NATURAL CONVECTIVE HEAT TRANSFER

FOR VERTICAL CYLINDERS WITH TRANSVERSE MASS FLUX
NATURAL CONVECTIVE HEAT TRANSFER
FOR VERTICAL CYLINDERS WITH TRANSVERSE MASS FLUX

by

JAGJIT KUMAR KHOSLA, B. Sc. Ch. E.,

Submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Master of Engineering.

Dept. of Chemical Engineering
McGill University
Montreal
April 10, 1967

© Jagjit Kumar Khosla 1968
A numerical solution for heat transfer in natural convection around a vertical cylinder with uniform surface temperature and uniform transverse mass flux was obtained for a fluid of $Pr = 0.7$, using a perturbation technique to the solution for a flat plate. Nonlinear, partial differential equations for momentum, energy and mass transfer were reduced to ordinary differential equations by introducing transverse flux and curvature variables. Heat transfer rates were found to decrease with blowing and increase with suction. Increase of curvature increases the rate of heat transfer. A method for calculating the transverse mass flow when it is linked with the driving force causing natural convection has been suggested.
ACKNOWLEDGEMENTS

The author wishes to express his thanks and appreciation for their contributions in many ways towards the success of this project, to

Dr. W.J.M. Douglas, for his guidance and constructive criticism,

The Pulp and Paper Research Institute of Canada, for their financial support and the use of Library and other facilities,

Dr. A. Held for his invaluable help in the solution of equations on computer,

The Staff and Graduate Students of the Chemical Engineering Department, for their suggestions during the course of this study.
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Co-ordinate system for the cylinder</td>
<td>17</td>
</tr>
<tr>
<td>2.</td>
<td>$\theta$ as a function of $\eta$</td>
<td>46</td>
</tr>
<tr>
<td>3.</td>
<td>$f'$ as a function of $\eta$</td>
<td>47</td>
</tr>
<tr>
<td>4.</td>
<td>$\theta$, $\theta'$ and $\theta''$ as functions of $\eta$</td>
<td>49</td>
</tr>
<tr>
<td>5.</td>
<td>$f'$, $f''$ and $f'''$ as functions of $\eta$</td>
<td>50</td>
</tr>
<tr>
<td>6.</td>
<td>Third order term as percentage of leading terms</td>
<td>51</td>
</tr>
<tr>
<td>7.</td>
<td>Heat transfer results for blowing</td>
<td>53</td>
</tr>
<tr>
<td>8.</td>
<td>Heat transfer results for suction</td>
<td>54</td>
</tr>
<tr>
<td>9.</td>
<td>Comparison of local heat transfer results for a cylinder with flat plate heat transfer results</td>
<td>55</td>
</tr>
<tr>
<td>10.</td>
<td>Comparison of local heat transfer results for a cylinder with and without transverse mass flux</td>
<td>56</td>
</tr>
<tr>
<td>11.</td>
<td>Average heat transfer results</td>
<td>60</td>
</tr>
<tr>
<td>12.</td>
<td>Transverse mass transfer rates with and without heat transfer</td>
<td>62</td>
</tr>
</tbody>
</table>
GENERAL INTRODUCTION

Much attention has been devoted in recent years to the development of generalized analysis of the transport of mass, momentum and energy. However, the interaction of two or more transport processes remains a difficult problem to generalize.

The present study treats the case of simultaneous convective transport of mass, momentum and energy associated with the interaction of two, mutually interacting velocities in the presence or absence of heat transfer. This aspect of the transport phenomena arises in a wide variety of practical applications.

Evaporation of acetone from cellulose acetate thread in dry extrusion, the sweat cooling of the electrodes in an electric arc furnace, and the drying of paper are some of the cases where interaction of natural convection and transverse mass flow is important. In pulp and paper industry the rate of drying of webs is a problem of great importance. Pressure, tension and speed of the drying rollers depend on the strength and hence the moisture contents of the paper sheet.

In aerodynamics the application of suction on an aerofoil to increase lift and decrease drag by preventing transition of the boundary layer from laminar to turbulent has been investigated.

Blowing of air through molten metal helps to keep the temperature of its surface lower than that which would cause the formation of undesirable oxides.
In polymerisation reactions, a controlled rate of injection of one or more of the reactants along the length of a flow reactor with porous walls may help to obtain the desirable molecular weight of the polymer.

The structural elements of turbojets, rocket engines, exhaust nozzles, gas turbine blades, the combustion chamber walls of fuel cells, and the surfaces of spacecraft during reentry are subjected to high temperatures. Injection of a cold fluid from the solid walls may be used to obtain the 'heat blocking effect', and thus protect the surface from thermal damage. This process is called transpiration cooling.

Air-conditioning, evaporation, condensation of vapours on a solid surface, diffusion of reactants and products towards and away from the catalyst particles in a packed bed, distillation, adsorption and absorption and in fact most chemical engineering operations involve in one way or the other interaction between two perpendicular fluxes.

The orientation of the present study of natural convective heat transfer with uniform transverse mass flux around a vertical cylinder has been to obtain a numerical solutions of the boundary layer type equations for momentum and energy transport. A third order perturbation technique along with the Kutta-Simpson method was employed to solve the non-linear partial differential equations of momentum and energy transport at a Prandtl number of 0.7.
Introduction:

The analogy between mass, momentum and heat transfer is applicable only to a very limited range of cases. The problem of combined transport of mass, momentum and energy has been solved analytically and numerically for some simple geometries and flow patterns. Since the interaction of two or more transport processes introduces many complications, simpler situations are studied first.

Most of the analytical and numerical solutions obtained for simultaneous heat and mass transfer have been for a flat plate immersed in a parallel flow environment. Transport processes at the stagnation point of blunt objects have been considered. The effect of variable fluid properties (8, 23) and as a further complication, diffusion thermo and thermo diffusion effects (25, 26, 28, 29, 35) have been included in some studies. The interaction of thermal radiation and convection has also been treated by some workers (5, 14, 27). Similarly, the effect of curvature has been found to be significant (1, 2, 19, 21, 32).

A review of the literature concerning simultaneous heat and mass transfer in natural convection has been given by Li (9). Here only the analytical and numerical methods employed by various workers for cases relevant to the present problem will be discussed.

Heat Transfer without Mass Transfer in Natural Convection

This section gives a summary of the various approaches which have been used for the case when there is no transverse mass flux at the surface. This is included for the sake of completeness since the effects of mass transfer
have followed the studies of pure heat transfer. It is suggested that the reader may prefer to proceed to the page 7 where heat and mass transfer in natural convection are discussed.

Pure free convective heat transfer to a vertical flat plate has been investigated by Schuh (17) and Ostrach (15, 16). Ostrach (16) gave the similarity solution of the momentum and energy equations for a large number of Prandtl numbers corresponding to liquid metals, gases, liquids and very viscous fluids (Pr = 0.01, .72, 1,2,10,100,1000). Local heat transfer coefficients have been expressed as proportional to (Gr.Pr)^{1/4}; for air the value of proportionality constant is 0.548.

Sparrow and Gregg (20) studied the case of laminar pure free convective heat transfer with uniform heat flux along the surface of a vertical flat plate. Similarity solutions have been given for the range of Prandtl numbers from 0.1 to 100 and extrapolated to 0.01. The heat transfer results are expressed in terms of a modified Grashof number, (Gr*)^{1/4} where Gr* = (Gr)(Nu). These results are in good agreement to those obtained by the Von Karman-Pohlhausen method. It has also been noted that local Nusselt numbers for uniform heat flux are about 10 to 15% higher than those for the case of uniform surface temperature.

Most of the numerical solutions assume that the fluid properties, except for density, do not change much in the range of temperatures encountered. More correctly, the variation of viscosity, thermal conductivity, and specific heat with temperature should be included. Sparrow and Gregg (23) proposed that the
product of density with viscosity, and of density with thermal conductivity is independent of temperature. For this case it has been shown that the variable property, momentum and energy equations become identical to those for constant property fluid. A reference temperature $T_r$ has been proposed, where

$$T_r = T_w - 0.38 (T_w - T_\infty).$$

When properties (with the exception of coefficient of thermal expansion $\beta = \frac{1}{T_\infty}$) are evaluated at $T_r$, the heat transfer results are very close to the exact solutions for a variety of gases having different physical property temperature relationships.

Similarity solution for a non-isothermal vertical flat plate in pure free convective heat transfer is given by Sparrow and Gregg (22). Two families of surface temperature variations are considered (i) $T_w - T_\infty = N x^n$, and (ii) $T_w - T_\infty = M e^{M x}$, where $T_w$ and $T_\infty$ are the wall and environment temperatures respectively, $x$ is the distance from the leading edge, and $N, n, M, m$ are constants.

A unified analysis giving necessary and sufficient conditions for all possible temperature and heat flux distributions at the wall for a vertical flat plate and a vertical cylinder for which similarity solutions exist are given by Yang (34). General similarity transformations are assumed as follows:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x},$$

$$\eta = y \phi(x, t),$$

$$f(\eta) = \frac{\psi(x, y, t)}{\phi_2(x, t)}, \quad \theta(\eta) = \frac{G(x, y, t)}{G(x, t)}$$

where $u$ and $v$ are dimensionless velocities, $x$ and $y$ are dimensionless...
distances in longitudinal and transverse directions, \( \psi \) is the stream function, and \( G \) is the Grashof number. Substituting these transformations into non-dimensionalized, unsteady state momentum and energy equations, two ordinary differential equations in terms of \( f \) and \( \Theta \) are obtained (differentiation is with respect to \( \eta \)). For a similarity solution to exist, the coefficients of \( f \) and \( \Theta \) and their derivatives must be constants. These constants when evaluated at the prescribed conditions of temperature and heat flux distribution give the expressions for \( \phi_1 \) and \( \phi_2 \). Yang lists the situations which yield similarity solutions.

Millsaps and Pohlhausen (12) have given the solutions for laminar free-convective fluid motion produced by a heated vertical cylinder for which the thermal distribution at the outer surface varies linearly with distance from the leading edge. This surface temperature distribution permits the following similarity solution:

\[
\psi = zf(r) \quad \phi = \frac{z}{r} h(r)
\]

where \( \psi \) is the stream function, \( z \) is distance from the leading edge, \( r \) is distance in radial direction, and \( \Theta \) is the Grashof temperature. Substitution of the above transformation along with \( r = \exp(t) \) into the momentum and energy equations, reduces them from non-linear partial differential equation to non-linear ordinary equations. The first estimation of the initial values was made by assuming parabolic temperature and velocity profiles, with subsequent corrections for additional accuracy.
Heat and Mass Transfer in Natural Convection:

An analytical solution for laminar free convection on a vertical flat plate for heat and mass transfer is given by Somers (18). Using the Von Karman-Pohlhausen method, the integral forms of the momentum, diffusion and energy equations were used to evaluate the constants. The inertia terms were neglected in order to simplify the mathematical treatment. Somers considered different thicknesses for the momentum, thermal, and the concentration boundary layers. As a simplification, he also considered the case for which the momentum boundary layer thickness is equal to either the concentration or to thermal boundary layer thickness, depending upon whether the convection currents are caused by a concentration or a temperature difference. With the assumption of similarity of velocity, concentration and temperature profiles in the boundary layer, the profiles were approximated by third order polynomials of \((y/\delta), (y/\delta'), (y/\delta'')\), respectively where \(y\) is distance in the transverse direction and \(\delta, \delta', \delta''\) are the respective boundary layer thicknesses. The variable \(u_x\) (maximum velocity), \(\delta'\) and \(\delta''\) were taken as functions of distance from the leading edge, \(x\), as follows

\[
\begin{align*}
  u_x &= \alpha_1 x^{1/2} \\
  \delta' &= \alpha_2 x^{1/4} \\
  \delta'' &= \alpha_3 x^{1/4}
\end{align*}
\]

where \(\alpha_1, \alpha_2, \alpha_3\) are the constants to be evaluated. The mass transfer Nusselt number, \(\text{Nu}_{\text{vx}}\), and heat transfer Nusselt number \(\text{Nu}_x\) were given as functions of the mass and heat transfer Grashof numbers \(\text{Gr}_{\text{vx}}\) and \(\text{Gr}_x\) respectively.

\[
\begin{align*}
  \text{Nu}_{\text{vx}} &= A(\text{Gr}_x + \text{Gr}_{\text{vx}}(\text{Pr}/\text{Sc})^{1/2})^{1/4} \\
  \text{Nu}_x &= B(\text{Pr}/\text{Sc})^{1/2} \text{Nu}_{\text{vx}}
\end{align*}
\]

where \(A\) and \(B\) are functions of \(\text{Pr}, \text{Sc}\) and concentration at the wall.
Mathers et. al. (11) have obtained the solution for a flat plate by solving the mass, momentum and energy transport equations on an analogue computer. Only the case of zero transverse velocity at the surface (applicable to the case of equal-molal counter diffusion or to heat transfer accompanied by small mass transfer rates) was considered. Since the inertia terms were neglected, the results are valid only for a Prandtl number of the order of 100 or higher, i.e. for liquids. The heat and mass transfer results were given in the form

\[
\begin{align*}
\text{Nu} &= 0.67 \left[ (Gr + (Pr/Sc)^{1/2} Gr') Pr \right]^{-1/4} \\
\text{Nu}' &= 0.67 \left[ (Gr' + (Sc/Pr)^{1/2} Gr) Sc \right]^{1/4}
\end{align*}
\]

where Nu, Nu', Gr, Gr' are the heat and mass transfer Nusselt numbers and Grashof numbers.

Wilcox (33) tried to improve on the solutions of Somers (18) and Mathers et. al. (11) who had both neglected the inertia terms. In the solution of the integral forms of the mass, momentum and energy equations for a vertical flat plate, Wilcox assumed velocity, concentration and temperature profiles of a more simplified form than those taken by Somers (18) in order that the inertia terms could be included. The cases of zero and non-zero wall velocities were considered, and for the latter, both the conditions of a constant wall temperature and an adiabatic wall were evaluated. The results for the zero wall velocity case agree to those given by Mathers et.al. (11) at high values of Sc and Pr, and when Lewis number (Sc/Pr) is approximately equal to one.
Similarity solutions for heat transfer to a vertical flat plate for the effect of mass transfer at $Pr = 0.73$ are given by Eichhorn (6). Similarity conditions require that the blowing or suction rate vary as the distance from the leading edge, $x$, raised to the power $(n-1)/4$, i.e. $v_w \propto x^{n-1}/4$, where $n$ in turn is the exponent in the power law surface temperature. For $n = 0$, this requirement implies that the blowing rate at the leading edge should be infinite. In practice, injection of a fluid through a porous plate with reasonable pressure drop is likely to produce a uniform transverse flux distribution over the surface. A uniform flux corresponds to $n = 1$, which would require the temperature distribution to be $T_w = T_\infty + Bx$. This temperature distribution does not correspond to any case of practical interest. Thus a similarity solution for uniform velocity at the surface is of no interest, but for the case of a uniform wall temperature, a similarity solution will be valid except near the leading edge.

Quite a different approach was tried by Nakamura (13), who used the method of successive integration and approximation for free convection on a vertical flat plate accompanied by vapour transfer. The zeroth approximation was taken from the solution for pure natural convective heat transfer given by Ostrach (16). Results were presented for different rates of evaporation and condensation for $Pr = 0.72$ and for $Sc = 0.5, 0.6$ and $0.72$.

The integral solutions given by Somers (18) and Wilcox (33) are valid only for a constant surface temperature. Mabuchi (19) extended this approach by accounting for the effect of Prandtl number and for the effect of the wall temperature
distribution $T_w = T_\infty + A x^n$ for the range $0 \leq n \leq 1$. The integral forms of conservation equations were solved with the help of assumed dimensionless velocity and temperature profiles expressed as a power series of $y/\delta$, the ratio of transverse distance to boundary layer thickness.

$$\frac{u}{u_x} = (y/\delta) - 3(y/\delta)^2 + 3(y/\delta)^3 - (y/\delta)^4$$

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{2}{1-w} (y/\delta) + \frac{6w}{1-w} (y/\delta)^2 + \frac{2-8w}{1-w} (y/\delta)^3 - \frac{1-3w}{1-w} (y/\delta)^4$$

where $u_x$ is the unknown non-dimensionalizing velocity which is eliminated in the process of solution, and $w$ is a parameter introduced to satisfy the wall temperature profile boundary condition. These approximate similar solutions impose restrictions on transverse velocity Reynolds number as follows:

For $n = 0$

$$\left| \frac{V_{w,x}}{v} \right| \leq 0.6 \left[ \frac{Gr_x}{4} \right]^{1/4} \quad \text{for} \quad Pr = 0.73$$

$$\left| \frac{V_{w,x}}{v} \right| \leq 0.5 \left[ \frac{Gr_x}{4} \right]^{1/4} \quad \text{for} \quad Pr = 1.0$$

For $n = 1$

$$\left| \frac{V_{w,x}}{v} \right| \leq 0.8 \left[ \frac{Gr_x}{4} \right]^{1/4} \quad \text{for} \quad Pr = 0.73$$

$$\left| \frac{V_{w,x}}{v} \right| \leq 0.65 \left[ \frac{Gr_x}{4} \right]^{1/4} \quad \text{for} \quad Pr = 1.0$$
where \( v_w \) is the transverse mass velocity, \( \nu \) is the kinematic viscosity, and \( Gr_x \) is the local Grashof number at a distance \( x \) from the leading edge.

The heat transfer solution for an isothermal wall at \( Pr = 0.73 \) is in good agreement with the exact solution given by Eichhorn (6) for the case of strong suction and moderate blowing. It has also been concluded that local Nusselt number depends on the upstream history of the surface temperature at constant blowing or suction. The ratio of local Nusselt numbers with and without transverse mass flux is relatively independent of the details of the upstream blowing or suction distribution for a constant surface temperature.

A perturbation technique has been employed by Sparrow and Cess (24) for natural convective heat transfer, with uniform blowing or suction, to a flat plate having a constant wall temperature. These conditions do not have a similarity solution. The stream function and dimensionless temperature were expanded as a power series in terms of a blowing parameter in the form of a dimensionless injection or suction velocity \( \frac{v_w x^{1/4}}{4\nu c} \). Only the first order approximation has been taken into account. For \( Pr = 0.72 \), the ratio of heat transfer with transverse velocity to non-blowing case is given by

\[
\frac{q}{q_0} = 1 - 2.77 \left( \frac{v_w x^{1/4}}{4\nu c} \right)
\]

Heat transfer increases with suction while decreases with blowing. Comparing these results with those given by Eichhorn (6), it is concluded that the boundary layer has 'poor memory' i.e. it is not very sensitive to the distribution of
transverse mass flux at the surface. Since the perturbation method used by Sparrow and Cess (24) is valid only for small values of the blowing parameter, this solution cannot give the asymptotic solution which exists for large suction rates. Transformations were suggested for the case when the power law distribution of temperature and of transverse mass flux at the surface are independent of each other.

Brdlik and Mochalov (3) solved the integral forms of momentum and energy equations for uniform temperature and uniform transverse mass flux at the surface of a vertical flat plate in a still atmosphere. The velocity and temperature profiles were assumed as fourth order polynomials of the ratio of transverse distance to the boundary layer thickness. These results, when compared with those of Sparrow and Cess (24), are found to be 10% higher for strong suction and 4.5% lower for strong blowing. Brdlik and Mochalov noted that these discrepancies might be reduced if higher order terms were added to the perturbation method. These also determined that the effect of transverse mass flux increased with an increase in Prandtl number.
Curvature Effect

Keeping all the system condition same, when a flat plate, which may be considered as a cylinder of infinite radius, is bent to a cylinder of finite radius, heat and mass transfer rates are changed. The ratio of boundary layer thickness to the cylinder radius, named as curvature parameter, has been used by Seban and Bond (19) as a criterion for the effect of curvature. Solutions for Prandtl number 0.715 have been obtained for laminar forced convection parallel to cylinder axis. Some numerical correction have been given by Kelly (7) to the solutions of Seban and Bond (19).

For laminar forced convection solutions for heat transfer to a cylinder for blowing or suction have been considered by Wanous and Sparrow (32) using the approach of Seban and Bond (19). It is observed that the effect of blowing or suction distribution at the surface does not influence very much the rates of heat transfer for blowing as well as for suction i.e. the boundary layer has poor memory.

Bourne and Davies (2) have obtained the solution for the case of laminar forced convection axial flow for very long and thin vertical cylinders. Effect of curvature and Prandtl number on local heat transfer rates has been shown by comparing heat transfer rates for the flat plate.

Cylinders with arbitrary cross-section have been considered by Bourne and Wardle (1) for forced convective heat transfer. Equivalent curvature parameter has been defined such that cross-section of elliptic cylinder is mapped conformally into that of right circular cylinders.
For natural convection Sparrow and Gregg (21) gave a laminar boundary layer solution for a vertical cylinder with uniform surface temperature, for Prandtl number of 0.72 and 1.0, using a perturbation technique. Not large derivations have been observed when comparing these results with those obtained by stagnation layer analysis. Limits on the curvature parameter have been defined within which flat plate solution may be applied to a cylinder without more than 5 percent error.

\[
\frac{2^{3/2}}{(Gr/4)^{1/4}} \left( \frac{x}{r_o} \right) \leq 0.11 \quad \text{for } Pr = 0.72
\]

where \( x \) is the distance from the leading edge and \( r_o \) is the radius of cylinder.

In the present study the effect of uniform transverse mass flux on a vertical cylinder in natural convection is obtained by solving the momentum, mass and energy equation numerically. This solution is extended to include the case when transverse mass flux rate is determined by the driving force causing natural convection.
ANALYSIS OF THE SYSTEM

Introduction

When the natural convective flow around a solid is altered by the addition of flow in transverse direction it is necessary to distinguish whether or not the transverse flow is directly linked to the heat transfer rate. The rate of transverse flow may be controlled independently in case of pressure injection or suction of the fluid through a solid with porous walls. In other cases, however, the transverse flux depends on the driving force causing natural convection, which would be the temperature difference in case of evaporation or condensation, and concentration difference in case of absorption or desorption. In case of injection or suction, the transverse flow at the surface is, for practical reasons, likely to be uniform, while for the case of evaporation and absorption the flux may be a function of position because of the coupling of the transport processes. In the present work the solution of the boundary layer equations is obtained for the case of uniform transverse flow interacting with natural convective flow at the surface of a vertical cylinder. This solution is further extended to the case where transverse flow is controlled by the driving force causing natural convection.

Since a flat plate may be regarded as a cylinder with infinite radius, the heat transfer solution for a cylinder may be expressed as an expansion series around the flat plate solution to give a third order correction for the curvature effect. A similarity solution, in which two velocity profiles $\bar{u}(x,\tau)$ at two different positions of $\tau$ differ only by a scale factor in $\bar{u}$ and $\bar{\tau}$, does not exist for the case of uniform transverse flow. The solution in the
present case therefore contains correction terms for the curvature effect, for the transverse flow effect, and for the interaction between the two.

**Mass, Momentum and Energy Equations for Uniform Transverse Flow.**

The co-ordinates used for a semi-infinite right circular cylinder of radius, a, suspended vertically in a constant temperature still atmosphere are indicated on Figure 1. The fluid velocities \( \bar{u} \) and \( \bar{v} \) are in the directions \( \bar{x} \) and \( \bar{r} \), respectively. The transverse velocity at the solid surface is \( \bar{v}_w \).

The environment and wall temperatures \( T_\infty \) and \( T_w \) respectively, and the fluid properties are assumed to be constant. For the variable fluid property case, a suitable reference temperature for their evaluation has been given by Sparrow and Gregg (23). The boundary layer thickness at the leading edge is taken as zero. The conservation of mass, momentum and energy can be represented by equations 1, 2, 3, respectively.

\[
\frac{\partial}{\partial x}(\bar{u} \bar{r}) + \frac{\partial}{\partial r} (\bar{v} \bar{r}) = 0 \tag{1}
\]

\[
\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial r} = g \beta (T_\infty - T_w) + \frac{\nu}{r} \frac{\partial}{\partial r} (r \frac{\partial \bar{u}}{\partial r}) \tag{2}
\]

\[
\bar{u} \frac{\partial T}{\partial x} + \bar{v} \frac{\partial T}{\partial r} = \frac{\alpha}{r} \frac{\partial}{\partial r} (r \frac{\partial T}{\partial r}) \tag{3}
\]

where \( g \) is the acceleration due to gravity, \( \beta \) is the volumetric coefficient of thermal expansion, \( \nu \) is the kinematic viscosity, and \( \alpha \left( \frac{k}{\rho c_p} \right) \) is the thermal diffusivity.

Equation 2 is a simplified form of the steady state \( \bar{x} \)-component,
FIG. 1
Co-ordinate System for the cylinder
Navier-Stokes equation for a Newtonian fluid around an axi-symmetrical body with zero rotational velocity. Also, the Prandtl boundary layer assumption has been applied i.e.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) > > \frac{\partial^2 u}{\partial x^2} \]

Equation 3 is a simplified form of the steady state thermal energy conservation equation, neglecting

(a) viscous dissipation
(b) compression or expansion effects
(c) thermo diffusion and diffusion thermo effects.

It has also been assumed that \( k \frac{\partial^2 T}{\partial x^2} \) is negligible which would not be true in the case of flow of molten metals where thermal conductivity of the fluid is very high.

For the purpose of obtaining geometric and thermal similarity, equation 1,2,3 are non-dimensionalized with the following definitions:

\[
\begin{align*}
\bar{r} &= \frac{r}{a}, \quad \bar{x} = \frac{x}{a}, \quad \bar{u} = \frac{u}{a}, \quad \bar{v} = \frac{v}{a} \\
\Theta &= \frac{T_\infty - T}{T_\infty - T_w}, \quad T = T_\infty - \Theta(T_\infty - T_w) 
\end{align*}
\]

where \( r \) and \( x \) are dimensionless lengths, \( u \) and \( v \), dimensionless velocities and \( \Theta \), a dimensionless temperature. Substitution of equations 4 and 5 in equations 1,2,3 gives:
\[ \frac{\partial}{\partial x} (ur) + \frac{\partial}{\partial r} (vr) = 0 \] (6)

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{g \beta a^3 (T_\infty - T)}{v^2} + \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) \] (7)

\[ u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial r} = \frac{1}{Pr} \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial \theta}{\partial r}) \] (8)

Boundary conditions associated with these equations are:

At \( \bar{r} = a \): \( r = 1, \bar{u} = u = 0 \) (the no slip condition),

\[ \bar{v} = \bar{v}_w, \quad v = v_w, \quad T = T_w, \quad \text{and} \quad \theta = 1 \] (9)

At \( \bar{r} = \infty \): \( r = \infty, \quad T = T_\infty, \quad \theta = 0, \quad \text{and} \)

\[ \bar{u} = u = 0 \) (still enviroment) \] (10)

Equations 7,8 are non-dimensionalized, non-linear partial differential equations representing momentum and energy transport.

Definition of the stream function according to equation 11 satisfies the mass conservation equation 6

\[ u = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r} \frac{\partial \psi}{\partial x} \] (11)

Substitution of equation 11 into the momentum and energy equations 7,8 results in equations 12,13.
\[
\frac{1}{r^2} \left( \frac{\partial^2 \psi}{\partial x^2} \frac{1}{\partial x} + \frac{1}{\partial x^2} \right) - \frac{1}{r^2} \left( \frac{\partial^2 \psi}{\partial r^2} \frac{1}{\partial r} + \frac{1}{\partial r^2} \right) = \Gamma + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right)
\]  
\hspace{1cm} (12)

\[
\frac{\partial \psi}{\partial r} \frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial r} = \frac{1}{Pr} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right)
\]  
\hspace{1cm} (13)

where \( \Gamma = \frac{g \beta (T_\infty - T)a^3}{\nu^2} \), \( Pr = \frac{\nu}{\alpha} \)

If \( G \) is defined as \( G = \left[ \frac{g \beta (T_\infty - T)a^3}{4 \nu^2} \right]^{1/4} \)

Then \( \theta = \frac{\Gamma}{4G^4} \).

**Transformation of Space Co-ordinates**

A transformation of space co-ordinates is made in order to introduce the curvature effect of the cylinder. The size of this curvature parameter, \( \xi \), will facilitate the comparison of heat transfer results for a cylinder with those for a flat plate.

Transformations 14, 15, 16 are used to change the space coordinates in the momentum and energy equations 12, 13, from \( x \) and \( r \), to \( \xi \) and \( \eta \) respectively.

\[
\psi = G x^{3/4} f(\eta, \xi) - v_w x
\]  
\hspace{1cm} (14)

\[
\eta = G x^{-1/4} (r^2 - 1)
\]  
\hspace{1cm} (15)

\[
\xi = \frac{x^{1/4}}{G}
\]  
\hspace{1cm} (16)
\[
\frac{\partial \eta}{\partial r} = 2Gr \ x^{-1/4}
\]  \hspace{1cm} (17)

\[
\frac{\partial \eta}{\partial x} = -\frac{\eta}{4 \ x} \quad \text{and} \quad \frac{\partial \xi}{\partial x} = \frac{\xi}{4 \ x}
\]  \hspace{1cm} (18)

Differentiation of \( \psi \) and \( \Theta \) with respect to \( x \) and/or \( r \) is expressed in terms of differentiation of \( f \) with respect to \( \eta \) and \( \xi \) in equations 17 through 26. Primes denote differentiation with respect to \( \eta \).

\[
\frac{\partial \psi}{\partial r} = Gf \ 3/4 \left[ \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial r} \right] = 2G^2 \ x^{-1/2} \frac{\partial f}{\partial \eta} = 2G^2 \ x^{1/2} \ f'
\]  \hspace{1cm} (19)

\[
\frac{\partial \psi}{\partial x} = Gf \ 3/4 \ x^{-1/4} + Gx \ 3/4 \left[ \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} \right] - v_w
\]

\[
\frac{\partial \psi}{\partial x} = \frac{3}{4} Gf \ x^{-1/4} - \frac{1}{4} G \eta x^{-1/4} f' + \frac{1}{4} \frac{\partial f}{\partial \xi} - v_w
\]  \hspace{1cm} (20)

\[
\frac{\partial^2 \psi}{\partial x \partial r} = \frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial r} \right) = G^2 x^{-1/2} (f' - \frac{1}{2} \eta f'' + \frac{1}{2} \xi \frac{\partial^2 \xi}{\partial \xi \partial \eta}) r
\]  \hspace{1cm} (21)

\[
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) = 4G^3 \ x^{1/4} \ f''
\]  \hspace{1cm} (22)

\[
\frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right) = 8G^4 \ x \left[ (1+\xi \eta) f'' + \xi \ f''' \right]
\]  \hspace{1cm} (23)

From equation 5

\[
\frac{T_\infty - T}{T_\infty - T_w} = \Theta(\eta, \xi)
\]
\[ \frac{\partial \theta}{\partial r} = \frac{\partial \theta}{\partial \eta} \left( \frac{\partial \eta}{\partial r} \right) = \theta' 2 \theta r^{-1/4} \] (24)

\[ \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) = 4 \theta^2 r^{-1/2} \left[ (1 + \xi \eta) \theta'' + \xi \theta' \right] \] (25)

\[ \frac{\partial \varrho}{\partial x} = \frac{\partial \varrho}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \varrho}{\partial \xi} \frac{\partial \xi}{\partial x} \]

\[ \frac{\partial \varrho}{\partial x} = \theta' \left( - \frac{1}{4} \frac{\partial \eta}{\partial x} \right) + \frac{\partial \varrho}{\partial \xi} \left( \frac{1}{4} \frac{\partial \xi}{\partial x} \right) \] (26)

Substitution of equations 17 through 26 in equations 12 and 13 results in equations 27 and 28, which are the transformed forms of the momentum and energy transport equations.

\[ 2(1 + \xi \eta) \theta'' + (2 \xi + 0.25 \xi^2) \frac{\partial f}{\partial \xi} - \xi v_w + 0.75 f \theta'' \]

\[ - 0.5(f')^2 - 0.25 f' \frac{\partial^2 f}{\partial \xi \partial \eta} + \theta = 0 \] (27)

\[ \frac{4}{Pr} (1 + \xi \eta) \theta'' + \left( \frac{4}{Pr} \xi + 1.5 f - 2 \xi v_w + 0.5 \xi \frac{\partial f}{\partial \xi} \right) \theta' \]

\[ - 0.5 f' \frac{\partial \theta}{\partial \xi} = 0 \] (28)

Substitution of equations 14, 15, 16 in equation 11 gives longitudinal and transverse velocities in space co-ordinates \( \gamma \) and \( \xi \).

\[ u = \frac{1}{r} \frac{\partial \psi}{\partial r} = 2 \theta^2 x^{1/2} f' \] (29)

\[ v = - \frac{1}{r} \frac{\partial \psi}{\partial x} = - \left[ 0.75 \theta G f x^{-1/4} - 0.25 G \theta x^{-1/4} f' \right] 

+ 0.25 \frac{\theta}{\theta} - v_w \frac{1}{r} \] (30)
The boundary conditions 9,10 can be written as

At \( \eta = 0 : \ r = 1, \ u = 0, \) hence \( f'(0) = 0 \)

\[ T = T_w, \] hence \( \Theta = 1 \)

If we choose \( f(0) = 0 \) and \( \frac{\partial f}{\partial \xi} \bigg|_{\eta=0} = 0 \)

then the condition \( v = v_w \) at \( r = 1 \) is satisfied in equation 30.

At \( \eta = \infty : \ u = 0 \). . . \( f'(\infty) = 0 \)

\( T = T_\infty \). . . \( \Theta(\infty) = 0 \).

**Perturbation Technique**

A perturbation technique is used to evaluate the heat transfer results for a cylinder in terms of correction factors applied to the solution for a flat plate.

From equation 16, \( \xi = \frac{\bar{x}}{\bar{G}} = \frac{1}{a} \left[ \frac{\bar{x}}{\left( \frac{Gr}{4} \right)^{1/4}} \right], \) where \( \frac{\bar{x}}{\left( \frac{Gr}{4} \right)^{1/4}} = 8 \)

is the boundary layer thickness at a distance \( \bar{x} \) from the leading edge and so \( \xi \) is the ratio of boundary layer thickness to the radius of the cylinder. It can be readily shown that the ratio of conduction heat transfer in a fluid layer around a vertical cylinder to that on a vertical flat plate (of the same width as the periphery of the cylinder) is a function of \( \frac{\text{Fluid layer thickness}}{\text{Radius of Cylinder}} \). Therefore \( \xi = \frac{8}{a} \) can be treated as a measure of the curvature effect caused by
the cylinder. For small values of \( \xi \), the functions \( f \) and \( \theta \) can be expanded in series by a third order perturbation technique, around the flat plate solution to accommodate the curvature effect.

\[
 f(\xi, \eta) = f_{oo}(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \xi^3 f_3(\eta) + \ldots \quad (32)
\]

\[
 \theta(\xi, \eta) = \theta_{oo}(\eta) + \xi \theta_1(\eta) + \xi^2 \theta_2(\eta) + \xi^3 \theta_3(\eta) + \ldots \quad (33)
\]

where \( f_{oo}(\eta) \) and \( \theta_{oo}(\eta) \) give the flat plate solution and \( f_1, f_2, f_3 \) and \( \theta_1, \theta_2, \theta_3, \ldots \) are functions created to account for the curvature effect. It has been assumed that the fourth and higher powers of \( \xi \) will not make any significant contribution to the values of \( f \) and \( \theta \). Substitution of equations 32 and 33 in equations 27 and 28, neglecting powers of \( \xi \) higher than three, and arranging in order of ascending powers of \( \xi \), results in equations 34 and 35.

\[
\xi^0 \left[ 2f_{oo}''' + .75f_{oo}'' - 0.5(f_{oo}')^2 + \theta_{oo}' \right] + \xi^1 \left[ 2f_1''' + 2f_{oo}'' + 2f_{oo}''' - \nu f_{oo}'' \right] \\
+ 0.75 f_{oo}'' + f_{oo}'' - 1.25f_1' + \nu f_{oo}' + \theta_1 \right] + \xi^2 \left[ 2f_2''' + 2f_2' + 2f_2'' - \nu f_2'' \right] \\
+ .75f_{oo}'' + f_{oo}'' + 1.25f_{oo}'' - .75(f_1')^2 - 1.5f_{oo}f_1' + \theta_2 \right] \\
+ \xi^3 \left[ 2f_3''' + 2f_3' + 2f_3'' - \nu f_3'' + .75f_{oo}''' + f_{oo}'' + 1.25f_{oo}f_1' + 1.5f_{oo}f_{oo}'' \right] \\
- 1.75f_{oo}f_1' - 1.75f_{oo}f_1' + \theta_3 \right] = 0 \quad (34)
\]
\[
\xi \left[ \frac{4}{\Pr} \theta''_{oo} + 1.5f_{oo} \theta'_{oo} \right] + \xi \left[ \frac{4}{\Pr} \theta''_{1} + \gamma \theta''_{1} + \theta'_{1} \right] - 2v \theta'_{w} + 1.5f_{1} \theta'_{1} + 1.5f_{2} \theta'_{2} \\
+ 0.5f_{1} \theta'_{2} - 0.5f_{2} \theta'_{1} + 2v \theta'_{1} + 2f_{1} \theta'_{1} + 2.5f_{2} \theta'_{2} \\
+ 1.5f_{1} \theta'_{2} - 0.5f_{1} \theta'_{3} - 0.5f_{2} \theta'_{1} + 2f_{1} \theta'_{1} + 2.5f_{2} \theta'_{2} \\
- 2v \theta'_{w} + 2f_{1} \theta'_{1} + 1.5f_{1} \theta'_{3} - 1.5f_{2} \theta'_{1} - 0.5f_{1} \theta'_{2} = 0 \quad (35)
\]

For the solution to be true for all values of \( \xi \), coefficients of various powers of \( \xi \) should individually be equal to zero. This results in eight equations 36 through 43. Since \( \gamma \) is the only variable, the primes in equations 36 through 43 and for all subsequent equations may be taken as denoting total differentiation.

\[
2f''_{oo} + 0.75f f''_{oo} - 0.5(f'_{oo})^2 + \theta''_{oo} = 0 \quad (36)
\]

\[
2f''_{1} + 2\gamma f''_{0} + 2f''_{oo} - v f''_{w} + 0.75f f''_{1} + f f''_{1} - 1.25f \theta''_{1} + \theta'_{1} = 0 \quad (37)
\]

\[
2f''_{2} + 2\gamma f''_{1} + 2f''_{2} - v f''_{w} + 0.75f f''_{2} + f f''_{2} + 1.25f f''_{2} - 0.75(f'_{1})^2 \\
- 1.5f' \theta''_{2} + \theta'_{2} = 0 \quad (38)
\]

\[
2f''_{3} + 2\gamma f''_{2} + 2f''_{3} - v f''_{w} + 0.75f f''_{3} + f f''_{3} + 1.25f f''_{3} + 1.5 f' f''_{3} \\
- 1.75 f' \theta''_{3} - 1.75f' \theta''_{2} + \theta'_{3} = 0 \quad (39)
\]
\[ 4\theta''_{oo} + 1.5(\text{Pr}) f_{oo} \theta'_{oo} = 0 \] \hfill (40)

\[ 4(\theta''_1 + \eta \theta''_{oo} + \theta'_{oo}) + \text{Pr}[- 2v \theta'_{oo} + 1.5f_1 \theta'_{oo} + 1.5f_{oo} \theta'_{1} + .5f_1 \theta'_{1} \nonumber \]

\[ - .5f_{oo} \theta'_{1} = 0 \; \hfill (41) \]

\[ 4(\theta''_2 + \eta \theta''_{oo} + \theta'_{oo}) + \text{Pr}[- 2v \theta'_{1} + 2f_1 \theta'_{1} + 2.5f_2 \theta'_{1} + 1.5f_{oo} \theta'_{2} \nonumber \]

\[ - .5f_{1} \theta'_{1} - f'_{oo} \theta'_{2} = 0 \; \hfill (42) \]

\[ 4(\theta''_3 + \eta \theta''_{oo} + \theta'_{oo}) + \text{Pr}[3f_3 \theta'_{oo} + 2.5f_2 \theta'_{1} - 2v \theta'_{2} + 2f_1 \theta'_{2} \nonumber \]

\[ + 1.5f_{oo} \theta'_{3} - 1.5f'_{oo} \theta'_{3} - f'_{1} \theta'_{2} - .5f'_{2} \theta'_{1} = 0 \; \hfill (43) \]

It is obvious that first equations 36 and 40 must be solved simultaneously, after which equations 37,41; 38,42; and 39,43 may be solved successively in pairs.

Equations 36 and 40 are non-linear simultaneous ordinary differential equations with two point boundary conditions. They represent the solution for heat transfer for a flat plate without transverse mass flux. Equations 37,38, 39,41,42,43 are inhomogeneous ordinary differential equations. For example, equation 37 is a third order equation in \( f_1 \). Since the inhomogeneous terms (not containing \( f_1 \) or \( \theta'_{1} \)) \( f_0 \) and its derivatives are now known functions of \( \eta \),
their values are substituted from the solution of equations 36 and 40. It can be easily shown from the properties of inhomogeneous equations that if \( f_{11} \) is a solution of a part of equation 37 (all terms except \( v_w f'' \)) when equated to zero i.e. equation 44

\[
2f''_{11} + 2\eta f''_{11} + 2f''_{11} + 0.75f''_{11} + f''_{11} + 1.25f'_{11} + \theta_{11} = 0 \quad (44)
\]

and that if \( v_w f_{12} \) is a solution of another part of equation 37 (the homogeneous part and the term \( v_w f'' \)) when that part is equated to zero, i.e. equation 45

\[
2f''_{11} + 0.75f''_{11} + f''_{11} + 1.25f'_{11} + \theta_{11} - v_w f''_{11} = 0 \quad (45)
\]

then \( f_{11} + v_w f_{12} \) is the solution of equation 37, i.e. \( f_1 = f_{11} + v_w f_{12} \) (46)

Using this approach one may obtain for equations 38, 39, 41, 42 and 43, the following definitions for \( f_2, f_3, \theta_1, \theta_2 \) and \( \theta_3 \)

\[
f_2 = f_{21} + f_{22} v_w + f_{23} v_w^2 \quad (47)
\]

\[
f_3 = f_{31} + f_{32} v_w + f_{33} v_w^2 + f_{34} v_w^3 \quad (48)
\]

\[
\theta_1 = \theta_{11} + \theta_{12} v_w \quad (49)
\]

\[
\theta_2 = \theta_{21} + \theta_{22} v_w + \theta_{23} v_w^2 \quad (50)
\]
\[
\theta_3 = \theta_{31} + \theta_{32}v_w + \theta_{33}v_w^2 + \theta_{34}v_w^3 \tag{51}
\]

where \( f_{11}', f_{12}', f_{21}', f_{22}', f_{31}', f_{32}', f_{33}', f_{34}' \) and \( \theta_{11}', \theta_{12}', \theta_{21}', \theta_{22}', \theta_{23}' \), \( \theta_{31}', \theta_{32}', \theta_{33}', \theta_{34}' \) all are functions of \( \gamma \) only.

Substitution of equations 46 through 51 in equations 32 and 33 results in

\[
f(\xi, \eta) = f_{oo}(\eta) + \xi(f_{11}(\eta) + v_w f_{12}(\eta)) + \xi^2(f_{21}(\eta) + v_w f_{22}(\eta)) + v_w^2 f_{23}(\eta) + \xi^3(f_{31}(\eta) + v_w f_{32}(\eta) + v_w^2 f_{33}(\eta) + v_w^3 f_{34}(\eta)) + \ldots \tag{52}
\]

\[
\theta(\xi, \eta) = \theta_{oo}(\eta) + \xi(\theta_{11}(\eta) + v_w \theta_{12}(\eta)) + \xi^2(\theta_{21}(\eta) + v_w \theta_{22}(\eta) + v_w^2 \theta_{23}(\eta)) + \xi^3(\theta_{31}(\eta) + v_w \theta_{32}(\eta) + v_w^2 \theta_{33}(\eta) + v_w^3 \theta_{34}(\eta)) + \ldots \tag{53}
\]

The substitution of equations 46 through 51 in equations 36 through 45, combined with the requirement that, for the solution to be true for all values of \( v_w \), the coefficients of all powers of \( v_w \) should individually be equal to zero, results in a set of twenty ordinary differential equations, (54) through (73).
\[ f_{oo}'''' = -0.5\{0.75 f_{oo} f_{oo}'' - 0.5(f_{oo}')^2 + \theta_{oo} \} \]  \hspace{1cm} (54) \\
\[ \theta_{oo}'''' = -0.25[1.5(Pr) f_{oo} \theta_{oo}'] \]  \hspace{1cm} (55) \\
\[ f_{11}'''' = -0.5\{ (0.75 f_{oo} f_{11}'' + f_{11} f_{11}'' - 1.25 f_{oo} f_{11}''' + \theta_{11}') + \]  \hspace{2cm} + 2 \eta (f_{oo}'''' + f_{11}''''') \} \]  \hspace{1cm} (56) \\
\[ \theta_{11}'''' = -0.5\{ Pr(2f_{11} \theta_{11}' + 1.5f_{oo} \theta_{11}' - 0.5f_{oo} \theta_{11}'') + 4 \eta(\theta_{oo}'''' + 4\theta_{11}') \} \]  \hspace{1cm} (57) \\
\[ f_{12}'''' = -0.5\{ (0.75 f_{oo} f_{12}'' + f_{12} f_{12}'' - 1.25 f_{oo} f_{12}''' + \theta_{12}') - f_{oo}'''' \} \]  \hspace{1cm} (58) \\
\[ \theta_{12}'''' = -0.25(Pr)[ (1.5f_{oo} \theta_{12}' - 0.5f_{oo} \theta_{12}'') + 2f_{12} \theta_{12}'') - 2\theta_{12}''] \]  \hspace{1cm} (59) \\
\[ f_{21}'''' = -0.5\{(-1.5f_{oo} f_{21}' + 1.25f_{21} f_{21}'' + 0.75f_{oo} f_{21}''') + f_{11} f_{11}''' + 2f_{11}'''' - \]  \hspace{1cm} - 0.75(f_{11}')^2 + 2 \eta f_{11}''' + \theta_{21}' \} \]  \hspace{1cm} (60) \\
\[ \theta_{21}'''' = -0.25\{ Pr: (2.5f_{21} \theta_{21}' - f_{21} \theta_{21}' + 1.5 f_{oo} \theta_{21}') + \]  \hspace{1cm} + 4 \eta(\theta_{12}'''' + 4\theta_{12}'') + Pr (-0.5 f_{11} \theta_{11}' + 2f_{11} \theta_{11}') \} \]  \hspace{1cm} (61)
\[ f'_{22} = -0.5 \left( (1.25f''_{oo22} - 1.5f'_{oo22} + \theta_{22} + 0.75f''_{oo22}) \\
+ f''_{122} + f_{112} - 1.5f'_{1112} + 2f''_{12} - f'_{11} \right) \] (62)

\[ \theta''_{22} = -0.25 \left[ Pr(2.5f''_{oo22} - f'_{oo22} + 1.5f'_{oo22}) \\
+ 4\gamma\theta''_{12} + f''_{112} - 2f''_{1211} - 0.5f'_{1112} - 0.5f'_{1211} - 0.5f'_{1211} \right] \] (63)

\[ f''_{23} = -0.5 \left( (\theta_{23} + 0.75f''_{oo23} - 1.5f'_{oo23} + 1.25f''_{oo23} - 0.75(f'_{12})^2 + f''_{122} - f''_{12} \right) \] (64)

\[ \theta''_{23} = -0.25(Pr)[(2.5f''_{oo23} - f'_{oo23} + 1.5f'_{oo23}) + 2f''_{1211} - 0.5f'_{1112} - 0.5f'_{1112}] \] (65)

\[ f''_{31} = -0.5 \left( (0.75f''_{oo31} - 1.5f'_{oo31} + 1.5f'_{oo31} + \theta_{31}) + 2\gamma f''_{21} + 2f''_{21} \\
+ f''_{1121} + 2f''_{2111} - 1.75f'_{2111} \right) \] (66)

\[ \theta''_{31} = -0.25[3f_{31} - f'_{oo31} + 1.5f'_{oo31} - 1.5f'_{oo31} + Pr(4\gamma\theta''_{21} - 0.5f'_{2111}] \] (67)
\[
\begin{align*}
\phi''_{32} &= -0.5\left[(-0.75f_{10}^{''} 30_{0}^{32} \cdot 1.75f_{10}^{''} 30_{0}^{32} + 1.5f_{20}^{''} 30_{0}^{32} + \Theta_{30}^{''} 30_{0}^{32}) + 2 \gamma \phi_{20}^{''} 30_{0}^{22} + 2f_{20}^{''} 30_{0}^{22} \\
&+ f_{10}^{''} 30_{0}^{22} + f_{10}^{''} 30_{0}^{22} + 1.25f_{12}^{''} 30_{0}^{12} + 1.25f_{20}^{''} 30_{0}^{22} - 1.75f_{12}^{''} 30_{0}^{12} - 1.75f_{20}^{''} 30_{0}^{22} \right] \\
(68) \\
\phi''_{32} &= -0.25\left[(3f_{32}^{10} 30_{0}^{32} + 1.5f_{32}^{10} 30_{0}^{32} - 1.5f_{32}^{10} 30_{0}^{32} \cdot 2f_{32}^{10} 30_{0}^{32} + 4f_{23}^{20} 30_{0}^{22} + Pr(2.5f_{21}^{10} 30_{0}^{22} \\
&+ 2f_{22}^{10} 30_{0}^{22} + 2f_{22}^{10} 30_{0}^{22} - 2f_{22}^{10} 30_{0}^{22} - 2f_{22}^{10} 30_{0}^{22} - f_{22}^{10} 30_{0}^{22} - f_{22}^{10} 30_{0}^{22} - f_{22}^{10} 30_{0}^{22} - 0.5f_{22}^{10} 30_{0}^{22} \right] \\
(69) \\
\phi''_{33} &= -0.5\left[(-0.75f_{33}^{''} 30_{0}^{33} \cdot 1.75f_{33}^{''} 30_{0}^{33} + 1.5f_{33}^{''} 30_{0}^{33} + \Theta_{33}^{''} 30_{0}^{33}) + 2 \gamma \phi_{22}^{''} 30_{0}^{22} + 2f_{22}^{''} 30_{0}^{22} \\
&+ f_{12}^{''} 30_{0}^{22} + f_{12}^{''} 30_{0}^{22} + 1.25f_{23}^{22} 30_{0}^{12} + 1.25f_{22}^{22} 30_{0}^{12} - 1.75f_{12}^{22} 30_{0}^{12} - 1.75f_{12}^{22} \right] \\
(70) \\
\phi''_{33} &= -0.25[Pr(3f_{33}^{10} 30_{0}^{33} + 1.5f_{33}^{10} 30_{0}^{33} - 1.5f_{33}^{10} 30_{0}^{33} + 4f_{23}^{20} 30_{0}^{22} + Pr(2.5f_{12}^{10} 30_{0}^{22} \\
&+ 2f_{23}^{22} 30_{0}^{22} + 2f_{22}^{22} 30_{0}^{22} - 2f_{22}^{22} 30_{0}^{22} - f_{22}^{22} 30_{0}^{22} - f_{22}^{22} 30_{0}^{22} - f_{22}^{22} 30_{0}^{22} - 0.5f_{22}^{22} 30_{0}^{22} \right] \\
(71) \\
\phi''_{34} &= -0.5\left[(-0.75f_{34}^{''} 30_{0}^{34} \cdot 1.75f_{34}^{''} 30_{0}^{34} + 1.5f_{34}^{''} 30_{0}^{34} + \Theta_{34}^{''} 30_{0}^{34}) - f_{23}^{''} 30_{0}^{23} + f_{23}^{''} 30_{0}^{23} + 1.25f_{23}^{23} \right] \\
- 1.75f_{12}^{23} 30_{0}^{22} \right] \\
(72)
\end{align*}
\]
The boundary conditions associated with these equations are:

At \( \eta = 0 \):

\[
\begin{align*}
& r = 1, \quad u = 0, \quad T = T_w \\
& f'(0) = f(0) = 0 \quad \theta(0) = 1 \\
\end{align*}
\]

Hence \( f'(0) = f'_{oo}(0) = f'_{11}(0) = f'_{12}(0) = f'_{21}(0) \ldots = f'_{33}(0) = f'_{34}(0) = 0 \) \hspace{1cm} (74)

\[
\begin{align*}
& f(0) = f_{oo}(0) = f_{11}(0) = f_{12}(0) = f_{21}(0) \ldots = f_{33}(0) = f_{34}(0) = 0 \\
& \theta_{oo}(0) = 1, \quad \theta_{11}(0) = \theta_{12}(0) \ldots = \theta_{33}(0) = \theta_{34}(0) = 0 \hspace{1cm} (75)
\end{align*}
\]

At \( \eta = \infty \):

\[
\begin{align*}
& u = 0, \quad T = T_\infty \\
& f'(\infty) = \Theta(\infty) = 0 \\
& f'(\infty) = f'_{oo}(\infty) = f'_{11}(\infty) = f'_{12}(\infty) \ldots = f'_{33}(\infty) = f'_{34}(\infty) = 0 \hspace{1cm} (77)
\end{align*}
\]

\[
\begin{align*}
& \Theta(\infty) = \Theta_{oo}(\infty) = \Theta_{11}(\infty) = \Theta_{12}(\infty) \ldots = \Theta_{33}(\infty) = \Theta_{34}(\infty) = 0 \hspace{1cm} (78)
\end{align*}
\]
Numerical Solution

Equations 54 through 73 are ordinary differential equations representing the fundamental momentum and energy transport in the axi-symmetric free convection boundary layer around a vertical cylinder with uniform transverse mass flow. These equations, being coupled, are solved in pairs at \( \text{Pr} = 0.7 \). Equations 54 and 55, which constitute the first pair, are non-linear equations, while the remainder are all inhomogeneous. The first group of terms on right hand side of equations 56 through 73 is the homogeneous part, the remainder being the inhomogeneous part. Solution of any pair of equations requires the values of the inhomogeneous terms from some or all of the previous pairs.

The solution of equations 54 through 73 has been carried out by numerical integration using the Kutta-Simpson method. The equations in each pair are respectively third and second order simultaneous differential equations. For numerical integration they are reduced to a set of five differential equations, each of first order. Five boundary conditions, 74 through 78, are available for the solution. However, the Kutta-Simpson method is applicable only to initial value problems. This forced us to choose two more initial conditions (i.e. \( f''(0) \) and \( \theta'(0) \)) arbitrarily. With these arbitrarily chosen values of \( f''(0) \) and \( \theta'(0) \), equations 54 and 55 are solved. This is repeated for different combinations of \( f''(0) \) and \( \theta'(0) \) until the conditions at infinity are matched. The numerical computation was done on an IBM 7044 computer.

For computation, some numerical value, \( \gamma_{\infty} \), of the independent variable, \( \gamma \),
is chosen to represent infinity. The implication of \( f'_{\infty}(\infty) = 0 \) is that the value of \( f'_{\infty} \) at \( \eta_{\infty} \) is zero and remains zero no matter how far beyond \( \eta_{\infty} \) we may go. This required that at and beyond \( \eta_{\infty} \), \( f''_{\infty}, f'''_{\infty} \) must also be zero. Similarly \( \Theta''_{\infty}, \Theta'_{\infty} \) along with \( \Theta_{\infty} \) must also be zero at and beyond \( \eta_{\infty} \).

Numerical integration with correctly chosen values of \( f''_{\infty}(0) \) and \( \Theta'_{\infty}(0) \) gives \( f_{\infty}, f'_{\infty}, f''_{\infty}, f'''_{\infty} \) and \( \Theta_{\infty}, \Theta'_{\infty}, \Theta''_{\infty} \) as functions of \( \eta \). These values are used in the solution of the succeeding pairs of equations.

Solution of each pair of the inhomogeneous equations 56 through 73 is obtained by a linear combination of the solutions of the homogeneous part equated to zero and the full equations. A set of values of \( f''_{ab}(0) \) and \( \Theta'_{ab}(0) \) \((a = 1, 2, 3; b = 1, 2, 3, 4)\) is chosen to satisfy one of the two conditions at infinity i.e. \( \Theta_{ab}(\infty) = 0 \). A linear combination of values of \( f''_{ab}(0) \) and \( \Theta'_{ab}(0) \), obtained for the homogeneous part and the equations containing all the terms, is made to satisfy \( f'_{ab}(\infty) = 0 \). The interested reader is referred to the details of the methods of solving of non-linear and inhomogeneous equations given in Appendix. Results have been calculated to eight significant figures after the decimal. The worst matching of the boundary conditions at infinity is four places after the decimal. This result is quite good, taking into consideration that the error gets accumulated as results from one set of equations are fed to succeeding sets.

The values of \( f''_{ab}(0) \) and \( \Theta'_{ab}(0) \) \((a = 0, 1, 2, 3; b = 0, 1, 2, 3, 4)\), which match the solution of equations 54 through 73 at \( \eta_{\infty} \) and beyond that, are given in Table I.
TABLE I

<table>
<thead>
<tr>
<th>( f''_{\alpha\beta}(0) )</th>
<th>( \phi'_{\alpha\beta}(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6789105</td>
<td>-0.2497558</td>
</tr>
<tr>
<td>0.11118057</td>
<td>-0.22487495</td>
</tr>
<tr>
<td>-0.08811011</td>
<td>0.17144587</td>
</tr>
<tr>
<td>-0.01239087</td>
<td>0.05675445</td>
</tr>
<tr>
<td>-0.02569557</td>
<td>0.00243219</td>
</tr>
<tr>
<td>-0.0369464</td>
<td>-0.03582649</td>
</tr>
<tr>
<td>0.01741969</td>
<td>-0.02799335</td>
</tr>
<tr>
<td>0.0482692</td>
<td>-0.00630005</td>
</tr>
<tr>
<td>0.05879534</td>
<td>0.02501713</td>
</tr>
<tr>
<td>0.01013797</td>
<td>0.00039481</td>
</tr>
</tbody>
</table>
Local Heat Transfer Results

Heat transfer results are presented in terms of the Nusselt number

\[ \text{Nu} = \frac{\bar{x}}{k/(h)} \],

the ratio of a length dimension to the fluid film thickness if the whole of the heat transfer were by conduction.

\[ \text{Nu} = \frac{\bar{x}}{k(T_\infty - T_w)} = \left( \frac{k}{a} \right) \frac{\partial \theta}{\partial r} \bigg|_{r=1} \]

\[ \text{Nu} = -\frac{\bar{x}}{a} \left( \frac{\partial \theta}{\partial \gamma} \right)_{\gamma=0} \left( \frac{\partial \eta}{\partial r} \right)_{r=1} = -25 \times 3^{1/4} \theta'(0) \quad (79) \]

\[ \left( \frac{\text{Nu}}{Gx^{3/4}} \right) = \left( \text{Nu} \right) \left( \frac{Gr}{4} \right)^{-1/4} = -2\theta'(0) \quad (80) \]

Substituting equation 53 in equation 80

\[ \left( \text{Nu} \right) \left( \frac{Gr}{4} \right)^{-1/4} = -2 \left[ \theta_{10}^1(0) + \xi (\theta_{11}^1(0) + v \Theta_{11}^1(0)) + \xi^2 (\theta_{21}^1(0) + v \Theta_{21}^1(0)) + v^2 \Theta_{23}^1(0) \right] \]

\[ + \xi^3 \left[ \Theta_{31}^1(0) + v \Theta_{32}^1(0) + v^2 \Theta_{33}^1(0) + v^3 \Theta_{34}^1(0) \right] \quad (81) \]

The product of \( \xi \) and \( \frac{v}{w} \) is \( F_w = \frac{\bar{v}_w \frac{\bar{x}}{v}}{(Gr)^{1/4}} \). It is proportional to the ratio of transverse flow Reynolds number, \( \frac{\bar{v}_w \frac{\bar{x}}{v}}{v} \), to the Grashof number.
Therefore equation 81 can be rewritten as

\[
(Nu) \left(\frac{Gr}{4}\right)^{-1/4} = - 2[\Theta'_{oo}(0) + \xi \Theta'_{11}(0) + \xi^2 \Theta'_{21}(0) + \xi^3 \Theta'_{31}(0)] + \\
\left\{ F_w \Theta'_{12}(0) + F_w^2 \Theta'_{23}(0) + F_w^3 \Theta'_{34}(0) \right\} + \left\{ \xi F_w \Theta'_{22}(0) + \xi^2 F_w \Theta'_{32}(0) + \xi^3 F_w \Theta'_{33}(0) \right\}
\]

(82)

The first term on the right hand side of equation 82 represents the surface temperature gradient along a flat plate for pure heat transfer. The second and third groups of terms represent respectively, the third order correction for the effect of curvature, and the correction for the transverse flow. The fourth group of terms gives the correction for the interaction between curvature and transverse flow effects.

For the case of pure heat transfer on a flat plate, \( \xi = 0, \ F_w = 0 \), and therefore:

\[
(Nu)_{F_w = 0} \left(\frac{Gr}{4}\right)^{-1/4} = - 2(\Theta'_{oo}(0)) = 0.4995 \approx 0.5
\]

or \( (Nu)_{F_w = 0} = 0.5 \left(\frac{Gr}{4}\right)^{1/4} \)

(83)

For pure heat transfer on a circular cylinder, \( F_w = 0 \).
Therefore the fractional increase in heat transfer due to the curvature is given by the second group of terms in the right hand side of equation (85)

\[
\frac{(\text{Nu})_{F=0}}{(\text{Nu})_{\xi=0}} = 1 + \left[ \frac{\theta'_{11}(0)}{\theta'_w(0)} + \frac{\theta'_{21}(0)}{\theta'_w(0)} + \frac{\theta'_{31}(0)}{\theta'_w(0)} \right]
\]  

(85)

The change in the rate of heat transfer due to transverse mass flow for a flat plate is given by equation 86

\[
\frac{(\text{Nu})_{\xi=0}}{(\text{Nu})_{F=0}} = 1 + \left[ \frac{\theta'_{12}(0)}{\theta'_w(0)} + \frac{\theta'_{23}(0)}{\theta'_w(0)} + \frac{\theta'_{34}(0)}{\theta'_w(0)} \right]
\]  

(86)

Similarly, for a circular cylinder, the relationship is:

\[
\frac{(\text{Nu})_{F=0}}{(\text{Nu})_{F=0}} = 1 + \left[ \frac{\theta'_{12}(0) + \theta'_{23}(0) + \theta'_{34}(0)}{\theta'_w(0)} + \frac{\theta'_{11}(0)}{\theta'_w(0)} + \frac{\theta'_{21}(0)}{\theta'_w(0)} + \frac{\theta'_{31}(0)}{\theta'_w(0)} \right]
\]  

(87)
Average Heat Transfer Rates

The heat transfer results in terms of Nusselt numbers given in the previous section are point values at a distance $\tilde{x}$ from the leading edge. Near the leading edge, the boundary layer being thin, the rate of heat transfer is higher than farther downstream. The heat transfer coefficient is therefore inversely proportional to some power of $\tilde{x}$. By taking average over a distance $L$ from the leading edge, an average heat transfer coefficient, $h_{av}$, is obtained as:

$$h_{av} = \frac{q''_{av}}{\Delta T} = \frac{1}{L} \int_0^L \frac{q''}{\Delta T} \, dx$$

$$h_{av} = \frac{1}{L \Delta T} \int_0^L 2k \Delta T \theta'(0) x^{-1/4} \, dx$$

$$h_{av} = \frac{2kG}{L} \int_0^L x^{-1/4} \left( \theta'_0(0) + \xi \theta'_1(0) + \xi^2 \theta'_2(0) + \xi^3 \theta'_3(0) + \right.$$\begin{align*} & F \theta'_w(0) + F^2 \theta'_w(0) + F^3 \theta'_w(0) +\xi F \theta'_w(0) + \xi^2 F \theta'_w(0) + \xi^3 F \theta'_w(0) + \\
& + \xi F^2 \theta'_w(0)\right) \, dx \\
\text{(88)}
\end{align*}

$$\left(\text{Nu}_{av}\right) = \frac{h_{av} L}{k} = -2 \frac{Gx}{4} \frac{1}{\Delta T} \left[ \frac{4}{3} \theta'_0(0) + (\xi \theta'_1(0) + F \theta'_w(0)) \right.$$\begin{align*} & + \frac{4}{5}(\xi^2 \theta'_2(0) + \xi F \theta'_w(0) + F^2 \theta'_w(0) + \frac{2}{3} \xi^3 \theta'_3(0) + \xi^2 F \theta'_w(0) + \xi^3 F \theta'_w(0) + \\
& + \xi^2 F^2 \theta'_w(0) + F^3 \theta'_w(0)\right]_x = L \\
\text{(89)}
\end{align*}
Mass Transfer with or without Heat Transfer

From an examination of the equations for heat and for mass transfer, it is apparent that under certain conditions the mass and energy transport equations are analogous. In the absence of a chemical reaction and thermo-diffusion and diffusion-thermo effects, the mass transport equation can be written as

\[
\nabla \cdot \rho \mathbf{u} + \nabla \cdot \rho \mathbf{u} = \frac{D}{\rho} \frac{\partial \rho}{\partial r} \left( r \frac{\partial \rho}{\partial r} \right)
\]

(90)

where \( c \) is the concentration of one of the components, \( D \) is the diffusivity, and \( \frac{\nu}{D} \) is the Schmidt number, \( Sc \). If a dimensionless concentration difference is defined as \( \phi = \frac{c_\infty - c}{c_\infty - c_w} \), equation 90 can be non-dimensionalized to equation 91.

\[
\nabla \cdot \rho \mathbf{u} + \nabla \cdot \rho \mathbf{u} = \frac{1}{Sc} \frac{1}{r} \frac{\partial \phi}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right)
\]

(91)

Equation 91 is similar to equation 7 except that \( Sc \) has replaced \( Pr \).

For the case of pure mass transfer from a vertical cylinder in a still atmosphere, the driving force is the concentration difference between the fluid near the surface and that far from it. This concentration difference is also responsible for the creation of free convection because of the density difference. Therefore, in the momentum equation, the body force takes the form \( g \left( \frac{\rho - \rho_\infty}{\rho_\infty} \right) \). If a Grashof number for mass transfer is defined as
\[
G_r = \frac{g x^3 (\rho_w - \rho_\infty)}{v^2 \rho_\infty}, \quad G = \left[ \frac{g a (\rho_w - \rho_\infty)}{4 v^2 \rho_\infty} \right]^{1/4}, \quad \xi^* = \frac{g a (3 - \rho_\infty)}{v^2 \rho_\infty},
\]

\[
\xi^* = \frac{x^{1/4}}{G} \quad \text{then the momentum transfer equation 92 is similar to equation 6.}
\]

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{g a (3 - \rho_\infty)}{v^2 \rho_\infty} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \quad (92)
\]

The solution for equations 91 and 92 will be the same as for equations 6 and 7 if \( Pr = sc \), if the surface concentration is uniform, and if the rate of transverse mass transfer at the surface is uniform along the surface. The latter condition is not usually met in the mass transfer case. In the heat transfer problem, the rate of flow of fluid at the surface in the transverse direction may be an independent variable. This is not the case in mass transfer, because the transverse velocity at the surface is determined by the concentration difference between the bulk and at the surface. Another case in which the transverse flux is not independent is that of evaporation of a liquid from the surface to the still atmosphere of its own vapours, since the rate of evaporation is coupled to the heat transfer rate by the latent heat of evaporation.

The rate of evaporation per unit time per unit area is given by

\[ \dot{V}_w = \dot{q''}/\rho \lambda, \quad \text{where } \dot{q''} \text{ is the rate of heat transferred from the superheated} \]
vapours in the surroundings to the surface covered with a layer of liquid at its boiling point, \( \lambda \) is the latent heat of evaporation, and \( \rho \) is the vapour density at a suitable temperature between that of bulk and at the surface.

\[
q'' = k \frac{\partial T}{\partial x_{T=a}} = - \frac{2Gk} {\alpha} \frac{\Delta T}{a^{1/4}} \theta'(0)
\]

\[
\overline{v}_w = -2 \left( \frac{G}{\alpha x^{1/4}} \right) \frac{k}{\rho \lambda} \theta'(0) = - \frac{2}{\alpha x} \nu \frac{k}{\nu \rho \lambda} \theta'(0)
\]

\[
\frac{\overline{v}_w}{\nu} = \frac{v_w}{\nu} = - \frac{2}{\xi} \left( \frac{k}{\rho \lambda} \right) \left( \frac{c}{\lambda} \right) \theta'(0)
\]

(93)

\[
v_w = - \frac{2}{\xi} \left( \frac{B}{P_r} \right) \theta'(0)
\]

(94)

A measure of the driving force for heat transfer is given by the dimensionless quantity, \( B = \frac{c_p \Delta T}{\lambda} \), which is the ratio of change in enthalpy of the evaporating vapours as they are heated from the saturation temperature to the bulk temperature, to the latent heat of evaporation.

The equation, analogous to 94, applicable to the case of mass transfer where it is the concentration difference, \( \Delta c \), which causes free convection is as follows:

\[
v_w^* = \frac{\overline{v}_w^*}{\nu} = - \frac{2}{\xi^*} \frac{B^*}{Sc} \theta'(0)
\]

(95)

where \( B^* = \frac{\Delta c}{\rho} \).
It has been shown by Sparrow and Cess (24) that the heat transfer results for a flat plate with constant transverse flow do not differ very much from those obtained by Eichhorn (6) for the case where \( \bar{v}_w \propto x^{-1/4} \). This conclusion will also be established by the present study. Sparrow and Cess termed these phenomena as 'the boundary layer has poor memory', i.e. the rate of heat transfer is not affected by the amount of upstream transverse mass flow. It is hoped that this independence of heat transfer of the transverse velocity distribution at the surface remains true for a cylinder also.

In view of these facts, it is proposed that no great error will be introduced if in equation 94, 95, the values of temperature and concentration gradients, \( \theta'(0) \) and \( \phi'(0) \), are substituted from the heat transfer solution for the case of uniform \( v_w \). Therefore equations 94 and 95 can be expanded into equations 96 and 97 respectively.

\[
\begin{align*}
v_w &= - \frac{2B}{\xi Pr} \left[ \theta'_{\infty}(0) + \xi (\theta'_{11}(0)+\theta'_{12}(0)v_w) + \xi^2 (\theta'_{21}(0)+\theta'_{22}(0)v_w+\theta'_{23}(0)v^2_w) \\
&\hspace{2cm} + \xi^3 (\theta'_{31}(0)+\theta'_{32}(0)v_w+\theta'_{33}(0)v^2_w+\theta'_{34}(0)v^3_w) \right] \\
&\hspace{2cm} + \xi(\phi'_{12}(0)+\phi'_{22}(0)v^2_w+\phi'_{32}(0)v^2_w+\phi'_{33}(0)v^3_w+\phi'_{34}(0)v^3_w) \\
&\hspace{2cm} + \phi'_{23}(0)v^2_w+\phi'_{31}(0)+\phi'_{32}(0)v^2_w+\phi'_{33}(0)v^2_w+\phi'_{34}(0)v^3_w \right] \\
&\hspace{2cm} + \xi^2 (\theta'_{21}(0)+\theta'_{22}(0)v_w+\theta'_{23}(0)v^2_w+\theta'_{24}(0)v^3_w) \\
&\hspace{2cm} + \xi^3 (\theta'_{31}(0)+\theta'_{32}(0)v_w+\theta'_{33}(0)v^2_w+\theta'_{34}(0)v^3_w+\theta'_{35}(0)v^3_w) \\
&\hspace{2cm} + \phi'_{23}(0)v^2_w+\phi'_{31}(0)+\phi'_{32}(0)v^2_w+\phi'_{33}(0)v^2_w+\phi'_{34}(0)v^3_w+\phi'_{35}(0)v^3_w) \right]
\end{align*}
\]

(96)

(97)

For some particular values of \( \xi \) and \( B \), a value of \( v_w \) can be found
which will satisfy equation 96. Thus plots of $v_w$ vs. $\xi$ with $B$ as a parameter will give the value of $v_w$ along the length of the cylinder at different values of $B$ (i.e. $\frac{C \Delta T}{\lambda}$). The same plot is also valid for equation 97 if $v_w$, $\xi$ and $B$ are replaced by $v_w^*$, $\xi^*$, $B^*$ respectively.
DISCUSSION OF RESULTS

Temperature and Velocity Profiles:

Temperature and velocity profiles in a boundary layer flow reveal much about the mechanisms governing the transport processes. This is especially helpful in complex transport processes for which it is difficult to visualize the effect of interaction of the processes occurring simultaneously. Equations 98, 99 give the temperature and velocity profiles for heat transfer with transverse mass flux around a vertical cylinder at Pr = 0.7.

\[
\theta = \theta_{oo} \left( \xi \theta_{11} + \xi^2 \theta_{21} + \xi^3 \theta_{31} \right) + \left( F_w \theta_{12} + F^2_w \theta_{23} + F^3_w \theta_{34} \right) \\
+ \left( F^2_w \theta_{22} + F^3_w \theta_{32} + F^4_w \theta_{43} \right) \tag{98}
\]

\[
u = 2C^2 \times 1/2 f' = 2C^4 \left[ f_{oo} + \left( F_{11} + \xi^2 f_{21} + \xi^3 f_{31} \right) \\
+ \left( F_w^2 f_{12} + F^2_w f_{23} + F^3_w f_{34} \right) \right] \tag{99}
\]

The effect of \( \eta \), the function of distance in the transverse direction, on \( \theta \) and \( f' \), the dimensionless temperature and longitudinal flow parameters respectively, is given in figures 2 and 3. Four combinations of \( \xi \) and \( F_w \), the curvature and the transverse flow parameters respectively, are considered for illustration.

Since suction causes a flow towards the surface, conditions in the near vicinity of the surface are in this case more strongly influenced by the bulk
Note the change in scale

\[ \theta \text{ as a Function of } \eta \]

<table>
<thead>
<tr>
<th>Curve</th>
<th>E</th>
<th>Fw</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0.2</td>
<td>0.0</td>
</tr>
<tr>
<td>b</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>c</td>
<td>0.2</td>
<td>0.5</td>
</tr>
<tr>
<td>d</td>
<td>0.3</td>
<td>0.75</td>
</tr>
</tbody>
</table>
FIG. 3

\( f' \) as a function of \( \gamma \)

- Note the change in scale.

<table>
<thead>
<tr>
<th>Curve</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
</tr>
<tr>
<td>4</td>
<td>0.8</td>
</tr>
<tr>
<td>5</td>
<td>1.0</td>
</tr>
</tbody>
</table>

- This table indicates different values for a parameter, adding context to the graph.
fluid conditions. With blowing, on the other hand, the temperature and fluid velocity at the surface influence the flow field for a considerably greater distance from the surface. Thus the temperature gradient at the surface is higher in case of suction, and lower for blowing, as compared to the gradient without transverse mass flux. Alternately, the effect of transverse mass flux may be described in terms of a reduction or an increase in the boundary layer thickness caused by suction or blowing.

Corresponding effects of transverse mass flux for the longitudinal flow variable, \( f' \), are shown in figure 3. The maximum value of \( f' \) and hence the maximum axial velocity in the boundary layer decreases with suction and increase with blowing. The disturbance shown in the curve d of figure 3 becomes more pronounced at higher values of \( F_W \) as the boundary layer becomes thicker. With excessive blowing the transverse flow will dominate the longitudinal flow caused by the density difference.

Accuracy of the Results.

For this analysis to be valid, an important requirement is that the boundary conditions at \( \eta = \infty \) be met. For \( \eta \approx 30 \) to be a satisfactory numerical approximation for \( \eta = \infty \), the physical conditions require that \( \Theta \) and \( f' \) be zero and remain so for all values of \( \gamma > 30 \), and that the higher derivatives of \( \Theta \) and \( f' \) follow the same requirement. Figures 4 and 5 illustrate for the particular value of \( \xi = 0.3 \) and \( F_W = 0.75 \), the way in which the boundary conditions for \( \eta = \infty \) are met at \( \eta = 30 \) or earlier. For all other values of \( \xi \) and \( f_W \), it is also true that \( \Theta = \Theta' = \Theta'' = 0 = f' = f'' = f''' \).
FIG. 6

Third order term as percentage of leading terms
Another factor affecting accuracy is the representation of the functions \( f \) and \( \Theta \) in series form in ascending powers of \( \xi \). In this expansion it was assumed that the sum of the terms containing fourth and higher powers of \( \xi \) was less than the term containing the third power of \( \xi \). Because it is difficult to prove this point directly, which would require the calculation of a large number of terms in the series, a limit was set on the system variables so that the third order term was small relative to the sum of the leading terms. The dependence of the third order term, as a percentage of the sum of the leading terms of the series, is plotted in figure 6 as a function of \( \xi \) and \( F_w \). Within the limits \(-1 < F_w < 1 \) and \( 0 < \xi < 0.85 \), it is observed that the third order term is never more than \( \pm 4\% \) of the sum of the leading terms. Because the results for cylinder have been obtained by a perturbation technique, the solution is not applicable to cylinders of high curvature, i.e. for \( \xi > 0.85 \), for which the neglected terms of the series would become significant. For a Grashof number of \( 4 \times 10^6 \), the limitation of \( \xi < 0.85 \) corresponds to cylinders of a length to diameter ratio up to 27. With respect to limit on transverse flow, it is of interest to note that if the limits for \( F_w \) were extended to \(-5 < F_w < 1.5 \), the third order term would become at the worst \(-10\% \) of the sum of the leading terms. The error caused by suction is smaller than that for the same rate of blowing because the boundary layer is thinner and more stable.

**Local Heat Transfer Results**

The local heat transfer results for blowing and for suction, given earlier as equation 82 are represented graphically on figures 7 and 8. The transverse mass flow parameter \( F_w \) is preferred to \( v_w = (\nabla u / \nu) \), when \( \xi \) is used as the
FIG. 7
Heat transfer results for blowing
Comparison of local heat transfer results for a cylinder with flat plate heat transfer results
Comparison of local heat transfer results for a cylinder with and without transverse mass flux.

FIG. 10
second parameter, because the radius appears both in $v_w$ and $\xi$, but not in $F_w$. The flow variable $F_w = (v_w \bar{x}/\nu)(Gr/4)^{-1/4}$ represents the ratio of the transverse flow Reynolds number to a function of Grashof number. The choice of $F_w$ and $\xi$ thus facilitates determination of the effect of curvature. However, it should be noted that $\xi$, also represents the effect of distance from the leading edge as in figure 12 where $\bar{x}$ does not appear in the ordinate.

The rate of heat transfer may be seen to increase for suction and decrease for blowing. A flow of fluid outward from the surface and originating at the surface temperature, acts to reduce the temperature gradient at the surface, and thus reducing the heat transfer rates. A flow of fluid towards the surface has, naturally, the opposite effect.

The curve for $\xi = 0$ in figures 7 and 8, which represents the heat transfer rate to a flat plate, differs at higher rates of blowing and suction from the rates given by Sparrow and Cess (24). Since Sparrow and Cess considered only a first order correction for the transverse flow, their results are lower than those obtained in the present work, for which a third order correction has been applied. The results obtained for a flat plate by Mabuchi (10) and by Brdlik and Mochalov (3) are in good agreement with the present work for blowing. For suction, however, our results are between those given by Sparrow (24) and those of Mabuchi (10), Brdlik and Mochalov (3). Mabuchi, and Brdlik and Mochalov assumed 'similar profiles' for velocity and temperature in the boundary layer, and for the transverse distance variable in both profiles, used a fourth order polynomial in terms of the ratio of transverse distance to boundary layer.
thickness. Since velocity and temperature profiles are in fact not 'similar' in the boundary layer, the perturbation technique was used by Sparrow and Cess (24) and this study. It is believed that the results of the present analysis is more accurate than those of the earlier studies.

When the flat plate results of the present study are compared with the exact solution given by Eichhorn (6) for the case of a transverse flux which is inversely proportional to the one-fourth power of the distance from the leading edge, good agreement is found for moderate rates of blowing and suction. Deviations are higher for suction than for blowing. However, the important conclusion is that the local heat transfer rate is influenced very little by the upstream transverse velocity distribution. This fact was first noticed by Sparrow and Cess (24).

The effect of curvature is indicated by the parameter, \( \xi = \left( \frac{1}{a} \right) \left( \frac{Gr}{4} \right)^{-1/4} \), the ratio of boundary layer thickness to the radius of the cylinder. It may be seen from figures 7 and 8 that for any value of \( F_w \), the rate of heat transfer increases as the curvature variable increases, the minimum value being for a flat plate.

A physical picture of the effect of curvature on heat transfer may be given by considering the value of heat flux for a flat plate at a particular surface and environment temperatures \( T_w \) and \( T_\infty \). As the surface acquires curvature in becoming a cylinder while the same \( T_w \) and \( T_\infty \) are maintained, the point values of heat flux at all points in the boundary layer would be decreased if the heat transfer at the wall were to remain constant. This reduction would result from the fact that for a cylinder the area for transverse flow of heat
or mass in the boundary layer increases in proportion to the radial distance. If the heat flux at all points in the boundary layer were to decrease, the temperature difference \( (T_\infty - T_w) \) would also decrease. Thus for the temperatures \( T_\infty \) and \( T_w \) to be maintained as curvature increases, it necessarily follows that the heat flux at the wall must increase. The magnitude of the increase is given on figures 7 and 8.

The combined effects of curvature, as discussed above, and of transverse mass flux, as discussed earlier, are illustrated on figure 9 as the ratio of heat transfer for a cylinder to that for a flat plate. An alternate representation, shown in figure 10, gives the ratio of heat transfer to a cylinder with transverse mass flux to that without mass transfer. It is apparent that while blowing (positive transverse mass flux) decreases heat transfer, positive curvature (convex) increases the rate. Alternately one may note that while negative transverse mass flux is suction acts to increase heat transfer, negative curvature decreases it.

In these terms therefore it can be stated that the effect of transverse mass flux is opposite to that of curvature.

Figure 10 also shows heat transfer results according to equation 84, for a cylinder without transverse mass flux. When compared with the results of Sparrow and Gregg (21) which are indicated, the agreement is seen to be good.

**Average Heat Transfer Results**

Heat transfer results, averaged over a length \( x = L \) according to
equation 89, are given in figure 11. The average value of heat transfer coefficient is higher than the local value at \( x = L \), because of the decrease of the local heat transfer coefficient with distance from the leading edge as the boundary layer thickness increases. However, the effects of \( \xi \) and \( F_w \) on the average Nusselt number are similar to that for the point values given on Figures 7 and 8.

**Mass Transfer with or without Heat Transfer.**

In the analysis presented to this point, a degree of freedom has existed between the variables, transverse velocity, and the heat transfer driving force, \( \Delta T \). However, if the source or the sink of transverse mass flow at the solid surface occurs by means of a phase change, then the transverse velocity and driving force become linked. In the case of coupled heat and mass transfer, the mass flux will usually not be constant over the surface. However, it has been pointed out earlier that the boundary layer has 'poor memory' i.e. the rate of heat transfer is not much affected by the transverse mass flux distribution at the surface. Thus it is proposed that where transverse mass flux distribution depends on the local rate of heat transfer, the solution for the uniform transverse mass flux distribution can be applied without large error.

Figure 12 gives the rate of transverse mass flux \( v_w \) as a function of \( \xi \) and \( B \) for heat and mass transfer and \( v_w^* \) as a function of \( \xi^* \) and \( B^* \) for pure mass transfer. Since \( \xi \), with ordinate \( v_w = (v_w a / \nu) \), represents longitudinal distance from the leading edge, figure 12 gives the rate of transverse mass flux as a function of length of the cylinder. In this analysis, the theoretical transverse mass flux increases without limit as the leading edge i.e. \( \xi = 0 \) is approached, since the boundary layer thickness is taken as zero at this point.
Downstream from the leading edge the boundary layer thickness increases in proportion to the one-fourth power of the distance from the leading edge. At higher values of $\xi$, the mass flux may be seen from figure 12 to approach an asymptotic value.

Li (9) has measured the rates of evaporation of water on flat plates in natural convection at high temperatures. For the evaporation rate he obtained with a 2 inch long plate in an environment at 374°C, which gives a Grashof number of $3.24 \times 10^5$, the value of the flow variable $F_w = 0.47$ may be calculated. This experimental value of $F_w$ is well within the range of validity of our theoretical results.

For the case when the heat transfer by radiation is significant, the rate of evaporation by radiation may be added to the right hand side of equation 96, and the value of $v_w$ may be found to satisfy this equation.
CONCLUSIONS

For heat transfer, by laminar natural convection with uniform transverse mass flux; for a vertical cylinder, a numerical solution has been obtained, which depicts the effect of curvature and transverse mass flux. Owing to the application of perturbation technique this solution is not valid for very long and thin cylinders.

It has been observed, that

(i) For all values of transverse mass flux, the effect of curvature, convex to the surroundings, is to increase the rate of heat transfer. Conversely, the rate of heat transfer is decreased if the curvature to the environment becomes concave.

(ii) For all values of curvature, from a flat plate to the highest value for which the solution obtained remains valid, the heat transfer rate is decreased by blowing (positive mass flux) and increased by suction (negative mass flux).

(iii) The effect of transverse mass flux is opposite to that of curvature.

(iv) The phenomena of 'poor memory' of the boundary layer has been confirmed for a flat plate and it is hoped to be true for a cylinder also. This provides a method for the estimation of transverse mass flow rates for a coupled heat and mass transfer process e.g. evaporation of a liquid. Evaporation by radiation can be taken into account as long as the mass transfer rate conforms to the limits imposed.
BIBLIOGRAPHY

Roman Symbols

a radius of the cylinder, feet

\( C_p \frac{\Delta T}{\lambda} \), dimensionless

B

\( \frac{\Delta c}{\rho_w} \), dimensionless

\( C_p \) specific heat at constant pressure, BTU/lb°F

c concentration of one of the components lb/ft³

\( \Delta c \) \( c_{\infty} - c_w \), lb/ft³

D diffusion coefficient ft²/hr

\( F_w \) blowing parameter \( (\bar{v} \bar{x}/\nu) (\frac{Gr}{4})^{1/4} \), dimensionless

\( f \) dimensionless dependent variable, defined in equation 32

\( f_a, f_{ab} \) dimensionless dependent variables, defined in equation 32,47,48.

\( G \) dimensionless constant, \( \left[ \frac{g^2(T_{\infty} - T_w) a^3}{4 \nu^2} \right]^{1/4} \)

\( G^* \) dimensionless constant \( \frac{g a^3 (\rho_w - \rho_\infty)}{4 \nu^2 \rho_w} \)

\( Gr \) Grashof number, \( \frac{g^2(T_{\infty} - T_w)x^3}{\nu^2} \), dimensionless

\( Gr^* \) Grashof number for mass transfer \( \left( \frac{g a^3 (\rho_w - \rho_\infty)}{\nu^2} \right)^{1/4} \)

g acceleration of gravity ft/hr².
h  heat transfer coefficient, BTU/hr ft² oF
k  thermal conductivity, BTU/hr ft oF
L  length of the cylinder
Nu Nusselt number, hL/k, dimensionless
Pr Prandtl number, Cpμ/k, dimensionless
q" heat transfer rate, BTU/hr ft²
r distance in radial direction, ft
r dimensionless distance in radial direction, ra
Sc Schmidt number, ν/D, dimensionless
T  temperature, oF
ΔT T∞ - Tw, oF
u velocity in x direction, ft/hr
u dimensionless velocity in x direction, ua/ν
v velocity in r direction, ft/hr
v dimensionless velocity in r direction, va/ν
x distance in longitudinal direction, ft
x dimensionless distance in longitudinal direction

Greek Symbols

α  thermal diffusivity, k/ρCp, ft²/hr
β  coefficient of thermal expansion, oF⁻¹
γ  gα(T∞ - T)a³
    μ²  , dimensionless
\[ \Gamma^* = \frac{g \alpha (T - T_w)}{\nu^2 \beta} \text{ dimensionless} \]

\( \eta \) dimensionless independent variable, defined in equation 15

\( \theta \) dimensionless dependent variable, \( (T_\infty - T)/(T_\infty - T_w) \)

\( \theta_a, \theta_{ab} \) dimensionless dependent variables, defined in equations 33,49,50,51.

\( \lambda \) latent heat of evaporation, BTU/lb

\( \mu \) absolute viscosity, lb/ft.hr

\( \nu \) kinematic viscosity, \( \text{ft}^2/\text{hr} \)

\( \xi \) curvature effect for heat transfer, defined in equation 16

\( \xi^* \) curvature effect for mass transfer, \( x^{1/4}/G^* \)

\( \rho \) density, lb/ft\(^3\)

\( \phi \) dimensionless dependent variable, \( (c_\infty - c)/(c_\infty - c_w) \)

\( \psi \) stream function, defined in equation 14.

**Subscripts**

\( w \) wall conditions

\( \infty \) environment conditions

\( \text{av.} \) average
APPENDIX

Simultaneous differential equations of order higher than one, with two point boundary conditions, may be solved by numerical integration on digital computers. The mass, momentum, and energy transport equations of this study were converted to a set of first order equations by defining the dependent variable and its derivatives (except the highest one) by new functions. Since initial values of the functions are required for a numerical integration method, some arbitrary but reasonable guess of the initial values of the functions is made, and the integration is carried out. The process is repeated with different initial conditions until the final conditions are satisfied. In the present case three step sizes, 0.05, 0.10 and 0.20, were used in the independent variable, \( \eta \), in the regions \( 0 \leq \eta < 10; 10 \leq \eta < 20; 20 \leq \eta < 30 \) respectively, because at higher values of \( \eta \), the dependent variables reach asymptotic values.

Since equations 74 through 78 give three initial conditions and two final conditions, two more initial conditions, \( f''(0) \) and \( \Theta'(0) \), have to be assumed such that the given final conditions, \( f''(\infty) \) and \( \Theta(\infty) \) are satisfied. Method of steepest descent was used to minimize the difference between final values \( f_\infty \) and \( \Theta_\infty \), obtained by solving equations with assumed initial conditions, and the desired final conditions, \( f'(\infty) \) and \( \Theta(\infty) \). Owing to the presence of local minima in the case of non-linear equations, the absolute minima was found by locating minima on the locus of \( (f'_\infty - f'(\infty))^2 \), such that the final condition, \((\Theta_\infty - \Theta(\infty))^2\) is always minimum.
For the case of inhomogeneous equations with a homogeneous part, one of the initial conditions, \( f''_{\text{ho}} \) was fixed arbitrarily, then the other initial condition, \( \theta'_{\text{ho}} \), was found so that \( (\theta_{\infty} - \theta(\infty))^2 \) was a minimum for the homogeneous part of the equations. This procedure was repeated for the full inhomogeneous equations to find \( \theta'_{\text{io}} \), fixing the value of \( f''_{\text{io}} \); so that \( (\theta_{\infty} - \theta(\infty))^2 \) was a minimum. Making a linear combination of the solutions of homogeneous and inhomogeneous equations, the desired initial conditions can be obtained as follows,

\[
f''(0) = \alpha f''_{\text{ho}} + f''_{\text{io}}; \quad \theta'(0) = \alpha \theta'_{\text{ho}} + \theta'_{\text{io}}
\]

where \( \alpha = \frac{f'_{\text{io}}}{f'_{\text{ho}}} \)

and \( \theta_{\infty}, f'_{\text{ho}} \) and \( \theta_{\infty}, f'_{\text{io}} \) are the final values for homogeneous and inhomogeneous equations respectively, obtained from the corresponding assumed initial values \( \theta'_{\text{ho}}, f''_{\text{ho}} \) and \( \theta'_{\text{io}}, f''_{\text{io}} \).