x < y$ if $\lambda_i < x_i$ for all $i$, where $x = (\lambda_1, \ldots, \lambda_n)$, $y = (\gamma_1, \ldots, \gamma_n) \in @R$. 
A norm on \( \mathbb{R} \) is \underline{monotone} if \( |x| < |y| \Rightarrow \|x\| < \|y\| \),
where \( |x| = (|\frac{x}{1}|) \), \( |\frac{x}{1}| \) denoting the ordinary absolute value on \( \mathbb{R} \).

If \( \{u_i\} \) is a basis for \( M \) define a map
\[
M \rightarrow \mathbb{R}
\]
by \( x = \sum \lambda_i u_i \rightarrow |x| = (|\lambda_i|), \ \lambda_i \in D \).

We have \( |x| = 0 \iff x = 0 \),
\( |\lambda x| = |\lambda| |x| \) for all \( \lambda \in D, x \in M \),
\( |x + y| \leq |x| + |y| \) for all \( x, y \in M \).

Define \( \|x\| = \| |x| \| \).

Then \( \| \| \) is a norm on \( M \).

**Corollary 1.**

If \( D \) is a valued ring, then for \( 1 \leq p < \infty \)
\[
\| (\lambda_i) \|_p = \left( \sum |\lambda_i|^p \right)^{\frac{1}{p}}
\]
is a norm on \( \mathbb{R} \), which may be called a \underline{Hölder norm}.

**Corollary 2.**

A Hölder norm on a free \( D \)-module whose dimension is greater
than \( 1 \) is non-Archimedean if and only if the valuation on \( D \) is non-Archimedean and \( p = \infty \).

**Remarks**

1. More generally, if \( M \) is a free \( D \)-module with basis \( \{u_i\} \) then
\[
\| \sum \lambda_i u_i \| = \max |\lambda_i| \|u_i\|, \text{ where } \{\|u_i\|\} \text{ is an arbitrary set of positive reals, is a norm on } M, \text{ which is non-Archimedean if and only if the valuation on } D \text{ is non-Archimedean.}
\]

2. The proof of theorem 1.1 will apply to the subspaces of \( K^\infty \) discussed by Monna.
3. If \( \mathbb{R} \) is ordered lexicographically, the function \( x \mapsto |x| \) becomes a non-real-valued norm on the free \( D \)-module \( M \). This norm will be non-Archimedean if the valuation on \( D \) is non-Archimedean.

The requirement that a normed module be torsion-free ceases to be necessary if condition (2) in the definition of a norm is replaced by a suitable weaker condition. In particular, we have the following (cf [22], p.613):

**Definition**

Let \( D \) be a valuated ring and \( M \) an arbitrary \( D \)-module. A non-Archimedean pseudonorm on \( D \) is a function \( x \mapsto \|x\| \) from \( D \) to \( \mathbb{R}^+ \) which satisfies the following conditions:

1. \( \|x\| = 0 \iff x = 0 \),
2. \( \|\lambda x\| \leq |\lambda| \|x\| \) for all \( \lambda \in D \), \( x \in M \),
3. \( \|x + y\| \leq \max (\|x\|, \|y\|) \) for all \( x, y \in M \).

We observe that a non-Archimedean pseudonorm on \( M \) does not imply a non-Archimedean valuation on \( D \).

A pseudonorm on a vector space \( E \) over a field with a non-trivial valuation (see § 2) is Köthe's "g - norm" (see [10], p.167) under which \( E \) is a topological vector space (cf [20], p.357). In the general case, a non-Archimedean pseudonorm on \( M \) defines a non-Archimedean metric \( d(x, y) = \|x - y\| \) under which \( M \) is at least a topological group.

Certain properties of non-Archimedean valuated fields and normed vector spaces are the direct consequences of the strong triangle inequality and the resulting non-Archimedean metric. These will hold for non-Archimedean pseudonormed \( D \)-modules (see [24], p.76; [20], p.353). In the list below we assume in item 7 that the valuation on \( D \) is non-Archimedean.
1. It follows from the strong triangle inequality that \( x, y \in M \)
with \( ||x|| > ||y|| \Rightarrow ||x \pm y|| = ||x|| \) ([24], p.73; [20], p.353).

2. Hence, \( x_n \rightarrow x \neq 0 \Rightarrow \exists n_0 \in \mathbb{N} \) such that \( ||x_n|| = ||x|| \) for \( n > n_0 \) ([15], p.1045; [20], p.353).

3. If \( \sum_{n=1}^{\infty} x_n \) converges, then \( \sum_{n=1}^{\infty} ||x_n|| < \max_{n} ||x_n|| \) ([8], p.165).

4. For every \( M \) there exists a completion \( \hat{M} \) ([14], p.68) such that \( \hat{W}_M = W \), where \( W_M = \{ ||x|| \mid 0 \neq x \in M \} \) and \( \hat{W} \) has a similar meaning for the extension of \( || \) to \( \hat{M} \) ([14], p.61).

5. If \( M \) is complete, then the series \( \sum_{n=1}^{\infty} x_n \) converges if and only if \( \lim_{n \to \infty} x_n = 0 \) ([24], p.75).

6. For \( a \in M \) and real \( \rho > 0 \) the sets \( \{ x \in M \mid ||x - a|| < \rho \} \), \( \{ x \in M \mid ||x - a|| < \rho \} \) are called respectively the closed and open spheres with center \( a \) and radius \( \rho \). However, all spheres are both open and closed in the topology determined by the pseudonorm. Further, each point of a sphere is its center and two spheres are either disjoint or one is contained in the other ([24], p.74).

7. If \( R \) is the valuation ring of \( D \), a sphere \( S \subset M \) is an \( R \)-sub-module of \( M \) if and only if \( 0 \in S \) (cf [18], p.532).

**Remark**

The \( R \)-submodules of a vector space \( E \) over a non-Archimedean valued field \( K \) are precisely those non-empty sets in \( E \) which Monna described as having property \( C \) and von Tiel calls \( K \)-convex (see [29], p.253).
The "K-convex null of a set \( S \subset E \) (p.254) is simply the \( \mathbb{R} \)-submodule of \( E \) generated by \( S \). Monna called K-convex sets of the form \( x_0 + S \), where \( S \) has property C and \( x_0 \) is a fixed vector in \( E \) ([18], p.532).

If \( N \) is a closed subspace of a vector space \( E \) over the real or complex field it is known that an \( F \)-norm of \( E \) determines an \( F \)-norm on the quotient space \( E/N \) (see [10], p.167). We prove an analogous result for non-Archimedean pseudonormed modules.

Theorem 1.2

Let \( D \) be a valuated ring, \( M \) a non-Archimedean pseudonormed \( D \)-module, and \( N \) a closed submodule of \( M \). Then

\[
\| x + N \| = \inf_{n \in N} \| x + n \|
\]

defines a non-Archimedean pseudonorm on the quotient module \( M/N \). If \( M \) is complete so is \( M/N \).

Proof:

We verify the three properties of a non-Archimedean pseudonorm.

(1) follows from the fact that \( N \) is closed (see [27], p.213).

(2') We note that for all \( \lambda \in D \) \( \lambda N \subset N \). Hence

\[
\| \lambda(x + N) \| = \| \lambda x + N \| = \inf_{n \in N} \| \lambda x + n \| < \inf_{n \in N} \| \lambda x + \lambda n \| = \| \lambda \| \| x + N \|
\]

(3') Let \( \| x + N \| \geq \| y + N \| \).

Then \( \forall n \in N \exists n' \in N \) such that \( \| x + n \| > \| y + n' \| \).

Hence, \( \| x + n + y + n' \| < \| x + n \| \) so that

\[
\|(x + N) + (y + N)\| = \| x + y + N \| = \inf_{n' \in N} \| x + y + n' \| = \inf_{n \in N} \| x + n + y + n' \| < \inf_{n \in N} \| x + n \| = \| x + N \|.
\]

For proof of the last assertion see [27].
Corollary 1.

If $M$ is a non-Archimedean pseudonormed $D$-module, $R$ the valuation ring of $D$, and $S$ a sphere in $M$, $0 \subset S$, then

$$\|S\| = 0, \quad \|x + S\| = \|x\|, \quad x \not\in S,$$

is a non-Archimedean pseudonorm on the $R$-module $M/S$.

Corollary 2.

If $E$ is a non-Archimedean normed vector space over a valuated field $K$ and $N$ is a closed subspace of $E$, then

$$\|x + N\| = \inf_{n \in N} \|x + n\|$$

defines a non-Archimedean norm on the quotient space $E/N$.

Remarks

1. An Archimedean pseudonorm on $M$ also induces a pseudonorm on $M/N$, which may be non-Archimedean.

2. Corollary 2 is false for normed modules.

Example

Consider $\mathbb{Z}$ as a normed $\mathbb{Z}$-module under ordinary absolute value or diadic valuation.

$2\mathbb{Z}$ is a closed submodule of $\mathbb{Z}$. In fact, in the latter case $2\mathbb{Z}$ is a sphere in $\mathbb{Z}$ centered at the origin. In either case the induced pseudonorm on $\mathbb{Z}/2\mathbb{Z}$ is given by

$$\|0\| = 0, \quad \|1\| = 1.$$ 

$\|\|$ is a non-Archimedean valuation on the field $K = \mathbb{Z}/2\mathbb{Z}$ and hence a non-Archimedean norm on the vector space $K$ over $K$. But $\|\|$ is not a norm on the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$, for

$$\|2 \cdot \bar{1}\| = \|0\| = 0 < |2| \|\bar{1}\| \neq 0.$$
§ 2. Non-Archimedean Norms and \( R \)-submodules of a Vector Space.

Let \( K \) be a non-Archimedean valuated field. Then \( K \) is the field of quotients of its valuation ring \( R = \{ \xi \in K \mid |\xi| < 1 \} \), while the valuation ideal \( P = \{ \xi \in K \mid |\xi| < 1 \} \) is a unique maximal ideal of \( R \). The set

\[
U = \{ \xi \in K \mid |\xi| = 1 \}
\]

is the group of units of \( R \), usually called the group of units of \( K \), and \( \bar{K} = R/P \) is a field, called the residue field of \( K \) ([8], p.81).

The set

\[
W_K = \{ |\xi| \mid 0 \neq \xi \in K \}
\]

is a multiplicative group, called the value group of \( K \). Moreover, for real \( e > 1 \) the map \( |\xi| \to -\log_e |\xi| \) is an isomorphism of \( W_K \) onto a subgroup of \( (\mathbb{R}^+, \cdot) \). Accordingly, the valuation on \( K \) is called trivial if \( W_K = \{ 1 \} \), dense if \( W_K \) is everywhere dense in \( \mathbb{R}^+ \), and discrete in the remaining case, i.e., if \( W_K \) is discrete in \( \mathbb{R}^+ - \{ 0 \} \) or, equivalently, if 0 is the only limit point of \( W_K \). The valuation on \( K \) is discrete if and only if there exists \( \pi \in K \) such that \( |\pi| = \max_{\xi \in P} |\xi| \) ([25], p.9).

Then \( W_K \) is the infinite cyclic group \( \{ |\pi|^i \mid i \in \mathbb{Z} \} \). In fact, if \( 0 \neq \xi \in K \), then \( \xi = \eta |\pi|^i \), \( \eta \in U \), \( i \in \mathbb{Z} \), so that \( \pi \) is a prime element of \( R \) ([14], p.107).

The topology of \( K \) is discrete if and only if its valuation is trivial ([14], p.62). For a field with a non-trivial valuation to be locally compact it is necessary and sufficient that it be complete with respect to a discrete valuation and have a finite residue field ([29,p.251]).
Let $E$ be a non-Archimedean normed vector space over $K$.

The **unit sphere** in $E$ is the $R$-module

$$S = \{x \in E \mid \|x\| < 1\}.$$  

$$W_E = \{\|x\| \mid 0 \neq x \in E\}$$

will be called the **value set** of the norm on $E$.

It may happen that $W_E \subset W_K$ and even $W_E = W_K$. This is the case, for example, with the norm $\| \|_\infty$ on $\Theta K$ defined by

$$\|\langle \xi_1 \rangle\|_\infty = \max_1 |\xi_1|, \ (\xi_1) \in \Theta K.$$  

If $W_E \subset W_K$ then $\forall x \in E \ x = \xi x_0, \ \xi \in K, \ |x_0| = 1$, so that the norm is completely determined by the set $\{x \in E \mid \|x\| = 1\}$ (cf [15], p.1045).

On the other hand, every non-Archimedean normed space contains a set of non-zero vectors $\{x_1\}$ such that $W_K \|x_i\| \cap W_K \|x_j\| = \emptyset$ for $i \neq j$. The strong triangle inequality gives

$$\|\sum_{i=1}^{n} \frac{1}{i} x_i\| = \max_{1 \leq i \leq n} \|\frac{1}{i} x_i\|, \ \frac{1}{i} \in K,$$

so that $\{x_1\}$ is an orthogonal family of vectors (see [22], p.603). It is clear that every orthogonal family is linearly independent. If such a family forms a basis for $E$, then $W_E = \bigcup W_K \|x_i\|$. In particular, if the valuation on $K$ is trivial, then $\|u_i\| \neq \|u_j\|$ for $i \neq j$ on a basis $\{u_i\}$ for $E$ implies that

$$\|\sum_{i=1}^{n} \xi_i u_i\| = \max_{1 \leq i \leq n} \|u_i\|, \ \xi_i \in K.$$  

$W_E$ does not possess properties comparable to the trichotomy for $W_K$. We shall say that a norm on $E$ is **dense** or **discrete** if $W_E$ is respectively dense in $\mathbb{R}^+$ or discrete in $\mathbb{R}^+ \setminus \{0\}$ (Monna and Springer...
in [22], p.606, adopt a more general definition of a discrete norm). A dense valuation on $K$ implies a dense norm on $E$, so that a discrete norm on $E$ implies a discrete valuation on $K$ whenever it is known that the latter is non-trivial ([15], p.1049). The converse statements hold if $E$ is finite dimensional.

When $E$ is finite dimensional, $E$ is complete with respect to its norm if $K$ is complete with respect to its valuation ([1], p.98) but this implication is not true in general. Nor does completeness of $E$ imply completeness of $K$. However, when $E$ is complete the scalar multiplication in $E$ over $K$ may be extended to scalar multiplication in the normed space $E$ over $\hat{K}$, where $\hat{K}$ is the completion of $K$ ([15], p.1054). If both $K$ and $E$ are complete, then $E$ is known as a non-Archimedean Banach space.

When the valuation on $K$ is non-trivial, the unit sphere in $E$ is compact or, equivalently, $E$ is locally compact if and only if $K$ is locally compact and $E$ is finite dimensional ([15], pp.1048-1053; [5], p.695), implying a complete and discrete valuation on $K$.

Let $E$ be a vector space over a field $K$ with a non-trivial non-Archimedean valuation. Monna ([18]) investigated the relationship between $K$-convex subsets of $E$ and non-Archimedean seminorms on $E$, i.e., real valued functions $p$ such that $p(\lambda x) = |\lambda|p(x)$ and $p(x + y) \leq \max(p(x), p(y))$ for all $x, y \in E$, $\lambda \in K$. For this purpose he chose a definition of a $K$-convex body in $E$ independent of the topology on $E$ (see [20], p.357). He also proved Kolmogorov's Criterion for non-Archimedean normed spaces (ibid, p.360). Here, however, he adopted
a topological definition of boundedness (p.353). We shall establish
a relation between non-Archimedean norms and $R$-submodules of $E$ using
algebraic concepts only.

For the remainder of this section (with the exception of
the last two paragraphs) $K$ will denote a field with a non-trivial non­
Archimedean valuation and $E$ a vector space over $K$ which may or may not
be normed. We shall need the following result of Monna's:

Lemma 2.1

Let $S$ be an $R$-submodule of $E$. Then for $\lambda, \beta \in K$ with
$|\lambda| > |\beta|$ $\lambda S$ is an $R$-submodule of $E$ and $\lambda S = \lambda S + \beta S$. Conversely,
$0 \in S$ and $\lambda S = \lambda S + \beta S$ for $\lambda, \beta \in K$ with $|\lambda| > |\beta|$ imply that
$S$ is an $R$-submodule of $E$.

Proof:

See [20], p. 354.

Definition

A set $S \subset E$ is said to be equilibrated if $\lambda x \in S$ for
all $x \in S$, $\lambda \in R$, i.e., if $\lambda \in R \Rightarrow \lambda S \subset S$.

Lemma 2.2

$S \subset E$ is equilibrated if and only if for $\lambda, \beta \in K$
with $|\lambda| > |\beta|$ $(\lambda + \beta)S \subset \lambda S$.

Proof:

Let $S$ be equilibrated and $x \in S$.

Then $(\lambda + \beta)x = \lambda(1 + \lambda^{-1}\beta)x$ with $|1 + \lambda^{-1}\beta| \leq \max (1, |\lambda^{-1}\beta|) = 1$. 
Hence \((1 + \lambda^{-1})x \in S\) and inclusion follows.

Conversely, let \(\lambda \in \mathbb{R}\).

Put \(\mu = \lambda - 1\) so that \(|\mu| < 1\).

Therefore \(\lambda S = (1 + \mu)S \subset 1.S = S\).

**Definition**

A set \(S \subset E\) is **absorbant** if for each \(x \in E\) there exists a real \(\epsilon > 0\) such that \(x \in \lambda S\) for all \(\lambda \in K\) with \(|\lambda| > \epsilon\).

**Lemma 2.3**

If \(S\) is an \(R\)-submodule of \(E\), then the following statements are equivalent:

(a) \(S\) is absorbant.

(b) For each \(x \in E\) there exists \(0 \neq \mu \in K\) such that \(x \in \mu S\).

(c) \(S\) contains a basis for \(E\).

**Proof:**

(a) \(\Leftrightarrow\) (b). See [3], p.6, remembering that \(S\) is an equilibrated set.

(a) \(\Rightarrow\) (c) because an absorbant set in \(E\) generates \(E\) (ibid., p.7).

(c) \(\Rightarrow\) (b). Let \(\{u_i\} \subset S\) be a basis for \(E\), \(x = \sum_{i=1}^{n} \lambda_i u_i \in E\), and

\[
|\lambda_k| = \max_{1 \leq i \leq n} |\lambda_i|.
\]

If \(|\lambda_k| = 0\), then \(\lambda_i = 0\) for all \(i\), and \(x \in S = 1.S\).

If \(|\lambda_k| \neq 0\), then \(\lambda_k \neq 0\) and \(x = \lambda_k \left( \sum_{i=1}^{n} \lambda_i^{-1} u_i \right)\). We have

\[
|\lambda_i^{-1} \lambda_i| < 1
\]

for all \(i\), \(\sum_{i=1}^{n} \lambda_i^{-1} \lambda_i u_i \in S\), and \(x \in \lambda_i S\).

**Corollary**

An \(R\)-submodule of \(E\) which contains an absorbant submodule is itself absorbant.

**Definition**

We shall say that a set \(S \subset E\) is **bounded** if for each \(0 \neq x \in E\)
there exists a real $\varepsilon > 0$ such that $\lambda x \notin S$ for all $\lambda \in K$ with $|\lambda| > \varepsilon$.

Clearly every subset of a bounded set is bounded. Further, boundedness with respect to any norm on $E$ (possibly Archimedean) implies boundedness in the above sense (but not conversely; see example on p.14).

**Lemma 2.4**

Let $E$ be a non-Archimedean normed space over $K$ and $S$ a sphere in $E$ centered at the origin. Then $S$ is an absorbant and bounded $R$-submodule of $E$.

**Proof:**

Clear.

**Corollary**

Every $R$-submodule of $E$ which is contained in a sphere and contains a sphere centered at the origin is absorbant and bounded.

**Remark**

When a normed space $E$ is infinite dimensional an absorbant and bounded $R$-submodule of $E$ need not be contained in a sphere, i.e., it may be unbounded with respect to the norm on $E$. Nor does it have to contain a sphere. In fact, its interior may be empty.

**Example** (cf [29], p.254).

Let $E = K$ be the space of sequences in $K$ with a finite number of non-zero terms. If the valuation on $K$ is dense choose $\rho \in K$ such that $|\rho| > 1$; if the valuation is discrete put $\rho = n^{-1}$. Then

$$
S_1 = \left\{ (\lambda_i) \in E \mid |\lambda_i| < |\rho|^{\frac{1}{i}}, \ i \in \mathbb{N} \right\}
$$

and

$$
S_2 = \left\{ (\lambda_i) \in E \mid |\lambda_i| < |\rho|^{-\frac{1}{i}}, \ i \in \mathbb{N} \right\}
$$

are absorbant and bounded $R$-submodules of $E$. But $S_1$ is unbounded and $S_2$ has
an empty interior with respect to the topology defined on $E$ by the non-Archimedian norm $\|(\lambda_i)\|_\infty = \max_i |\lambda_i|$. 

**Theorem 2.1**

Let $E$ be a vector space over $K$ and $S$ an absorbant and bounded $K$-submodule of $E$. Define $\| \|_S$ on $E$ by $\|x\|_S = \inf_{x \in \lambda \in S} |\lambda|$ for all $x \in E$. Then:

(a) $\| \|_S$ is a non-Archimedian norm on $E$.

(b) $\{x \in E \mid \|x\|_S < 1\} \subset S \subset \{x \in E \mid \|x\|_S < 1\}$.

(c) If the valuation on $K$ is discrete, then $W_E \subset W_K$ and $S = \{x \in E \mid \|x\|_S < 1\}$.

**Proof:**

(a) The first norm property (§1, p. 2) is an immediate consequence of the boundedness of $S$. The second property is evident. It remains to prove the strong triangle inequality.

Let $x, y \in E$ with $\|x\|_S \geq \|y\|_S$.

Then for all $\lambda \in K$ with $x \in \lambda S$ there exists $\mu \in K$ with $y \in \mu S$ and $|\lambda| \geq |\mu|$.

Now, $x + y \in \lambda S + \mu S = \lambda S$ by lemma 2.1.

So $\|x + y\|_S \leq \|x\|_S = \max (\|x\|_S, \|y\|_S)$ (cf [18], p 534).

(b) Since $\| \|_S$ is a seminorm, see [18], p 535.

(c) The first result follows from the properties of $W_K$, for the second, see [29], p 256.

**Remark**

An Archimedian norm on $E$ cannot be defined in the above manner by any subset of $E$. For, if $S$ is to define a norm on $E$, we should have at least:

(i) $S$ is equilibrated.

(ii) $(\lambda + \beta)S = \lambda S + \beta S$ for all $\lambda, \beta \in K$. 
15.

The first condition gives for $|\alpha| > |\beta|

$$(\alpha + \beta)S \subset \alpha S \subset \alpha S + \beta S,$$

while the second forces equality. Hence $S$ is an $R$-module in $E$ so that any norm defined by $S$ will be non-Archimedean.

Every non-Archimedean norm on a vector space $E$ over $K$ determines a unit sphere which is an absorbant and bounded $R$-submodule of $E$. Conversely, every absorbant and bounded $R$-submodule $S$ of $E$ defines a non-Archimedean norm $\| \|$ on $E$. Further, if the valuation on $K$ is discrete, then $S$ is the unit sphere for $\| \|_S$. It does not follow, however, that if $S$ is the unit sphere for a non-Archimedean norm $\| \|$ on $E$ then $\|x\|_S = \|x\|$ for all $x \in E$. In fact, we already know that this cannot happen in the discrete case unless the value set of the norm on $E$ is contained in the value group of $K$. The actual state of affairs resembles closely the situation described by Monna for seminorms (see [18], p.53).

**Lemma 2.5**

Let $\| \|$ be a non-Archimedean norm on $E$, $S$ the corresponding unit sphere, and $\| \|_S$ the norm on $E$ determined by $S$. Then:

(a) If the valuation of $K$ is dense, $\|x\| = \|x\|_S \forall x \in E$.

(b) If the valuation of $K$ is discrete, $\|n\| \|x\|_S < \|x\| < \|x\|_S, x \in E$.

(c) In the discrete case $\|x\| = \|x\|_S \forall x \in E$ if and only if the value set of $\| \|$ is contained in the value group of $K$.

**Proof:**

We have for $x \in E, \lambda \in K$

$$\|x\| < |\lambda| \iff x \in \lambda S \text{ so that } \|x\| < \inf_{x \in \lambda S} |\lambda| = \|x\|_S.$$
If the valuation of $K$ is dense,

$$\|x\| < \|x\|_S \Rightarrow \exists \lambda \in K \text{ such that } \|x\| < |\lambda| < \inf_{x \in \lambda S} |\lambda|.$$  

Then $x \not\in \lambda S$ and $x \not\in \lambda S\setminus$. Hence (a).

If the valuation of $K$ is discrete,

$$\inf_{x \in \lambda S} |\lambda| = |\pi|^n \text{ for some } n \in \mathbb{Z}.$$ 

If $\|x\| < |\pi|^{n+1}$, then $x \in \pi^{n+1}S$ with $|\pi^{n+1}| < \inf_{x \in \lambda S} |\lambda|$, which is a contradiction. Thus

$$|\pi|^{n+1} < \|x\| < |\pi|^n$$

and both (b) and (c) follow immediately.

Remark

Instead of the unit sphere the open sphere $S_o = \{x \in E \mid \|x\| < 1\}$ may be used to establish the norm $\|\|_S$ on $E$ (cf [18], p.530). We then have:

(a') $\|x\| = \|x\|_S \forall x \in E$, if the valuation of $K$ is dense.

(b') $|\pi| \|x\|_{S_o} \leq \|x\| \leq |\pi| \|x\|_{S_o}$, if the valuation of $K$ is discrete.

(c') In the latter case, $W_x \subseteq W_K$ $\Rightarrow$ $|\pi| \|x\|_{S_o} = \|x\|$.

We recall that two norms $\|\|_1$ and $\|\|_2$ on a vector space $E$ are said to be equivalent if there exist two positive reals $a$ and $b$ such that

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \forall x \in E.$$  

Lemma 2.6

Let $\|\|_1$ and $\|\|_2$ be a norm on $E$ determined by the absorbant and bounded $R$-submodules $S_1$ and $S_2$ of $E$ such that $S_1 \subset \subset S_2$, $\alpha \in K$. Then

$$\|x\|_2 \leq |\alpha| \|x\|_1 \forall x \in E.$$  

Proof:

We have

\[ x \in \lambda S_1 \Rightarrow x \in \lambda \gamma S_2, \quad \lambda \in K. \]

Hence

\[ \|x\|_2 = \inf_{x \in S_2} |x| = \inf_{x \in \lambda S_2} |\lambda x| = |\lambda| \inf_{x \in S_1} |x| = |\lambda| \|x\|_1. \]

Corollary

\[ S_1 \subseteq \lambda S_2 \subseteq \beta S_1, \quad \beta \in K \Rightarrow \| \|_1 \text{ and } \| \|_2 \text{ are equivalent.} \]

Theorem 2.2

Let \( \| \| \) be a non-Archimedian norm on \( E \) and \( M \) an \( R \)-submodule of \( E \) such that \( \{ x \in E \mid \|x\| < 1 \} \subseteq M \subseteq \{ x \in E \mid \|x\| < 1 \} \). Then \( M \) determines a non-Archimedian norm on \( E \) equivalent to \( \| \| \). In fact, if the valuation on \( K \) is dense, \( \|x\|_M = \|x\| \) for all \( x \in E \); if the valuation of \( K \) is discrete, then \( \forall x \in E \mid n\|x\|_M < \|x\| < \|x\|_M \).

Proof:

The given inclusion relation implies that \( M \) is absorbant and bounded so that \( \| \|_M \) is a non-Archimedian norm on \( E \).

Thus the equality in the dense case and the inequality in the discrete case follows from Monna's results (see [18], p 537).

Remarks

1. Equivalence in both cases may also be proved as follows:

With the notation of lemma 2.5 and the remark which follows it, we have \( nS \subseteq S_0, \quad |n| < 1 \), in the case of discrete valuation and \( nS = S_0 \) when the valuation on \( K \) is discrete. Thus, the given inclusion and the corollary to lemma 2.6 imply that \( \| \|_M \) is equivalent to \( \| \|_S \) which is equivalent to \( \| \| \) by lemma 2.5.

2. The equivalence of \( \| \| \) and \( \| \|_S \) in the case of discrete valuation
(as well as the equality of $S$ and the unit sphere of $\| \cdot \|_S$) has been mentioned by Fleischer ([8], p 168). Serre's equivalent norm for the same case defined by $\| x \|' = \inf \{ |\lambda| \in K \mid \| x \| < |\lambda| \}$ ([26], p 69) is our $\| \cdot \|_S$.

We conclude this section with some remarks concerning the case of trivial valuation on the field $K$. Obviously the definitions of absorbant and bounded sets may be extended to a vector space $E$ over such a field. However, since now the $R$-submodules of $E$ are precisely the subspaces of $E$, the only absorbant $R$-submodule of $E$ is $E$ itself. Since $E$ is bounded it defines a non-Archimedian norm $\| \cdot \|_E$ whose value set is clearly $\{1\}$. Conversely, if $E$ is a normed space with $W_E = \{1\}$, then $E$ is its own unit sphere and $\| x \| = \| x \|_E$ for all $x \in E$.

With the help of a concept of Serre's (ibid.) these observations may be combined with previous results (lemma 2.5) to obtain the following theorem:

**Theorem 2.3**

If $E$ is a non-Archimedian normed space over a valued field $K$ and $S$ is the unit sphere in $E$, then $\| x \| = \| x \|_S$ for all $x \in E$ if and only if $W_E \subseteq \overline{W}_K$, where $\overline{W}_K$ is the closure of $W_K$.

§3. **Linear Transformations and Normed Algebras.**

Let $K$ be a non-Archimedian valued field, $E_1$ and $E_2$ vector spaces over $K$, and $T$ a linear transformation from $E_1$ to $E_2$. Since a linear transformation is an $R$-module homomorphism, the image $T(S_1)$ of an $R$-module $S_1 \subseteq E_1$ and the inverse image of an $R$-module $S_2 \subseteq E_2$ are also $R$-modules (cf [18], p 533; [20], p 355).
19.

Let norms $\| \|_1$ and $\| \|_2$ be defined on $E_1$ and $E_2$ respectively. $T$ is said to be bounded if there exists a real $c > 0$ such that $\| T(x) \|_2 \leq c \| x \|_1$ for every $x \in E_1$. The following theorem is an exact counterpart of a well known result for normed spaces over $\mathbb{R}$ or $\mathbb{C}$ (see [27], p.219):

**Theorem 3.1**

Let $E_1$ be a non-Archimedean normed space over a field $K$ with a non-trivial valuation. Let $E_2$ be an arbitrary normed space over $K$, $T$ a linear transformation from $E_1$ to $E_2$, and $S$ the unit sphere in $E_1$. Then the following statements are equivalent:

(a) $T$ is bounded.

(b) $T$ is continuous.

(c) $T(S)$ is bounded with respect to the norm on $E_2$.

**Proof:**

(a) $\Rightarrow$ (b) and (c).

Clear.

(b) $\Rightarrow$ (c).

Since the valuation on $K$ is non-trivial, there exists $\rho \in K$ with $|\rho| > 1$.

If $T(S)$ is unbounded, then for each $n \in \mathbb{N}$ there exists $x_n \in S$ with $\| T(x_n) \|_2 > |\rho|^n$.

Let $y_n = \rho^{-1} x_n$. Then $\| y_n \|_1 = \| x_n \|_1 \| |\rho|^n$ $< 1 = |\rho|^n$, so that $y_n \to 0$ as $n \to \infty$.

But $\| T(y_n) \|_2 = \frac{\| T(x_n) \|_2}{|\rho|^n} > 1$ for all $n \in \mathbb{N}$.

Thus $T(y_n) /\to 0$, contradicting the continuity of $T$.

(c) $\Rightarrow$ (a).
Suppose \( T(S) \) is contained in a closed sphere centered at the origin with radius \( k \).

Clearly the result holds if \( x = 0 \).

For \( 0 \neq x \in E, x \in \lambda S, \lambda \in K \Rightarrow \|x\|_1 < |\lambda|, \) so that \( \lambda^{-1}x \in S \).

Hence, \( \frac{\|T(x)\|_2}{|\lambda|} < k \) and \( \|T(x)\|_2 < k |\lambda| \).

Therefore, \( \|T(x)\|_2 < k \inf_{x \in \lambda S} |\lambda| = k \|x\|_S \).

The required result follows from the equivalence of \( \|x\|_1 \) and \( \|x\|_S \).

In fact, \( \|T(x)\|_2 < c \|x\|_1 \) for all \( x \in E \) where \( c = k \) if \( W_{E_1} \subset \overline{W}_K \) and \( c < \frac{k}{|\pi|} \) in the remaining case.

**Remark**

The equivalence of (a) and (b) was proved by Monna without the assumption that \( \|\| \) is non-Archimedian ([15], pp 1134-1135). The equivalence of (b) and (c) is essentially a special case of a theorem proved by van Tiel for certain classes of \( K \)-convex topological spaces (see [29], p.269).

**Let** \( \mathcal{B}(E_1, E_2) \) be the space of bounded linear transformations from \( E_1 \) to \( E_2 \), where for the moment we permit the valuation on \( K \) to be trivial. Then

\[
\|T\|_{1,2} = \sup_{0 \neq x \in E_1} \frac{\|T(x)\|_2}{\|x\|_1} = \inf_{x \in E_1} \{ c \mid \|T(x)\|_2 < c \|x\|_1 \}, \quad T \in \mathcal{B},
\]

defines a norm on \( \mathcal{B}(E_1, E_2) \), called the bound of \( T \), such that

\[
\|T(x)\|_2 < \|T\|_{1,2} \|x\|_1 \quad \text{for all} \quad x \in E_1.
\]

If \( W_{E_1} \subset \overline{W}_K \), then for all \( T \in \mathcal{B} \)

\[
\|T\|_{1,2} = \sup_{\|x\|_1 = 1} \|T(x)\|_2.
\]
Another norm on $\mathcal{B}$ is given by
\[
\|T\|_S = \sup_{x \in S} \|T(x)\|_2 = \inf_{x \in S} \{k \mid \|T(x)\|_2 < k\}, \quad T \in \mathcal{B}
\]
We have $\|T\|_S < \|T\|_1,2$ for all $T \in \mathcal{B}$.

If the valuation on $K$ is non-trivial, then $\mathcal{B}(E_1, E_2)$ is the space of continuous linear transformations from $E_1$ to $E_2$ and the two norms on $\mathcal{B}$ are equivalent. In fact,
\[
W_{E_1} < W_K \Rightarrow \|T\|_S = \|T\|_1,2 \text{ for all } T \in \mathcal{B} \text{ (cf [26], p.71), so that} \]
\[
\|T(x)\|_2 < \|T\|_S \|x\|_1 \text{ for all } x \in E_1 \text{ (cf [21], p.124).}
\]
Both norms on $\mathcal{B}$ will be non-Archimedean if $\|\cdot\|_2$ is non-Archimedean. If the valuation on $K$ is non-trivial and $W_{E_1} \subset W_K$, $\mathcal{B}$ will be a complete normed space whenever $E_2$ is complete, for the proof in [27], p.221, will hold in this case (cf [26]). Finally, if the norm on $E_2$ is discrete, then there exists $0 \neq x \in S$ such that
\[
\|T(x)\|_2 = \|T\|_S.
\]
Since a discrete norm on $E_2$ implies $W_K = W_K$, the above assertions yield the following result (cf [21], [26]):

**Theorem 3.2**

Let $E_1$ and $E_2$ be non-Archimedean normed spaces over a field $K$ with non-trivial valuation such that $W_{E_1} \subset W_K$.

Let $\mathcal{B}(E_1, E_2)$ be the space of continuous linear transformations from $E_1$ to $E_2$. Then
\[
\|T\| = \sup_{\|x\|_1 < 1} \|T(x)\|_2, \quad T \in \mathcal{B}
\]
defines a non-Archimedean norm on $\mathcal{B}(E_1, E_2)$ such that
\[
\|T(x)\|_2 < \|T\| \|x\|_1 \text{ for all } x \in E_1.
\]
If $E_2$ is a Banach space, so is $\mathcal{B}(E_1, E_2)$. If the norm on $E_2$ is discrete,
then \[ \|T\| = \sup_\|x\|_1 = 1 \|T(x)\|_2, \quad T \in \mathcal{B}, \]

and there exists \( x_0 \in E_1 \) with \( \|x_0\| = 1 \) such that

\[ \|T(x_0)\|_2 = \|T\|. \]

**Remark**

Since Theorem 3.1 is trivially true if \( W_{E_1} = W_{E_2} = \{1\} \), Theorem 3.2 will also hold in this case.

**Corollary 1.**

Let \( K \) be a field with a non-dense valuation, \( E \) a non-Archimedean normed space over \( K \) such that \( W_E \subset W_K \), and \( f \) a continuous functional on \( E \).

Then

\[ \|f\| = \sup_{\|x\| = 1} |f(x)| \]

is a non-Archimedean bound for \( f \) so that

\[ |f(x)| \leq \|f\| \|x\| \text{ for all } x \in E. \]

Further, there exists \( x_0 \in E \) with \( \|x_0\| = 1 \) such that \( |f(x_0)| = \|f\| \).

**Definition**

Let \( A \) be an algebra over \( K \). A non-Archimedean vector norm \( \|\| \) on \( A \) will be called a non-Archimedean algebra norm if it satisfies the additional conditions:

\( (4) \) \[ \|xy\| \leq \|x\| \|y\| \text{ for all } x, y \in A, \]

\( (5) \) \[ \|1\| = 1. \]

\( A \) is known as a non-Archimedean Banach algebra if it is a non-Archimedean Banach space.

We note that property \( (5) \) implies \( W_K \subset W_A \).
If \( \mathcal{B}(E) \) is the algebra of bounded linear operators on a non-Archimedean normed space \( E \), then
\[
\|T\|_1 = \sup_{\|x\| \leq 1} \|T(x)\| , \quad T \in \mathcal{B},
\]
defines a non-Archimedean algebra norm on \( \mathcal{B}(E) \) (for proof of (l) see [23], p.76). We thus have an additional corollary to theorem 3.2:

**Corollary 2.**

Let \( E \) be a non-Archimedean vector space over a field \( K \) such that \( W_E \subset W_K \). Let \( \mathcal{B}(E) \) be the algebra of continuous linear operators on \( E \). Then
\[
\|T\| = \sup_{\|x\| \leq 1} \|T(x)\| , \quad T \in \mathcal{B},
\]
defines a non-Archimedean algebra norm on \( \mathcal{B}(E) \) such that \( W_{\mathcal{B}} \subset W_K \) and
\[
\|T(x)\| \leq \|T\|\|x\| \text{ for all } x \in E.
\]
\( \mathcal{B}(E) \) is a non-Archimedean Banach algebra if \( E \) is a non-Archimedean Banach space. If the norm on \( E \) is discrete, then
\[
\|T\| = \sup_{\|x\| = 1} \|T(x)\| , \quad T \in \mathcal{B},
\]
and there exists \( x_0 \in E \) with \( \|x_0\| = 1 \) such that
\[
\|T(x_0)\| = \|T\|.
\]
In this case, \( W_{\mathcal{B}} = W_K \).

We shall see that an algebra over a non-Archimedean valued field may well have an Archimedean norm. This cannot happen, however, if the algebra is commutative and its norm is multiplicative. In fact, we have a somewhat stronger result:

**Theorem 3.3**

Let \( A \) be a commutative algebra over a non-Archimedean valued field and \( \| \| \) a norm on \( A \) such that \( \|x^2\| = \|x\|^2 \) for all \( x \in A \). Then \( \| \| \) is non-Archimedean.
Proof:

By induction, \( \|x^{2^n}\| = \|x\|^{2^n} \) for every \( n \in \mathbb{N} \).

Let \( a, b \in A \) with \( \|a\| > \|b\| \). Then

\[
\|a + b\|^{2^n} = \|(a + b)^{2^n}\| = \|\sum_{i=0}^{n} (2^n)^ia^{2^n-1}b^i\|
\]

\[
\leq \sum_{i=0}^{n} \|(2^n)^ia^{2^n-1}b^i\|, \text{ by property (3)},
\]

\[
= \sum_{i=0}^{n} \|(2^n)^ia^{2^n-1}b^i\|, \text{ by property (2)},
\]

\[
\leq \sum_{i=0}^{n} \|a^{2^n-1}b^i\|, \text{ since } \| \text{ is non-Archimedean},
\]

\[
\leq \sum_{i=0}^{n} \|a\|^{2^n-1}b^i, \text{ by property (4)},
\]

\[
\leq (2^n + 1)\|a\|^{2^n}.
\]

Hence, \( \|a + b\| \leq (2^n + 1)^{1/2^n} \|a\| \) for every \( n \in \mathbb{N} \).

Letting \( n \to \infty \) we get

\( \|a + b\| \leq \|a\| = \max (\|a\|, \|b\|) \).

Lemma 3.1

If \( A \) is a non-Archimedean normed \( K \)-algebra and \( S \) is the unit sphere in \( A \), then \( S \) is an \( R \)-subalgebra of \( A \), where \( R \) is the valuation ring of \( K \). The set

\( Q = \{ x \in A \mid \|x\| < 1 \} \)

is an algebra ideal of \( S \).

Proof:

Clear.

Theorem 3.4

With the notation of lemma 3.1, let \( \mathcal{R}(S) \) be the Jacobson radical of \( S \). Then:
(a) If $x \in A$ and $\sum_{n=0}^{\infty} x^n$ is convergent, then $\sum_{n=0}^{\infty} x^n$ is the inverse of $1 - x$ and $\|(1 - x)^{-1}\| \leq \max_{n} \|x^n\|$.

(b) If $x \in \mathbb{Q} \cap \mathcal{R}(S)$, then $\sum_{n=0}^{\infty} x^n$ converges to $(1 - x)^{-1} \in S$. In fact, $\|1 - x\| = \|(1 - x)^{-1}\| = 1$.

(c) If $A$ is complete, then $\mathbb{Q} \subset \mathcal{R}(S)$.

Proof:

All the assertions are immediate consequences of lemma 3.1, item 3 of §1, and the following known results:

1. If $A$ is a normed algebra over a valued field, $x \in A$, and $\sum_{n=0}^{\infty} x^n$ converges, then $1 - x$ is invertible and $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. Conversely, if $\|x\| < 1$ and $1 - x$ is invertible, then $\sum_{n=0}^{\infty} x^n$ converges to $(1 - x)^{-1}$. If $A$ is complete and $\|x\| < 1$, then $1 - x$ has an inverse in $A$ ([14], pp 75-76).

2. The radical of a ring $S$ is the largest ideal $I$ such that, for all $x \in I$, $1 - x$ is a unit ([12], p 57).
Norms and Bounds on Finite Dimensional Spaces

§ 4. Norms on \( \mathbb{K}^m \). Duality.

Let \( \mathbb{K} \) be a valued field and \( E \) a finite dimensional vector space over \( \mathbb{K} \). If \( \mathbb{K} \) is complete, then all norms on \( E \) are equivalent ([1], p.95). If in addition the valuation of \( \mathbb{K} \) is non-trivial, then all \( m \)-dimensional normed spaces over \( \mathbb{K} \) are topologically isomorphic ([3], p.27; for non-Archimedean spaces see [5], p.695). For each is topologically isomorphic to \( \mathbb{K}^m \) normed with \( \| \cdot \|_\infty \) and hence to \( \mathbb{K}^m \) with the product topology ([4], pp 67, 69).

On the other hand, without special assumptions about \( \mathbb{K} \), all the Hölder norms on \( E \) are equivalent to \( \| \cdot \|_\infty \). So is any non-Archimedean norm for which \( E \) has an orthogonal basis \( \{ u_i \}_{i=1}^m \) ([17], p.464). For then

\[
x = \sum_{i=1}^m \alpha_i u_i \Rightarrow \| x \| = \max_{1 \leq i \leq m} \| \alpha_i u_i \|.
\]

If \( \| x \|_\infty = \max_{1 \leq i \leq m} | \alpha_i | = | \alpha_k | \), we have

\[
\| x \|_\infty \| u_i \| \leq | \alpha_k | \| u_k \| < \max_{1 \leq i \leq m} \| \alpha_i u_i \| < \| x \|_\infty \max_{1 \leq i \leq m} \| u_i \|.
\]

Again, when \( E \) is finite dimensional over a valued field \( \mathbb{K} \), convergence to zero with respect to any norm on \( E \) is equivalent to coordinatewise convergence, for the proof referred to in [28], p.253, will hold here. Hence all linear transformations from \( E \) to an arbitrary normed space over \( \mathbb{K} \) are continuous and bounded. The boundedness is obvious when the valuation on \( \mathbb{K} \) is trivial and follows from the remark after theorem 3.1 in the non-trivial case. We calculate the bound of a linear
transformation between finite dimensional non-Archimedean normed spaces with orthogonal bases.

Theorem 4.1

Let $E_1$ and $E_2$ be finite dimensional non-Archimedean normed spaces over $K$ with orthogonal bases $\{u_j\}_{1 \leq j \leq n}$ and $\{v_i\}_{1 \leq i \leq m}$. Let $T$ be a linear transformation from $E_1$ to $E_2$. Then the bound of $T$ is given by

$$\|T\|_{1,2} = \max_{i,j} \frac{\|\alpha_{ij}\|\|v_i\|_2}{\|u_j\|_1}$$

where $(\alpha_{ij})$ is the matrix of $T$ with respect to the given bases.

Proof:

Let $0 \neq x = \sum_j \frac{1}{j} u_j$, $\|x\|_1 = \max_j \|\frac{1}{j} u_j\| = \|\frac{1}{n} k u_k\|$.

Then $T(x) = \sum_{i,j} \alpha_{ij} \frac{1}{j} v_i$ and

$$\|T(x)\|_2 = \max_{i,j} \|\alpha_{ij} \frac{1}{j} v_i\| = \|\alpha_{pq}\| \|\frac{1}{q} v_q\|_2 = \|\alpha_{pq}\| \|v_p\|_2$$, say.

We have

$$\frac{\|T(x)\|_2}{\|x\|_1} = \frac{\|\alpha_{pq}\| \|\frac{1}{q} v_q\|_2}{\|\frac{1}{n} k u_k\|_1}$$

$$= \frac{\|\alpha_{pq}\| \|\frac{1}{q} v_q\|_2}{\|\frac{1}{n} k u_k\|_1}$$

$$\leq \frac{\|\alpha_{pq}\| \|v_p\|_2}{\|u_q\|_1}$$

Thus

$$\sup_{0 \neq x \in E_1} \frac{\|T(x)\|_2}{\|x\|_1} = \max_{i,j} \frac{\|\alpha_{ij}\|\|v_i\|_2}{\|u_j\|_1} \leq \frac{\|\alpha_{pq}\| \|v_p\|_2}{\|u_q\|_1}$$, say.

Now put $x = u_s$. 
Then,
\[
\frac{\|T(u_g)\|_2}{\|u_g\|_1} = \frac{\sum \langle i,s \rangle v_i}{\|u_g\|_1} = \max \frac{\sum \langle i,s \rangle v_i}{\|u_g\|_1} = \frac{\sum \langle r,s \rangle v_r}{\|u_g\|_1}
\]
So,
\[
\|T\|_{1,2} = \max \frac{\sum \langle i,s \rangle v_i}{\|u_g\|_1} = \max \frac{\sum \langle i,s \rangle v_i}{\|u_g\|_1}
\]

Corollary
Let \( E \) be a non-Archimedean normed vector space over \( K \) with an orthogonal basis \( \{u_i\}_{1 \leq i \leq m} \). Let \( T \) be a linear operator on \( E \). Then the bound of \( T \) is given by
\[
\|T\|_{1,j} = \max_{i,j} \frac{\sum \langle i,j \rangle u_i}{\|u_j\|}
\]
where \( (\langle i,j \rangle) \) is the matrix of \( T \) with respect to the given basis. If also \( W_E \subset \overline{W}_K \), then
\[
\|T\|_S = \max_{i,j} \frac{\sum \langle i,j \rangle u_i}{\|u_j\|}
\]
Let \( K \) be a valued field. An algebra norm on \( K^m \) will be called a matrix norm. We note that this definition implies that \( \|I\| = 1 \). Since \( K^m \) may be identified with the algebra of linear operators on \( K^m \), every vector norm on \( K^m \) in determining an operator bound also determines a matrix norm
\[
\lub \langle A \rangle = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}, \ A \in K^m
\]
which may be called a matrix bound.
From now on we take $E = \mathbb{R}^n$. Lower case Latin letters will denote column vectors while ordinary capitals will stand for matrices.

If a vector and a matrix norm satisfy the condition
\[ \|Ax\| \leq \|A\| \|x\| \]
for every $x$ and $A$, then the two norms are said to be consistent. A bound is of course consistent with the corresponding vector norm. The following results known for $\mathbb{R}$ and $\mathbb{C}$ remain valid for $K$.

For any matrix norm and for the bound associated with a consistent vector norm
\[ r(A) \leq \text{lub } (A) \leq \|A\|, \]
where $r(A)$ is the spectral radius of the matrix $A$ ([9], p.45). If $y$ is a fixed non-zero vector, then for any matrix norm
\[ \|x\| = \|xy'\| \]
defines a consistent vector norm ([13]; in [9] the Hermitian transpose is used throughout). It follows that for a matrix norm to be a bound it is sufficient that the norm be minimal (cf [13]). The converse has been proved by Ljubic for real matrices (ibid). The following theorem is weaker than the corresponding results in $\mathbb{R}^n$.

**Theorem 4.2**

Let $K$ be a valued field and $\|\|$ a norm on $K^n$. For all $x \in K^n$ let
\[ \|x\|^* = \sup_{y \neq 0} \frac{|x'y|}{\|y\|} \].

Then:
(a) $\|\|^*$ is a norm on $K^n$.
(b) $\|\|^*$ is non-Archimedean if and only if the valuation of $K$ is non-Archimedean.
(c) $|x'y| \leq \|x\|^* \|y\|$.
Proof:
All the assertions follow from the fact that $\| \cdot \|^* \text{ is the bound of a linear functional generated by } x$.

**Corollary 1.** (see [13]).

For the bound associated with $\| \cdot \|$

$$\text{lub } (xy') = \|x\| \|y\|^*.$$  

**Proof:** (cf [13]):

$$\text{lub } (xy') = \sup_{z \neq 0} \frac{\|xy'z\|}{\|z\|} = \|x\| \sup_{z \neq 0} \frac{\|y'z\|}{\|z\|} = \|x\| \|y\|^*.$$  

**Corollary 2.** (cf [9], p. 43).

For all $x, y \in K^m$, $A \in K^m$

$$|x' Ay| \leq \|x\|^* \|y\| \text{ lub } (A) = \text{lub } (yx') \text{ lub } (A).$$  

**Definition**

Since $\| \cdot \|^*$ is a norm on the dual space of $K^m$ it will be called the dual norm of $\| \cdot \|$.

**Remarks**

1. As distinct from the situation in $\mathbb{R}^m$ in general duality is not an involutory relation. To see this we need only consider an Archimedean norm on $K^m$ when $K$ is non-Archimedean. For then $(\| \cdot \|)^* = \| \cdot \|$ contradicts (b).

2. It follows from (c) that $\|e_i\| \|e_i\|^* \neq 1$ Thus, if $S$ and $S^*$ are the unit spheres for $\| \cdot \|$ and $\| \cdot \|^*$ respectively,

$$e_1 \in S \cap S^* \iff \|e_i\| = \|e_i\|^* = 1.$$  

If this condition is fulfilled for all $i$, then for all $i, j$

$$\text{lub } (e_i^* e_j^*) = 1 \text{ and } |a_{ij}| < \text{lub } (A).$$
Lemma 4.1 (cf [13])

Let \( K \) be a valued field. If norms \( \| \|_1 \) and \( \| \|_2 \) are defined on \( K^m \), then for the corresponding bounds

\[
\text{lub}_1(A) = \text{lub}_2(A) \iff \|x\|_1 = a\|x\|_2 \text{ for some } a > 0.
\]

Proof:

One implication is obvious and the other follows from corollary 1 to theorem 4.2.

Lemma 4.2

Let \( K \) be a valued field. Let \( \| \| \) be a norm on \( K_m \) and suppose that for each matrix \( A \) there exist vectors \( x \) and \( y \) with \( \|yx^T\| = 1 \) such that \( |x^TAy| = \|A\| \). Then \( \| \| \) is minimal. In fact, \( \| \| \) is a bound.

Proof:

Let \( \| \|_1 \) be a matrix norm such that \( \|A\|_1 < \|A\| \) for all \( A \). Then for the bound associated with a vector norm consistent with \( \| \|_1 \) we have

\[
\text{lub}(yx^T) < \|yx^T\|_1 < \|yx^T\| = 1.
\]

Hence by corollary 2 to theorem 4.2,

\[
|x^TAy| < \text{lub}(yx^T) \text{lub}(A) < \text{lub}(A) < \|A\|_1 < \|A\|
\]

and the given condition forces

\[
\text{lub}(A) = \|A\|_1 = \|A\| \text{ for all } A.
\]

It is known that when \( K = \mathbb{R} \) or \( K = \mathbb{C} \) non-zero vectors may be found for which equality holds in corollary 2 to theorem 4.2 (see [9]). If these are suitably normalized, then the conditions of lemma 4.2 are satisfied and we obtain an alternative proof of Ljubić's theorem:
Theorem 4.3

When \( K = \mathbb{R} \) or \( K = \mathbb{C} \), a matrix norm on \( \mathbb{K}^m \) is a bound if and only if it is minimal.

It is known that every Archimedean valued field is topologically isomorphic to a subfield of \( \mathbb{C} \) valued with the ordinary absolute value ([28], p.246). Accordingly in this case it is customary to take \( K = \mathbb{R} \) or \( K = \mathbb{C} \). Further, every norm on \( \mathbb{R}^m \) or \( \mathbb{C}^m \) is determined by its unit sphere which is an equilibrated convex body ([6], p.108; [9], p.41). Theorem 2.3 shows that this statement remains true for certain norms on \( \mathbb{K}^m \) when \( K \) is non-Archimedian, where however \( S \) is an absorbant and bounded \( \mathbb{R} \)-module. Writing \( \| \cdot \| = \| \cdot \|_S \) we have in all cases

\[
\text{lub} (A) = \sup_{x \in S} \|Ax\|_S = \text{lub}_S (A).
\]

We also have (cf [9]):

Lemma 4.3

Let \( K \) be a non-Archimedian valued field, \( K = \mathbb{R} \) or \( K = \mathbb{C} \). Let \( S_1 \subset \mathbb{K}^m \) be an absorbant and bounded \( \mathbb{R} \)-module if \( K \) is non-Archimedian and an equilibrated convex body in the other cases. If the matrix \( P \) is non-singular, then for \( x \in \mathbb{K}^m, A \in \mathbb{K}^m \)

\[
S_2 = PS_1 \Rightarrow \|x\|_{S_2} = \|P^{-1}x\|_{S_1} \quad \text{and} \quad \text{lub}_{S_2}(A) = \text{lub}_{S_1}(P^{-1}AP).
\]

Proof:

We may assume that \( K \) is non-Archimedian. Then \( S_2 \) is an absorbant and bounded \( \mathbb{R} \)-submodule of \( \mathbb{K}^m \). Further,

\[
\forall \lambda \in K, x \in \lambda S_2 \iff P^{-1}x \in \lambda S_1.
\]

Hence,

\[
\inf_{x \in \lambda S_2} |\lambda| = \inf_{P^{-1}x \in \lambda S_1} |\lambda|.
\]
Also, \( \sup_{x \neq 0} \frac{\|Ax\|_{S_2}}{\|x\|_{S_2}} = \sup_{x \neq 0} \frac{\|P^{-1}Ax\|_{S_2}}{\|x\|_{S_2}} = \sup_{y \neq 0} \frac{\|P^{-1}Ay\|_{S_1}}{\|y\|_{S_1}}. \)

The required results are now obtained by a reference to definitions.

**Corollary**

\( S_2 = \mathcal{K} S_1, \mathcal{K} \in K \Rightarrow \|x\|_{S_1} = \|\mathcal{K}x\|_{S_2} \) and \( \operatorname{lub}_{S_1}(A) = \operatorname{lub}_{S_2}(A). \)

Let \( K \) be a non-Archimedian valuated field. Consider the non-Archimedian norm on \( K^n \) defined by

\[ \|x\|_{\infty} = \max_i \left| \frac{x_i}{1} \right|. \]

The unit sphere of \( \| \|_{\infty} \) is the cube \( N = \{ x \in K^n \mid \frac{x_i}{1} < 1, 1 < i < m \} \).

We have \( W_{K^n} = W_K \) so that by theorem 2.3

\[ \|x\|_{\infty} = \|x\|_N \]

for all \( x \in K^n \).

It follows from § 3 that

\[ \operatorname{lub}_N(A) = \sup_{\|x\|_N = 1} \|Ax\|_N, \]

while theorem 4.1 gives

\[ \operatorname{lub}_N(A) = \max_{i,j} \left| \frac{x_{ij}}{1} \right|. \]

**Definition**

When the valuation of \( K \) is non-Archimedian we shall describe as **natural** the norm and bound defined above.

The natural norm is **self-dual**. For, if \( \|y\|_N = 1 \), then

\[ |x'y| < \max_i \left| \frac{x_i}{1} y_i \right| < \max_i \left| \frac{x_i}{1} \right| = \left| \frac{x}{1} \right|, \]

say,

and equality is attained for \( y = e_k \). Since

\[ W_{K^n} = W_K \Rightarrow \|x\|_N^* = \sup_{\|y\|_N = 1} |x'y|, \]

we have

\[ \|x\|_N^* = \left| \frac{x}{1} \right| = \|x\|_N. \]
Further,
\[ \|x\|^\star = |x'_k x| = |x'e_k| = \|x\|_N \]
indicates that for each \(x\) and for each \(y\) there exists a non-zero vector such that equality holds in (c) of theorem 4.2.

Again, if \( \max \sum \frac{x_i y_j}{\gamma_{ij}} = |\gamma_{rs}| \), then
\[ |x'_k A e_k| = |\gamma_{rs}| = \text{lub}_N(A) \]
so that by lemma 4.2 the natural bound is minimal.

Finally, \( \| \|_N \) is the only non-Archimedean norm on \( K^m \) for which (with the notation of the remark after theorem 4.2) \( e_i \in S \cap S^\star \) for all \( i \). For, if \( |\gamma_i| = \max \| \gamma_i \| \),
\[ |\gamma_i| = |x'_k y| \leq \|e_k\|^\star \|y\| = \|y\| \leq \max \|\gamma_i e_i\| = \max \|\gamma_i\|. \]
Hence \( \|y\| = \max \|\gamma_i\| = \|y\|_N \) for all \( y \in K^m \).

If \( G = \text{diag} (\gamma_1, \gamma_2, \ldots, \gamma_m) \), \( \gamma \neq 0 \), then
\[ GN = \{ x \in K^m | \|x\|_G \leq |\gamma_i|, 1 \leq i \leq m \} \]
also defines a norm on \( K^m \) and a corresponding bound on \( K^m \), which may be called a \( g \)-norm and a \( g \)-bound respectively (see [9], p 145). By lemma 4.3 we have
\[ \|x\|_{GN} = \max \|x_i\|_{GN} \text{ and } \text{lub}_{GN}(A) = \max |\sum \gamma_{ij} y_j|. \]
It may be shown that the dual of \( \| \|_{GN} \) is given by
\[ \|x\|_{GN^{-1}} = \max |\gamma_i x_i|. \]
Remark

The non-Archimedean bound which corresponds to the natural norm should be compared with the cubic bound for real matrices (see [6], p 108).

On the other hand, a slight modification of the calculation in [6] shows that, even when $K$ is non-Archimedean, the bound associated with the Archimedean norm $\|x\| = \sum_{i=1}^{m} |\delta_{i1}|$ on $K^m$ is given by the same formula as in the real case. For its dual, however, we have

$$\|x\| = \max_{i} |\delta_{i1}| = \|x\|_N,$$

verifying that duality need not be involutory.

Let $K$ be a valuated field. If $\| \|$ is a matrix norm, then for the bound associated with a consistent vector norm we have

$$1 \leq \lub (e_i e_j^t) < \|e_i e_j^t\| \quad \text{for all } i,$$

so that

$$1 \leq \max_{i,j} \|e_i e_j^t\| = n(J) \quad \text{(theorem 4.2)}.$$

Further, for all $i, j$

$$\|\kappa_{i,j}\| \leq \lub (e_i e_j^t) \lub (A) < \|e_j e_i^t\| \|A\| \leq n(J) \|A\|.$$

Therefore,

$$\max_{i,j} \|\kappa_{i,j}\| \leq \|A\|.$$

If $\| \|$ is non-Archimedean, then

$$\|A\| \leq \max_{i,j} \|\kappa_{i,j} e_i e_j^t\| = n(J) \lub (A) \lub (N).$$

We have proved the following theorem:
Theorem 4.4

If $\| \| \|$ is a non-Archimedian matrix norm, then

$$\frac{\ln_n(A)}{n(J)} < \|A\| < n(J) \ln_n(A)$$

where $n(J) = \max_{i,j} \|e_i e_j\|$.

Corollary

A non-Archimedian matrix norm is the natural bound if and only if $\max_{i,j} \|e_i e_j\| = 1$.

§ 5. The Polar Norms and Bounds on $K^m$

Let $K$ be a valued field. The polar of a set $S \subset K^m$ is defined by

$$S' = \{ x \in K^m \mid u \in S \Rightarrow |x'u| < 1 \}.$$ 

The following lemma is analogous to a result in [9], p. 422.

Lemma 5.1

If the valuation of $K$ is non-Archimedian and $S$ is an absorbant set in $K^m$, then $S'$ is an absorbant and bounded $R$-submodule of $K^m$.

Proof:

Let $u \in S$.

Then for $x, y \in S'$ and $\lambda, \mu \in R$

$$|\lambda x + \mu y'u| < \max (|\lambda| |x'u|, |\mu| |y'u|) < 1$$

so that $\lambda x + \mu y' \in S'$ and $S'$ is an $R$-submodule of $K^m$.

To show that $S'$ is absorbant, let $x \in K^m$ and $x'u = \mu \in K$.

If $\mu = 0$, $x \in S' = 1 \cdot S'$; otherwise $x \in \mu S'$ (see lemma 2.3).
Again, let $0 \neq x \in K^m$. Since $S$ is absorbent there exists $u \in S$ such that $x'u \neq 0$. Let \( \lambda = \frac{1}{|x'u|} \). Then for all $\lambda \in K$ with $|\lambda| > \lambda$

\[
|(\lambda x)'u| = |\lambda| |x'u| = \frac{|\lambda|}{\lambda} > 1
\]

so that $\lambda x \notin S'$ and $S'$ is bounded.

Now suppose that $S$ itself is an absorbent and bounded $R$-module and that a non-Archimedean norm $\| \| = \| \|_S$ is defined on $K^m$. Then $W_K \subseteq W_K$ and

\[
\|x\| = \sup_{y \in S} |x'y|, \quad x \in K^m.
\]

Moreover, $S'$ is the unit sphere for $\| \|_{\| \|}$ and the set of values of $\| \|_{\| \|}$ is contained in $W_K$. Hence

\[
\|x\|_{\| \|} = \|x\|_{S'}, \quad \text{for all } x \in K^m.
\]

If the valuation of $K$ is non-dense, then by corollary 1 to theorem 3.2

\[
\|x\|_{S'} = \sup_{\|y\|_{S'} = 1} |x'y|
\]

and for each $x \in K^m$ there exists a vector $y_0$ with $\|y_0\|_S = 1$ such that

\[
|x'y_0| = \|x\|_{S'}, \|y_0\|_S.
\]

In this case duality is an involutory relation, for we also have

\[
\|y\|_S = \sup_{\|x\|_{S'} = 1} |x'y|.
\]

To see this put $\|y\|_{\| \|} = \sup_{\|x\|_{S'} = 1} |y'x| = \sup_{\|x\|_{S'} = 1} |x'y|$. Then by (c) of theorem 4.2 $\|y\|_{\| \|} \leq \|y\|_S$. Further, there exists $x_0 \in K^m$ with $\|x_0\|_{S'} = 1$ such that $\|y\|_{\| \|} = |x_0'y|$. Therefore,
Let $K$ be a non-Archimedean valued field and $S$ an absorbant and bounded $R$-submodule of $K^m$. Let

$$\|x\|_{S'} = \sup_{y \in S} |x'y|, x \in K^m.$$ 

Then:

(a) $\|\|_{S'}$ is a non-Archimedean norm on $K^m$ such that

$$|x'y| \leq \|x\|_{S'} \|y\|_S$$

for all $x, y \in K^m$.

(b) If the valuation on $K$ is non-dense, then

$$\|x\|_{S'} = \sup_{\|y\|_S = 1} |x'y|$$

and

$$\|y\|_S = \sup_{\|x\|_{S'} = 1} |x'y|.$$ 

Further, for each $x \in K^m$ there exists a vector $y_0$ with $\|y_0\|_S = 1$ and for each $y \in K^m$ there exists a vector $x_0$ with $\|x_0\|_{S'} = 1$ such that equality is attained in (a).

**Corollary 1**

If the valuation of $K$ is non-dense, then the bound determined by $S$ is minimal. In fact, for each matrix $A$ there exist vectors $x$ and $y$ with $\text{lub}_S(yx') = 1$ such that

$$|x'y| = \text{lub}_S(A).$$

**Proof:**

By (b), for each $y \in K^m$ there exists a vector $x_0$ with $\|x_0\|_{S'} = 1$ such that $|x_0'y| = \|Ay\|_S$. By corollary 1 to theorem 3.2 there exists $y_0$
with \( \|y_0\|_S = 1 \) such that \( \|Ay_0\|_S = \text{lub}_S(A) \). Further, \( \text{lub}_S(y_0x'_0) = \|y_0\|_Sx'_0 \), by corollary 1 to theorem 4.2.

**Corollary 2**

If the valuation of \( K \) is non-dense, then the bound of \( A \) with respect to \( S \) is the bound of \( A' \) with respect to \( S' \).

**Proof:**

See [9].

We observe that it has been possible not only to extend the area of application of Ljubic's theorem but also to obtain different necessary and sufficient conditions. These may be summarized as follows:

**Theorem 5.2**

Let \( K \) be a valuated field. For a matrix norm on \( K_n \) to be a bound it is sufficient that for each \( A \in K_n \) there exist vectors \( x \) and \( y \) with \( \|yx'\| = 1 \) such that

\[
|x'Ay| = \|A\|.
\]

This condition is necessary in the following cases:

- (a) \( K = \mathbb{R} \)
- (b) \( K = \mathbb{C} \)
- (c) \( K \) is non-Archimedean and the matrix bound is natural,
- (d) \( K \) is non-Archimedean with a non-dense valuation and the bound is associated with a non-Archimedean norm on \( K_n \) for which \( W_{K(m)} \subset W_K \).

**Remark**

For complex matrices \( |x'Ay| \) may be replaced by \( |x^H Ay| \) or \( \text{Re} x^H Ay \) while \( yx' \) becomes \( yx^H \).

The next theorem should be compared with weaker results for complex matrices (see [9], p.46).
Theorem 5.3

Let K be a field with a non-trivial non-Archimedean valuation and let K contain all the characteristic roots of \( A \in K_n \). Then there exists an absorbant and bounded \( R \)-submodule \( S \subset K^n \) such that if \( r(A) = 0 \) \( \text{ lub}_S(A) \) is arbitrarily small and if \( r(A) \neq 0 \) \( \text{ lub}_S(A) = r(A) \).

Proof:

In view of lemma 4.3 we need consider only the Jordan canonical form of \( A \) and in fact only the Jordan block belonging to \( \lambda \), where \( \lambda \) is a characteristic root of \( A \) such that \( |\lambda| = r(A) \). Let \( W(\varepsilon) = \text{diag}(1, \varepsilon, \varepsilon^2, \ldots) \), where \( 0 < |\varepsilon| < |\lambda| \) if \( \lambda \neq 0 \) and \( |\varepsilon| > 0 \) otherwise. Then

\[
W(\varepsilon) = \begin{pmatrix}
\lambda & 0 & \ldots & 0 & 0 \\
1 & \lambda & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & \lambda \\
\end{pmatrix}
\]

has natural bound equal to \( |\varepsilon| \) if \( \lambda = 0 \) and to \( |\lambda| \) if \( \lambda \neq 0 \).

Let \( K \) be a valued field. To preserve uniformity of terminology we shall call

\[
U = \{ \varepsilon \in K \mid |\varepsilon| = |\varepsilon^{-1}| = 1 \}
\]

the group of units of \( K \). For any norm on \( K_n \) the condition number of a non-singular matrix \( A \) will be defined by

\[
c(A) = \|A\| \|A^{-1}\|.
\]

Definition

Let \( S \) be the unit sphere of a norm on \( K_n \). A matrix \( B \in S \) will be called \( S \)-unitary if it has an inverse in \( S \).

It is clear that \( B \) is \( S \)-unitary if and only if \( \|B\| = \|B^{-1}\| = 1 \) and that \( S \)-unitary matrices form a multiplicative group \( U \) which may be
called the $\mathcal{U}$-unitary group of the norm on $K_m$. If $B \in \mathcal{U}$, then $c(B) = 1$ and $\|Bx\| = \|x\|$ for every consistent vector norm on $K^m$. Conversely, if $B$ is non-singular and $\|Bx\| = \|x\|$ for all $x$, then $B$ belongs to the $\mathcal{U}$-unitary group of the associated bound.

**Lemma 5.2**

Let the matrix $B$ be non-singular and have a non-empty spectrum. Then $B \in \mathcal{U}$ for a norm on $K_m$ if and only if $c(B) = 1$ and the spectrum of $B$ is contained in $\mathcal{U}$.

**Proof:**

Let $\lambda \in K$ be a characteristic root of $B$.

$B \in \mathcal{U}$ implies that

$$1 = \frac{1}{\|B^{-1}\|} \leq |\lambda| < \|B\| = 1 \text{ so that } \lambda \in \mathcal{U}.$$ 

Conversely we have

$$\|B\| > |\lambda| = 1 \text{ and } \|B^{-1}\| > \frac{1}{|\lambda|} = 1$$

so that $c(B) = 1$ forces equality.

**Lemma 5.3**

Let $C$ be a non-singular matrix with a non-empty spectrum. Then for any norm on $K_m$ $c(C) = 1$ if and only if $C = \lambda B$ for some $0 \neq \lambda \in K$ and $B \in \mathcal{U}$.

**Proof:**

Let $\{\lambda_i\}_{1 \leq i \leq k}$ be the spectrum of $C$ such that $|\lambda_k| \leq \ldots \leq |\lambda|$. 
$c(C) = 1$ implies that
\[
\|C\| = \frac{1}{\|C^{-1}\|} = \lambda_1 \leq \cdots \leq \lambda_k \leq \|C\|
\]
forcing equality.

Let $\lambda \in K$ such that $\|\lambda\| = \lambda_i$ for $1 \leq i \leq k$. Then by lemma 5.1
\[
B = \lambda^{-1}C \subseteq \mathcal{U}.
\]

Since the converse is obvious, the result follows.

**Theorem 5.4**

Let $K$ be a non-Archimedean valued field, $K = \mathbb{R}$ or $K = \mathbb{C}$. Let $\|\| \|$ be a norm on $K^m$ with the proviso in the first case that $\|\| \|$ be non-Archimedean with $W_{K^m} \subseteq W_K$. Then a non-singular matrix $C = \omega B$, $0 \neq \omega \in K$, $B \in \mathcal{U}$ for the associated bound on $K^m$ if and only if
\[
lub (C^{-1}A \gamma C) = lub (A) \quad \text{for all} \quad A \in K^m.
\]

**Proof:**

Since the conclusion is obvious when $C$ has the required form, it suffices to prove the converse.

Let $S$ be the unit sphere in $K^m$. Then in all cases $\|x\| = \|x\|_S$ and
\[
lub (A) = lub_S (A) \quad \text{for all} \quad x \in K^m, \quad A \in K^m.
\]
By lemma 4.3 and similar results in the other cases, we have
\[
lub_S (A) = lub_{CS} (A).
\]
Hence by lemma 4.1
\[
\|x\|_S = a\|x\|_{CS}, \quad a > 0.
\]
Further, when $K$ is non-Archimedean lemma 4.3 shows that the value set of $\|\|_{CS} \subseteq W_K$. Hence in all cases, $a = |\lambda|$ for some $\lambda \in K$.

Thus
\[
CS = \lambda S.
\]
43.

It follows that

\[ \text{lub}_S(C) = \sup_{x \in S} \|Cx\|_S = |\lambda| \quad \sup_{x \in S} \|x\|_S = |\lambda| \]

Similarly \( \text{lub}_B(C^{-1}) = \frac{1}{|\lambda|} \). Hence \( c(C) = 1 \) and the result follows from

Lemma 5.3 (it is easy to see that the constant involved is actually \( \lambda \)).

**Lemma 5.4**

Let \( K \) be a complete non-Archimedian field, \( \| \| \) a non-Archimedian norm on \( K_m \) and \( \mathcal{A} = \{ A \in K_m : \| A \| < 1 \} \). Then

\[ \mathcal{G} = \{ A \in K_m : A = I - H, H \in \mathcal{A} \} \]

is a normal subgroup of \( \mathcal{U} \).

**Proof:**

\( K_m \) is complete and so \( \mathcal{G} \subset \mathcal{U} \) by theorem 3.4.

Let \( B_1, B_2 \in \mathcal{G} \). Then:

\[ B_1B_2 = (I - H_1)(I - H_2) = I - H_1 - H_2 + H_1H_2 \]

with

\[ \| H_1 + H_2 - H_1H_2 \| \leq \max(\| H_1 \|, \| H_2 \|) < 1. \]

Again,

\[ B_1^{-1} = I - C, \text{ where } C = I - (I - H_1)^{-1} \]

and

\[ \| C \| = \| \sum_{i=1}^{\infty} H_1^i \| \leq \max \| H_1 \|^i \| H_1 \| < 1. \]

The normality of \( \mathcal{G} \) is clear.

**Remark**

With an obvious extension of the definitions this lemma will hold in any complete non-Archimedian algebra.

**Theorem 5.5**

Let \( K \) be a complete non-Archimedian field with \( R, P \) and \( U \) as defined in § 2. Let \( A \) be a matrix in \( K_m \) and \( I - A = (\beta_{ij}) \) with \( \beta_{ij} \in P \) for
all $i$, $j$. Then $A$ is non-singular with a spectrum contained in $U$. Further, all the entries of $A^{-1}$ are in $R$ and $A^{-1} = I - C$ with the spectrum of $C$ contained in $P$.

Proof:
If the natural bound $\text{ub}_N\left(\langle \lambda_{ij}\rangle \right) = \max_{i,j} |\lambda_{ij}|$ is defined on $K_m$, then $(\beta_{ij}) \in \mathcal{Q}$ and $A \in \mathcal{U}$ by the previous lemma.

This proves the assertion concerning the entries of $A^{-1}$, while an application of lemma 5.2 yields the spectral property of $A$ (as well as of $A^{-1}$).

Again, $C \in \mathcal{Q}$ and so $r(0) < \text{ub}_N(C) < 1$.

§ 6. $p$-Adic Numerical Analysis.

Let the field $K$ be complete with respect to a non-trivial non-Archimedean valuation and let $A$ be a matrix over $K$. By § 1 the geometric series $\sum_{i=0}^{\infty} A^i$ converges with respect to a non-Archimedean norm if and only if $A^i \to 0$. Since convergence in $K_m$ is equivalent to coordinate-wise convergence, the situation is exactly the same as for real matrices (see [6], p.113). Here too $A^i \to 0$ if and only if $r(A) < 1$ in the algebraic closure of $K$. Further, if $r(A) < 1$, then there exists a non-Archimedean norm on $K_m$ such that $\|A\| < 1$ (theorem 5.3). In this event, by theorem 3.4 $I - A$ is non-singular and $(I - A)^{-1} = \sum_{i=0}^{\infty} A^i$. Put

$$S_n = \sum_{i=0}^{n} A^i,$$

so that

$$(I - A)^{-1} - S_n = \sum_{i=n+1}^{\infty} A^i = (I - A)^{-1} A^{n+1}.$$
In fact, we have obtained the following estimate for the rate of convergence of a geometric series:

**Theorem 6.1**

If \( \| \| \) is a non-Archiimedean matrix norm and \( \| A \| < 1 \), then
\[
\| (I - A)^{-1} - \sum_{i=0}^{n} A^i \| = \| A^{n+1} \| < \| A \|^{n+1}.
\]

**Remark**

The above estimate is sharper than the corresponding estimate for real or complex matrices ([6]; [9], p.54). Actually we are dealing with a special case, for we also have \( \| (I - A)^{-1} - S_n \| < \| (I - A)^{-1} \| \| A \|^{n+1} \) (see [9]). In our case, however, \( \| (I - A)^{-1} \| = 1 \). The same observation applies to the next theorem.

**Theorem 6.2** (cf [9], p.55).

Let \( A \) be a non-singular matrix and \( C \) an approximation to \( A^{-1} \).

Let \( H = I - AC \), \( \| H \| < 1 \), where \( \| \| \) is non-Archiimedean. Then:

(a) \( \| A^{-1} \| = \| C \| \)
(b) \( \| A^{-2} - C \| < \| CH \| \)
(c) \( \| A - C^{-1} \| < \| HA \| \)

**Proof:**

\( I - H \) is non-singular and \( \| I - H \| = \| (I - H)^{-1} \| = 1 \).

Now, (a) follows from \( A^{-1} = C(I - H)^{-1} \),
\[
C = A^{-1}(I - H);
\]
(b) from \( A^{-1} = C + CH(I - H)^{-1} \);
and (c) from \( C^{-1} = A + (I - H)^{-2}HA \).
Methods of Successive Approximation

Let $x^*$ be an approximate solution for a system of linear equations

$$Ax = h, \quad A \text{ non-singular.}$$

Define the error and residual vectors respectively by

$$s = x - x^* \quad \text{and} \quad r = h - Ax^* = As.$$ 

For any consistent non-Archimedian vector and matrix norms we have

$$\|s\| < \|0\| \|r\|,$$

where $C$ is an approximation to $A^{-1}$ such that $\|H\| < 1$.

In a method of successive approximation for the solution of such a system, given an initial approximation $x_0$, a sequence of approximating vectors is formed by the recursion formula

$$x_{n+1} = x_n + C_n r_n,$$

where $C_0, C_1, \ldots$ is a certain sequence of matrices. We note that

$$x_{n+1} - x_n = s_n - s_{n+1}.$$ 

If

$$H_n = I - C_n A$$

then

$$s_{n+1} = H_n s_n, \quad r_{n+1} = AH_n A^{-1} r_n.$$ 

Hence,

$$\|s_{n+1}\| \leq \|H_n\| \|s_n\|$$

and a sufficient condition for convergence is $\|H_n\| < 1 \quad \forall n$. 
For non-Archimedean norms this condition implies that

\[ \|x_{n+1} - x_n\| = \|s_n\| \]

so that

\[ \|s_{n+1}\| \leq \|H_n\| \|x_{n+1} - x_n\| \] (cf [9], p.24).

In the classical method of successive approximation

\[ C_n = I, \quad H_n = I - A = H \text{ for all } n. \]

\[ x_{n+1} = Hx_n + h \]

\[ = H^{n+1}x_0 + \sum_{i=0}^{n} H^i h. \]

If \( \|H\| < 1 \), we have in the non-Archimedean case

\[ \|s_{n+1}\| = \|(I - H)^{-1}h - \sum_{i=0}^{n} H^i h - H^{n+1}x_0\| \]

\[ = \|\sum_{i=n+1}^{\infty} H^i h - H^{n+1}x_0\| \]

\[ \leq \max\left(\|H\|^{n+1} \|h\|, \|H\|^n \|x_0\|\right) \] (cf [6], p.183).

If \( x_0 = h \), then

\[ x_n = \sum_{i=0}^{n} H^i h \] (p.185), and

\[ \|s_n\| = \|\sum_{i=n+1}^{\infty} H^i h\| = \|H^{n+1}h\| = \|H\|^n \|h\|. \]

Again, consider methods of successive approximation for finding the inverse of a non-singular matrix. If \( A \) is such a matrix and \( x_0 \) is an initial approximation to \( A^{-1} \), error and residual matrices are defined by

\[ S_n = A^{-1} - x_n \quad \text{and} \quad R_n = AS_n = I - AX_n. \]
All such methods depend on the possibility of finding a sequence of matrices $C_n$ such that the sequence $(X_n)$ tends to $A^{-1}$ as a limit, where
\[ X_{n+1} = X_n + C_n R_n. \]
If $C_n = X_n$, we have
\[ X_{n+1} = X_n (I + R_n). \]

If $C_n = X_n$, we have
\[ R_{n+1} = R_n^2, \] and
\[ S_{n+1} = S_n A S_n. \]

Hence for any norm,
\[ \| R_{n+1} \| < \| R_n \|^2 \quad \text{and} \quad \| S_{n+1} \| < \| A \| \| S_n \|^2, \]
and the process converges quadratically if either $\| R_n \| < 1$ or $\| S_n \| < 1$ for any norm $\| \cdot \|$, \(\forall n ([9], p. 95)\).

In fact, since
\[ R_n = R_o^n \quad \text{and} \quad S_n = A^{-1} R_n = X_o (I - R_o)^{-1} R_o^n, \]
we have, if $\| R_o \| < 1$
\[ \| S_n \| = \| X_o R_o^n \| < \| X_o \| \| R_o \|^n \quad (cf \ [6], p. 159). \]

Let $K = \mathbb{Q}_p$ be the field of $p$-adic numbers with the usual $p$-adic valuation. We shall express the elements of $\mathbb{Q}_p$ in decimal notation ([1], p. 36) and note that for $a \in \mathbb{Q}_p$ with $|a|_p < 1$ the number of zeros after the decimal point equals $\text{ord} \ a = -\log_p |a|_p$

is the ordinal of $a$. As vector and matrix norms we shall use the natural norm $\| x \| = \max_i |x_i|_p$ and the corresponding natural bound $\| A \| = \max_{i,j} |a_{ij}|_p$.

We shall extend the concept of the ordinal function to these norms and observe that for any method of successive approximation
\[ \text{ord} \ s_n = 1, \quad n > n_0, \]
gives the number of correct decimal places in approximating vector $x_n$ and that a similar statement applies to matrices. To fix our ideas we shall take $p = 5$.

**Example 1**

Consider the following system of 5-adic equations:

\[
\begin{align*}
1.32 & \cdot 5^{-1} - .02 & \cdot 5^{-2} - .12 & \cdot 5^{-3} = .43 \\
-0.02 & \cdot 5^{-1} + 1.40 & \cdot 5^{-2} - .04 & \cdot 5^{-3} = .03 \\
-0.12 & \cdot 5^{-1} - .04 & \cdot 5^{-2} + 1.30 & \cdot 5^{-3} = 1.12
\end{align*}
\]

It will be seen that the coefficient matrix of the system satisfies the hypothesis of theorem 5.5 and is therefore non-singular. The table below presents the solution of this system by Gaussian elimination (see [6], p.130).

**Single - Division Scheme**

<table>
<thead>
<tr>
<th>1.32</th>
<th>-0.02</th>
<th>-0.12</th>
<th>0.43</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.02</td>
<td>1.40</td>
<td>-0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>-0.12</td>
<td>-0.04</td>
<td>1.30</td>
<td>1.12</td>
</tr>
<tr>
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<td>-0.02</td>
<td>-0.14</td>
<td>0.12</td>
</tr>
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<td>0.2310</td>
<td>1.342032</td>
<td>1.11033</td>
</tr>
<tr>
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<td>1.342032</td>
<td>1.11033</td>
<td></td>
</tr>
<tr>
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<td>1.342032</td>
<td>1.11033</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.04</td>
<td>0.31231</td>
<td></td>
</tr>
<tr>
<td>1.342032</td>
<td>1.110300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.342032</td>
<td>1.110300</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.322413</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.020241</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.042020</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Example 2

The equation in example 1 may be written in the following form:

\[ \begin{align*}
\frac{\xi_1}{2} &= 0.22 \frac{\xi_1}{2} + 0.02 \frac{\xi_2}{2} + 0.12 \frac{\xi_3}{2} + 0.43 \\
\frac{\xi_2}{2} &= 0.02 \frac{\xi_1}{2} + 0.14 \frac{\xi_2}{2} + 0.04 \frac{\xi_3}{2} + 0.03 \\
\frac{\xi_3}{2} &= 0.12 \frac{\xi_1}{2} + 0.04 \frac{\xi_2}{2} + 0.24 \frac{\xi_3}{2} + 1.12 \\
\end{align*} \]

The solution by the classical method of successive approximation may be arranged as follows ([7], p.125):

\[ x_{n+1} = Hx_n + h. \]

### Method of Successive Approximation

<table>
<thead>
<tr>
<th>$H$</th>
<th>.22</th>
<th>.02</th>
<th>.12</th>
</tr>
</thead>
<tbody>
<tr>
<td>.02</td>
<td>.14</td>
<td>.04</td>
<td></td>
</tr>
<tr>
<td>.12</td>
<td>.04</td>
<td>.24</td>
<td></td>
</tr>
</tbody>
</table>

\[ \begin{align*}
\chi_0 &= h = 0.43 \\
\chi_1 &= 0.001040 + 0.021401 + 1.321011 \\
\chi_2 &= 0.044233 + 0.020034 + 1.323244 \\
\chi_3 &= 0.042013 + 0.020201 + 1.322334 \\
\chi_4 &= 0.042013 + 0.020243 + 1.322413 \\
\chi_5 &= 0.042021 + 0.020241 + 1.322413 \\
\end{align*} \]
We note that
\[ \text{ord } s_n > -\log_6 \|H\| \|h\| = 6 \]
so that \( x_n \) must be correct to 5 decimal places at least. A glance at
the previous result shows that this is indeed the case. An alternative
method of successive approximation (ibid, p.124) yields an exact measure
of the accuracy of each approximating vector.

**Method of Successive Approximation**

Formula: \[ x_n = \sum_{i=0}^{n} H^i h \]

<table>
<thead>
<tr>
<th>( h )</th>
<th>.43</th>
<th>.03</th>
<th>1.12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Hh )</td>
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<td>.040401</td>
<td>.201011</td>
</tr>
<tr>
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<td>.004023</td>
<td>.002233</td>
</tr>
<tr>
<td>( H^3h )</td>
<td>.003224</td>
<td>.000220</td>
<td>.000112</td>
</tr>
<tr>
<td>( H^4h )</td>
<td>.000000</td>
<td>.000003</td>
<td>.000021</td>
</tr>
<tr>
<td>( H^5h )</td>
<td>.000013</td>
<td>.000000</td>
<td>.000000</td>
</tr>
<tr>
<td>( H^6h )</td>
<td>.000004</td>
<td>.000000</td>
<td>.000000</td>
</tr>
</tbody>
</table>

| \( x_5 \)  | .042021 | .020241 | 1.322413 |
| \( x_6 \)  | .042020 | .020241 | 1.322413 |
Here,

\[ \text{ord } s_5 = \text{ord } h \cdot h = 6, \]

while

\[ \text{ord } s_6 \geq -\log_5 \|h\| \|h\| = 7. \]

Thus \( x_5 \) and \( x_6 \) must be correct to 5 and 6 decimal places respectively. The latter is precisely the accuracy of example 1, for rounding errors do not occur in arithmetical operations with \( p \)-adic numbers.

**Example 3**

The coefficient matrix \( A \) of example 1 may be inverted by Gaussian elimination. To 6 decimals we get

\[
A^{-1} = \begin{pmatrix}
1.222421 & .020320 & .113120 \\
.020320 & 1.000110 & .004002 \\
.113120 & .004002 & 1.20130
\end{pmatrix}
\]

As the initial approximation for an iterative solution we shall take

\[
X_0 = \begin{pmatrix}
1.22 & .02 & .11 \\
.02 & 1.10 & .04 \\
.11 & .04 & 1.24
\end{pmatrix}
\]

so that

\[
R_0 = \begin{pmatrix}
.003224 & .000124 & .002404 \\
.000214 & .000404 & .001323 \\
.002114 & .001204 & .000223
\end{pmatrix}
\]

Thus

\[ \text{ord } s_1 \geq -\log_5 \|X_0\| \|R_0\|^2 = 6, \]
and

\[ X_1 = X_0 + X_0 R_0 \]

\[
\begin{pmatrix}
1.222423 & .020323 & .113124 \\
.020323 & 1.100114 & .044002 \\
.113124 & .044002 & 1.240130
\end{pmatrix}
\]

is correct to 5 decimal places (cf [6], p.160).
Bibliography


