Spectral Multiplicity Theory in Nonseparable Hilbert Spaces: A Survey

by

Kevin A. Linder

A Thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science.

Department of Mathematics and Statistics
McGill University
Montréal

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ABSTRACT

Spectral multiplicity theory solves the problem of unitary equivalence of normal operators on a Hilbert space $\mathcal{H}$ by associating with each normal operator $N$ a multiplicity function, such that two operators are unitarily equivalent if and only if their multiplicity functions are equal. This problem was first solved in the classical case in which $\mathcal{H}$ is separable by Hellinger in 1907, and in the general case in which $\mathcal{H}$ is nonseparable by Wecken in 1939. This thesis develops the later versions of multiplicity theory in the nonseparable case given by Halmos and Brown, and gives the simplification of Brown's version to the classical theory. Then the versions of Halmos and Brown are shown directly to be equivalent. Also, the multiplicity function of Brown is expressed in terms of the multiplicity function of Halmos.
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RÉSUMÉ

La théorie de multiplicité spectrale répond le problème d'équivalence unitaire d'opérateurs normaux sur un espace de Hilbert $\mathcal{H}$ en associant à chaque opérateur normal $N$ une fonction de multiplicité tel que deux opérateurs sont en équivalence unitaire si et seulement si leurs fonctions de multiplicité sont égales. Ce problème fut résolu en premier lieu par Hellinger en 1907 dans le cas classique où $\mathcal{H}$ est séparable, et par Wecken en 1939 dans le cas général où $\mathcal{H}$ est non-séparable. Cette thèse développe les versions plus récentes de la théorie de multiplicité dans le cas non-séparable selon Halmos et Brown, et présente la simplification de la version de Brown au cas classique. Nous démontrons directement que les versions de Halmos et Brown sont équivalentes. De plus, la fonction de multiplicité de Brown est exprimée en fonction de la fonction de multiplicité de Halmos.
Acknowledgements

I would like to thank my supervisor, Professor J.R. Choksi, for his guidance and constructive criticisms. I would also like to thank FCAR for their generous financial assistance during the past year. I am indebted to Cynthia Parkinson for typing the thesis and all of the alterations, to Pierre Giguere for translating the abstract into French, and to my parents, Roger and Wendy, for their support and understanding. Lastly, I must thank my darling Lianne and Clio, for without their love and strength I would have been lost.
# TABLE OF CONTENTS

## CHAPTER I: THE PRELIMINARIES

<table>
<thead>
<tr>
<th>SECTION</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Spectral Measures and Spectral Integrals</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>Cyclic Subspaces and Cyclic Projections</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>Finite Measures</td>
<td>9</td>
</tr>
</tbody>
</table>

## CHAPTER II: HALMOS' VERSION OF MULTIPLICITY THEORY

<table>
<thead>
<tr>
<th>SECTION</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Halmos' Multiplicity Function</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>Commutators and Double Commutators</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>Rows and Columns</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>Cyclic Projections Revisited</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>Separable Projections</td>
<td>23</td>
</tr>
<tr>
<td>6</td>
<td>Measures and Columns of Cyclic Projections</td>
<td>25</td>
</tr>
<tr>
<td>7</td>
<td>Singular (or Orthogonal) Systems</td>
<td>29</td>
</tr>
<tr>
<td>8</td>
<td>The Multiplicity of a Projection Doubly Commuting with E</td>
<td>33</td>
</tr>
<tr>
<td>9</td>
<td>The Multiplicity Function of a Normal Operator</td>
<td>37</td>
</tr>
</tbody>
</table>

## CHAPTER III: BROWN'S VERSION OF MULTIPLICITY THEORY

<table>
<thead>
<tr>
<th>SECTION</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The $\sigma$-ideals of Finite Measures</td>
<td>42</td>
</tr>
<tr>
<td>2</td>
<td>Cyclic Projections and $\sigma$-ideals</td>
<td>47</td>
</tr>
</tbody>
</table>
SECTION 3: Subspaces from σ-ideals 48
SECTION 4: Stacks and Multiplicity 51
SECTION 5: Brown's Multiplicity Function 56
SECTION 6: The Separable Case 58
SECTION 7: The Relationship between $m_H(N)$ and $m_B(N)$ 62

REFERENCES 68
CHAPTER I: THE PRELIMINARIES

Section 1: Introduction

Two linear operators \( A \) and \( B \) on Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \) respectively are said to be **unitarily equivalent** if there exists a unitary transformation \( U: \mathcal{H} \rightarrow \mathcal{K} \) such that 

\[
UAU^{-1} = B.
\]

If \( A \) and \( B \) are unitarily equivalent operators we write \( A \equiv B \). Multiplicity theory addresses the following question: When are two operators unitarily equivalent? In order to answer this question in a useful way we would like to provide a "complete set of unitary invariants" for every operator. That is, we wish to associate with each operator \( X \) an object \( m(X) \) such that, given operators \( A \) and \( B \), \( A \equiv B \) if and only if \( m(A) = m(B) \).

Even in the special case where the dimension of \( \mathcal{H} \) is finite the question is a difficult one. However, if attention is confined to normal matrices then the problem is solved by diagonalization. That is, a complete set of unitary invariants for a normal \( n \times n \) matrix is given by all of its eigenvalues, counting multiplicities.

A generalization of this result (for self-adjoint operators) to the case of a separable Hilbert space was first given by Hellinger [8] in 1907. There are many sources for the theory for normal operators in the separable case, including Stone [11], and more recently Dunford and Schwartz [5] and Conway [3], to name only a few.

The problem of unitary equivalence of normal operators in the most general case of a nonseparable Hilbert space was first solved by Wecken [12] (for self-adjoint operators) in 1939. We shall develop the later treatments by Halmos [6] and Brown [2]. If we let
$m_H(N)$ and $m_B(N)$ denote the objects associated with the normal operator $N$ given by Halmos and Brown respectively, it will follow that if $N$ and $N'$ are normal operators then $m_H(N) = m_H(N')$ if and only if $m_B(N) = m_B(N')$. This may be seen more directly, however, as we shall show.

In this discussion all operators are assumed to be linear and bounded. We adopt the notation of [2] and mention the notation of [6] (where it differs) in parentheses, except in the following case: Given measures $\mu$ and $\nu$ on a measurable space $(M, \mathcal{S})$, if $\mu$ is absolutely continuous with respect to $\nu$ then we write $\mu \ll \nu$. We take the Radon-Nikodym theorem for granted.

If $\mu \ll \nu$ and $\nu \ll \mu$ then we write $\mu \sim \nu$ (this equivalence relation is denoted by $\mu \equiv \nu$ in [6]). If $\mu$ and $\nu$ are singular (or orthogonal) then we write $\mu \perp \nu$. If $\{\mu_i\}$ is a family of measures on $(M, \mathcal{S})$ such that $\mu_i \perp \mu_k$ whenever $i \neq k$ then we say that $\{\mu_i\}$ is a singular (or orthogonal) family of measures.

We use $\Box$ to denote the end of a proof.

Section 2: Spectral Measures and Spectral Integrals

Recall that given a measurable space $(M, \mathcal{S})$ and a Hilbert space $\mathcal{H}$ a spectral measure is a set function $E: \mathcal{S} \to L(\mathcal{H})$ taking each $M \in \mathcal{S}$ to a projection $E(M)$ on $\mathcal{H}$, such that

i) $E(M) = I$, and

ii) If $\{M_n\}$ is a countable, disjoint sequence $\subseteq \mathcal{S}$ then

$$E(\bigcup_n M_n) = \sum_n E(M_n) .$$

We assume the basic properties of spectral measures (see for example chapter II of [6]).
For a given spectral measure \( E \) and given \( x, y \in \mathcal{H} \), we let \( \mu_{x, y} \) denote the complex measure defined by \( \mu_{x, y}(M) = (E(M)x, y) \), for all \( M \in \mathcal{S} \), but we abbreviate \( \mu_{x, x} \) by \( \mu_x \) (\( = \rho(x) \) in [6]). If an operator \( A \) commutes with \( E(M) \) for all \( M \in \mathcal{S} \), we say that \( A \) commutes with \( E \). If an operator \( B \) commutes with every operator that commutes with \( E \), we say that \( B \) doubly commutes with \( E \).

If \( E \) is a spectral measure and \( f \) is a bounded, measurable function: \( \mathcal{M} \to \mathbb{C} \) then the **spectral integral** \( \int f \, dE \) is defined to be the unique operator \( A \) on \( \mathcal{H} \) such that for all \( x, y \in \mathcal{H} \),

\[
(Ax, y) = \int f(\lambda) \, d(E(\lambda)x, y) = \int f(\lambda) \, d\mu_{x, y}(\lambda).
\]

We assume the spectral theorem for normal operators (see [6: Sections 38-44]):

**Theorem 1.2.1:** Given a normal operator \( N \) on a Hilbert space \( \mathcal{H} \), there exists a unique spectral measure \( E_N \), called the spectral measure of \( N \), from the class of all Borel sets of \( \mathbb{C} \) to \( \mathcal{L}(\mathcal{H}) \), such that

\[
N = \int \lambda dE_N(\lambda).
\]

Moreover,

i) \( E_N(\mathbb{C} \setminus \sigma(N)) = 0 \), where \( \sigma(N) \) is the spectrum of \( N \),

ii) for every bounded Baire function \( f \) on \( \sigma(N) \), \( f(N) = \int f \, dE_N \), and

iii) for every \( B \) in \( \mathcal{L}(\mathcal{H}) \), \( B \) commutes with \( N \) if and only if \( B \) commutes with \( E_N \).

If \( E \) and \( E' \) are two spectral measures from \( \mathcal{S} \) to \( \mathcal{L}(\mathcal{H}) \) and \( \mathcal{L}(\mathcal{K}) \) respectively then we say that \( E \) and \( E' \) are unitarily equivalent if there exists a unitary transformation
U: $\mathcal{H} \to \mathcal{K}$ such that for all $M \in \mathcal{S}$ we have $UE(M)U^{-1} = E'(M)$. Again, if $E$ and $E'$ are unitarily equivalent spectral measures we write $E \equiv E'$.

The following is a simple but important consequence of the spectral theorem:

Theorem 1.2.2: Given normal operators $N$ and $N'$ on $\mathcal{H}$ and $\mathcal{K}$ respectively,

$$N \equiv N' \text{ if and only if } E_N \equiv E_{N'}.$$  

Proof: Suppose there exists a unitary transformation $U: \mathcal{H} \to \mathcal{K}$ such that for all $M \in \mathcal{S}$

$$UE_N(M)U^{-1} = E_{N'}(M).$$

Then given $x, y \in \mathcal{K}$ and $M \in \mathcal{S}$,

$$\mu'_{x, y}(M) = (E_{N'}(M)x, y) = (UE_N(M)U^{-1}x, y) = (E_N(M)U^{-1}x, U^{-1}y) = \mu_{U^{-1}x, U^{-1}y}(M).$$

Hence for all $x, y \in \mathcal{K}$,

$$(N'x, y) = \int \lambda d\mu'_{x, y}(\lambda) = \int \lambda d\mu_{U^{-1}x, U^{-1}y}(\lambda) = (NU^{-1}x, U^{-1}y) = (UNU^{-1}x, y),$$

and so $N' = UNU^{-1}$.

Conversely, if there exists a unitary transformation $U: \mathcal{H} \to \mathcal{K}$ such that

$N' = UNU^{-1}$ then define a set function $F: \mathcal{S} \to L(\mathcal{H})$ by

$$F: M \mapsto UE_N(M)U^{-1}, \text{ for all } M \in \mathcal{S}.$$  

Then $F$ is a spectral measure since $E_N$ is, and for all $x, y \in \mathcal{K}$, and all $M \in \mathcal{S}$,

$$(F(M)x, y) = (E_N(M)U^{-1}x, U^{-1}y).$$

Hence for all $x, y \in \mathcal{K}$,

$$(N'x, y) = (UNU^{-1}x, y) = (NU^{-1}x, U^{-1}y) = \int \lambda d(E_N(\lambda)U^{-1}x, U^{-1}y) = \int \lambda d(F(\lambda)x, y).$$

By the uniqueness of the spectral measure of $N'$, $F = E_{N'}$.  

Thus the spectral theorem allows us to replace the original question by the following:

When are the spectral measures of two normal operators equivalent? To answer this question we shall break up spectral measures of normal operators into relatively simple pieces of the following sort:

Given a finite measure space \((M, \mathcal{S}, \mu)\), the spectral measure \(E: \mathcal{S} \to L(L_2(\mu))\) defined by \(E(M)f = \chi_M f\), for all \(M \in \mathcal{S}\) and all \(f \in L_2(\mu)\), is called the standard (or canonical) spectral measure on \(L_2(\mu)\).

**Theorem 1.2.3:** If \(\mu\) and \(\nu\) are finite measures on \((M, \mathcal{S})\) then \(\mu \sim \nu\) if and only if the standard spectral measure on \(L_2(\mu)\) is unitarily equivalent to the standard spectral measure on \(L_2(\nu)\).

**Proof:** Denote the standard spectral measures on \(L_2(\mu)\) and \(L_2(\nu)\) by \(E_{\mu}\) and \(E_{\nu}\) respectively. If \(U^{-1}E_{\nu}U = E_{\mu}\) for some unitary transformation \(U: L_2(\mu) \to L_2(\nu)\) then
\[
U^{-1} \int \lambda dE_{\nu}(\lambda) U = \int \lambda dE_{\mu}(\lambda) \quad \text{as in theorem 1.2.2.}
\]
In fact, for any bounded, measurable function \(f\) on \(M\) we have
\[
U^{-1} \int fdE_{\nu} U = \int fdE_{\mu}.
\]
Hence \(U\) is multiplication by \(U(1)\), since \(U\) agrees with multiplication by \(U(1)\) on bounded functions in \(L_2(\mu)\) (a dense subset of \(L_2(\mu)\)), and since multiplication by \(U(1)\) is a closed transformation. Now
\[
\|U(\chi_M)\|_{L_2(\nu)}^2 = \|U(1)\|_{L_2(\mu)}^2 \chi_M^2 d\nu = \|U(1)\|_{L_2(\mu)}^2 d\nu \quad \text{for all } M \in \mathcal{S}.
\]
But since \(U\) is distance-preserving,
\[
\|U(\chi_M)\|_{L_2(\nu)}^2 = \|\chi_M\|_{L_2(\mu)}^2 = \mu(M),
\]
so
\[ \int_M |U(1)|^2 \, du = \mu(M), \text{ for all } M \in S. \] Hence if \( \nu(M) = 0 \) then \( \mu(M) = 0 \), i.e. \( \mu \ll \nu \).

Similarly \( \nu \ll \mu \), so that \( \nu \sim \mu \).

Conversely, if \( \nu \sim \mu \) then there exists a function \( f \) on \( M \) such that for all \( M \in S \),

\[ \mu(M) = \int_M |f|^2 \, du. \]

So define \( U : L^2(\mu) \to L^2(\nu) \) by \( U(g) = fg \).

Since \( \|U(g)\|^2_{L^2(\nu)} = \int_M |f|^2 |g|^2 \, du = \int_M |fg|^2 \, d\mu = \|g\|^2_{L^2(\mu)} \), we conclude \( U \) is distance-preserving. It remains to show that \( U \) is onto:

If \( g \in L^2(\nu) \) is orthogonal to \( U(L^2(\mu)) \) then \( g \perp U(\chi_M) \), for all \( M \in S \). Hence

\[ (g, U(\chi_M))_{L^2(\nu)} = \int_M |gf|^2 \, du = \int_M |f|^2 \, du = 0, \text{ for all } M \in S. \]

Hence \( gf = 0 \) a.e. \( \nu \). But \( f \neq 0 \) a.e. \( \nu \), since \( \mu \sim \nu \), so \( g = 0 \) a.e. \( \nu \). So \( U(L^2(\mu)) = L^2(\nu) \). So \( U \) is a unitary transformation: \( L^2(\mu) \to L^2(\nu) \). Since \( U^{-1}E_\nu(M)U = E_\mu(M) \), for all \( M \in S \), \( E_\nu \equiv E_\mu \).

More generally, given a singular family \( \{\mu_j\} \) of measures on \( S \) and a family \( \{c_j\} \) of cardinal numbers, if for all \( j \) \( \mathcal{H}_j \) denotes the direct sum of \( c_j \) copies of \( L^2(\mu_j) \) then the standard spectral measure on \( \mathcal{H} = \bigoplus_j \mathcal{H}_j \) is the spectral measure \( E : S \to L(\mathcal{H}) \) defined by \( E(M) \{f_{jk}\} = \{\chi_M f_{jk}\} \), for all \( M \in S \) and all \( \{f_{jk}\}_{k=1}^{c_j} \in \mathcal{H}_j \), for all \( j \).

Note that here each \( L^2(\mu_j) \), viewed as a subspace of \( \mathcal{H} \), is invariant under every \( E(M) \).

Hence the projection of \( \mathcal{H} \) onto each \( L^2(\mu_j) \) commutes with \( E \).

Section 3: Cyclic Subspaces and Cyclic Projections

Given a vector \( x \in \mathcal{H} \) and a spectral measure \( E \), the cyclic subspace generated by \( x \), denoted by \( Z_x \), is the subspace of \( \mathcal{H} \) spanned by the vectors \( E(M)x, M \in S \).
Theorem 1.3.1: \( Z_x \) is the smallest subspace of \( \mathcal{H} \) containing \( x \) which reduces \( E \), i.e. which reduces \( E(M) \), for all \( M \in \mathcal{S} \).

Proof: \( Z_x \) contains \( x \), since \( E(M) = 1 \). Also, for all \( M, N \in \mathcal{S} \),

\[
E(M)E(N)x = E(M \cap N)x \in Z_x,
\]
so that \( Z_x \) is invariant under each \( E(M) \), and hence \( Z_x \) reduces each \( E(M) \).

To see that \( Z_x \) is the smallest subspace having these properties, note that if \( B \) is a subspace containing \( x \) and reducing \( E \) then for all \( M \in \mathcal{S} \) \( E(M)x \in B \). Hence \( Z_x \subseteq B \).

The cyclic projection \( Z_x \) generated by \( x \in \mathcal{H} \) is the projection of \( \mathcal{H} \) onto \( Z_x \). (We avoid the term "cycle" since in [6] it is an abbreviation for "cyclic projection", while in [2] it stands for "cyclic subspace.") By virtue of theorem 1.3.1 \( Z_x \) is the smallest projection \( P \) such that

i) \( Px = x \), and

ii) \( P \) commutes with \( E \).

The following theorem shows how to decompose spectral measures using cyclic subspaces and projections:

Theorem 1.3.2: Given a spectral measure \( E \) on \( \mathcal{H} \) and a vector \( x \in \mathcal{H} \), there exists a unique unitary transformation \( U: L_2(\mu_x) \rightarrow Z_x \) such that \( U^{-1}EU \) is the standard spectral measure on \( L_2(\mu_x) \).

Proof: Consider \( U: \chi_M \rightarrow E(M)x \), for all \( M \in \mathcal{S} \). Note that \( U(1) = U(\chi_M) = x \). Extend the domain of \( U \) to include all simple functions by requiring that \( U \) be linear (the linearity
of $E$ ensures that $U$ is well-defined). Then $U$ maps a dense subset of $L_2(\mu_x)$ onto a dense subset of $Z_x$. Note that $U$ is distance-preserving, since for all $M \in \mathcal{S}$,

$$\|U(\chi_M)\|_\mathcal{S}^2 = \|E(M)x\|_\mathcal{S}^2 = (E(M)x, x)_{\mathcal{S}} = \mu_x(M) = \|\chi_M\|_{L_2(\mu_x)}^2.$$

Hence $U$ extends to a unitary mapping: $L_2(\mu_x) \to Z_x$.

Now given $M, M' \in \mathcal{S}$,

$$U^{-1}E(M)U(\chi_M) = U^{-1}E(M)E(M')x = U^{-1}E(M \cap M')x = \chi_M \cap M' = \chi_M \cdot \chi_{M'}.$$

So $U^{-1}E(M)U$ is multiplication by $\chi_M$ on characteristic functions, hence on simple functions, hence on all of $L_2(\mu_x)$.

$U$ is unique, since if $V$ is also such a unitary transformation then $V^{-1}U$ is a unitary operator: $L_2(\mu_x) \to L_2(\mu_x)$ such that $U^{-1}VE_{\mu_x}V^{-1}U = E_{\mu_x}$, where $E_{\mu_x}$ is the standard spectral measure on $L_2(\mu_x)$. But we saw in theorem 1.2.3 that in this case $V^{-1}U$ is multiplication by $V^{-1}U(1)$. So since $V^{-1}U(1) = V^{-1}(x) = 1$, $V^{-1}U = 1$. Similarly,

$$U^{-1}V = 1.$$ Hence $U = V$.

We are already in a position to decompose spectral measures, using theorem 1.3.2:

Theorem 1.3.3: Given a spectral measure $E$ on $\mathcal{H}$, there exists a family $\{\mu_j\}$ of finite measures on $(\mathcal{M}, \mathcal{S})$ such that $E$ is unitarily equivalent to the direct sum of the standard spectral measures on $\sum_j \Theta L_2(\mu_j)$.

Proof: Apply Zorn's lemma to obtain a maximal family $\{Z_x\}$ of singular cyclic subspaces of $\mathcal{H}$. Then $\mathcal{H} = \sum_j \Theta Z_x$, else $\mathcal{H} \setminus \sum_j \Theta Z_x$ contains a nonzero vector $x$, and so contains $Z_x$, contradicting the maximality of $\{Z_x\}$.

The result follows from applying theorem 1.3.2 to each $\mu_x = \mu_j$. 

O
In order to provide a complete set of unitary invariants, however, we require a more
delicate decomposition, which in turn requires several additional concepts.

Section 4: Finite Measures

Given a measure $\mu$ on $(M, S)$ and $M \in S$, define the measure $\mu |_M (= \mu_M)$ by

$$\mu |_M (N) = \mu (M \cap N), \text{ for all } N \in S.$$ 

Theorem 1.4.1: Given a measure $\mu$ on $(M, S)$ and $M, N \in S$, $\mu |_M \sim \mu |_N$ if and only if the symmetric difference $M \Delta N$ has $\mu$ measure 0.

Proof: If $\mu(M \Delta N) = 0$ then $\mu(M \setminus N) = 0$.

Hence if $\mu |_N (G) = 0$ for some $G \in S$,

$$\mu |_M (G) = \mu (G \cap M) = \mu (G \cap M \cap N) + \mu (G \cap (M \setminus N)) = \mu (G \cap M \cap N)$$

$$= \mu |_N (G \cap M) = 0.$$ 

So $\mu |_M \ll \mu |_N$. Similarly $\mu |_N \ll \mu |_M$, since $\mu(N \setminus M) = 0$.

Conversely, if $\mu |_M \sim \mu |_N$ then since $\mu |_M (N \setminus M) = \mu (M \cap (N \setminus M)) = 0$ and

$$\mu |_N (M \setminus N) = 0, \mu |_N (N \setminus M) = \mu (N \setminus M) = 0 \text{ and } \mu |_M (M \setminus N) = \mu (M \setminus N) = 0,$$

that is $\mu(M \Delta N) = 0$.

We are concerned here with finite measures of the form $\mu_x$ for some spectral measure $E$, where $\mu_x(.) = (E(.)x, x)$, so let
\[ \mathcal{M} = \{ \mu \mid \mu \text{ is a finite measure on } (M, \mathcal{S}) \}. \]

Note that \( \mathcal{M}/\sim \) is partially ordered by \( \ll \), where \([\mu] \ll [\nu]\) if and only if \( \mu \ll \nu \). We shall write \( \mu \) for \([\mu]\) and \( \mathcal{M} \) for \( \mathcal{M}/\sim \), on the understanding that definitions will hold only up to equivalence.

Theorem 1.4.2: Given \( \mu \in \mathcal{M}, \{ \nu \in \mathcal{M} \mid \nu \ll \mu \} \) (interpreted via the preceding remark) is order isomorphic with the measure ring of \( \mu \), \( \mathcal{S}/\Delta(\mu) \) (where \( M,N \in \mathcal{S} \) are equivalent if and only if \( \mu(M \Delta N) = 0 \)).

Proof: Suppose \( \nu \in \mathcal{M} \) such that \( \nu \ll \mu \), and let \( \nu(.) = \int f \, d\mu \). Setting

\[ M_f = \{ z \in M \mid f(z) > 0 \}, \int f \, d\mu = 0, \text{ for all } M \in \mathcal{S}. \text{ Hence } \nu(M) = \int f \, d\mu, \text{ for all } M \in \mathcal{S}. \text{ Hence } \nu(M) = 0 \text{ if and only if } \mu(M_f \cap M) = \mu|_{M_f}(M) = 0, \text{ that is, } \nu \sim \mu|_{M_f}. \text{ Hence by theorem 1.4.1, } F: \mathcal{M} \to M_f \text{ is an order isomorphism of } \{ \nu \in \mathcal{M} \mid \nu \ll \mu \} \text{ with } \mathcal{S}/\Delta(\mu). \]

The supremum of \( \mu_1 \) and \( \mu_2 \in \mathcal{M} \), denoted by \( \mu_1 \vee \mu_2 \), is defined as 'the' finite measure \( \mu \) such that

i) \( \mu_1 \ll \mu \) and \( \mu_2 \ll \mu \), and

ii) if \( \mu_1 \ll \nu \) and \( \mu_2 \ll \nu \) then \( \mu \ll \nu \).

Of course \( \mu \) is defined only up to equivalence: \( \mu_1 + \mu_2 \) will do, for example, as a representative of \( \mu_1 \vee \mu_2 \).

More generally, given an arbitrary family \( \{ \mu_j \} \subseteq \mathcal{M} \), if there exists a measure \( \mu \in \mathcal{M} \) such that
i) \( \mu_j \ll \mu \), for all \( j \), and 

ii) if \( \mu_j \ll \nu \), for all \( j \), then \( \mu \ll \nu \),

then the supremum of \( \{ \mu_j \} \), denoted by \( \nu J \mu_j \), exists and equals \( \mu \). The supremum of a countable family \( \{ \mu_n \} \subseteq \mathcal{M} \) always exists: Take \( \mu = \sum \frac{1}{2^n} \frac{\mu_n}{\mu_n(M)} \), where the summation is over only those \( \mu_n \) such that \( \mu_n(M) > 0 \).

A family \( \{ \mu_j \} \subseteq \mathcal{M} \) is said to be bounded if there exists a measure \( \mu \in \mathcal{M} \) such that \( \mu_j \ll \mu \), for all \( j \).

Theorem 1.4.3: Any bounded family \( \{ \mu_j \} \subseteq \mathcal{M} \) has a supremum. Moreover, in this case \( \{ \mu_j \} \) has a countable subfamily \( \{ \mu_{j_k} \} \) such that \( \nu J \mu_j = \nu J \mu_{j_k} \).

Proof: Since \( \{ \mu_j \} \) is bounded by \( \mu \), say, there exists a family \( \{ N_j \} \subseteq \mathcal{S} \) such that

\[ \mu_j = \mu \big|_{N_j} \text{ for all } j \text{, as in theorem 1.4.2.} \]

Setting \( \alpha = \sup \{ \mu \big( \bigcup_{j \in J} N_j \big) \ | \ J \text{ is finite} \} \),

there exists a sequence \( \{ M_n \} \) of finite unions of the \( N_j \) such that

\[ \lim_{n \to \infty} \mu(M_n) = \alpha. \text{ Let } M = \bigcup_n M_n. \]

Given \( j', \mu(N_j \cup M_n) \leq \alpha \), for all \( n \), by definition of \( \alpha \). Hence \( \mu(N_j \cup M) = \alpha \). But \( \mu(M) = \alpha \), so \( \mu(N_j \setminus M) = 0 \). Hence \( \mu \big|_{N_j} \ll \mu \big|_{M_n} \), that is, \( \mu_j \ll \mu \big|_{M_n} \).

However, given \( \nu \in \mathcal{M} \) such that every \( \mu_j \ll \nu \), every \( \mu \big|_{M_n} \ll \nu \), and hence \( \mu \big|_{M} \ll \nu \). So \( \mu \big|_{M} = \nu \mu_j = \nu \mu \big|_{M_n} \). Of course, \( \{ N_j \big|_{N_j \in \text{some } M_n} \} \) is a countable subfamily of \( \{ N_j \} \), and \( \nu \big( \mu \big|_{N_j} \big|_{N_j \in \text{some } M_n} \big) = \nu \mu \big|_{M_n} = \nu \mu_j. \)
It follows from theorem 1.4.3 that any bounded, singular family \( \{ \mu_j \} \subseteq \mathcal{M} \) is equal to the countable subfamily \( \{ \mu_k \} \) such that \( \forall \mu_j = \forall k \mu_j_k \), and hence is countable.

If \( \{ \mu_j \} \) is a nonempty family \( \subseteq \mathcal{M} \) then the infimum of \( \{ \mu_j \} \), denoted by \( \wedge_j \mu_j \), is defined as the measure \( \in \mathcal{M} \) that is equal (up to equivalence) to

\[
\forall \{ \mu \in \mathcal{M} | \forall \mu \ll \mu_j, \text{ for all } j \}. 
\]

The existence of the infimum of \( \{ \mu_j \} \) is guaranteed by theorem 1.4.3. Also, observe that given \( \mu, \nu \in \mathcal{M}, \mu \perp \nu \) if and only if \( \mu \wedge \nu = 0 \).

For the record, then, \( \mathcal{M} \) is a boundedly complete, \( \sigma \)-complete lattice. It is also distributive, and is relatively complemented:

**Theorem 1.4.4:** (Lebesgue decomposition) Given \( \mu, \nu \in \mathcal{M} \), there exist \( \mu_0, \mu_1 \in \mathcal{M} \) such that \( \mu_0 \nu \mu_1 \sim \mu, \mu_0 \perp \nu, \) and \( \mu_1 \ll \nu \).

**Proof:** Since \( \nu \ll \nu \mu \), there exists \( N \in \mathfrak{S} \) such that \( \nu \sim (\nu \mu)_{\nu} \).

Let \( \mu_0 = (\nu \mu)_{\mathcal{M} \setminus N} \) and \( \mu_1 = \nu \wedge \mu \). Clearly \( \mu_0 \perp \nu \) and \( \mu_1 \ll \nu \).

Moreover, \( \mu_0 \ll \mu \) and \( \mu_1 \ll \mu \). So suppose that \( \mu_0 \ll \lambda \) and \( \mu_1 \ll \lambda \).

If \( \lambda(M) = 0 \) then \( (\nu \mu)((\mathcal{M} \setminus N) \cap M) = 0 \), and \( (\nu \wedge \mu)(M) = 0 \).

Hence \( (\nu \mu)(M) = (\nu \mu)(N \cap M) = \nu(M) \), and so \( \mu(M) = 0 \). So \( \mu \ll \lambda \).

This completes the treatment of finite measures as far as Halmos is concerned. Brown now discusses \( \sigma \)-ideals of measures, with an eye to defining the object \( m_B(N) \) as a function sending cardinal numbers to \( \sigma \)-ideals of \( \mathcal{M} \). Halmos, however, defines the object \( m_H(N) \) as a function taking elements of \( \mathcal{M} \) to cardinal numbers. Therefore he deliberately avoids discussing \( \sigma \)-ideals. Of course we shall discuss them, but only after we develop the version of multiplicity theory found in [6].
CHAPTER II: HALMOS' VERSION OF MULTIPLICITY THEORY

Section 1: Halmos' Multiplicity Function

Recall that for a finite dimensional Hilbert space the set of all eigenvalues of a normal operator \( N \), together with their multiplicities, form a complete set of unitary invariants. That is, the function \( m(N) : \mathbb{C} \to \mathbb{Z}^+ \) defined by \( m(N)(z) = \) the multiplicity of \( z \) as an eigenvalue of \( N \), if \( z \) is an eigenvalue of \( N \), and \( = 0 \), if \( z \) is not an eigenvalue of \( N \), is a suitable object for the solution to our problem: Given normal operators \( M, N \), \( M \equiv N \) if and only if \( m(M) = m(N) \). For an infinite dimensional Hilbert space, not only must the range of \( m(N) \) include all cardinal numbers not exceeding the dimension of the Hilbert space, but also the domain should be a set of measures, according to Halmos.

A multiplicity function is a function \( m : \mathcal{M} \to \{\text{cardinal numbers}\} \) such that

i) if \( \mu = 0 \) then \( m(\mu) = 0 \),

ii) if \( 0 \neq \nu \ll \mu \) then \( m(\mu) \leq m(\nu) \), and

iii) if \( \{\mu_n\} \) is a bounded, (hence countable) singular family of nonzero finite measures, then \( m(\bigvee_n \mu_n) = \min_n \{m(\mu_n)\} \).

Given \( \mu \in \mathcal{M} \), if, whenever \( 0 \neq \nu \ll \mu \), \( m(\mu) = m(\nu) \) then we say that \( \mu \) has uniform multiplicity (with respect to \( m \)).

Theorem 2.1.1: Given a multiplicity function \( m \) on \( \mathcal{M} \) and \( 0 \neq \mu \in \mathcal{M} \), there exists
\[ \eta \in \mathcal{M} \] such that \( 0 \neq \eta \ll \mu \) and \( \eta \) has uniform multiplicity.

Proof: Suppose that \( \mu \) does not have uniform multiplicity, and let
\[ \nu_0 = V\{ \nu \in \mathcal{M} \mid \nu \ll \mu \text{ and } m(\nu) < m(\mu) \}. \]
By theorems 1.4.3 and 1.4.4 we may express \( \nu_0 \) as \( V\{ \nu_n \} \) where \( \{ \nu_n \} \) is a countable, singular family \( \subseteq \mathcal{M} \), and so \( m(\mu) < m(\nu_0) \). So apply theorem 1.4.4 to \( \mu \) and \( \nu_0 \) to obtain \( \mu_0 \perp \nu_0 \) and \( \mu_1 \ll \nu_0 \ll \mu \) such that \( \mu_0 \nu \mu_1 \sim \mu \). Hence
\[ \mu_0 \nu \nu_0 \sim \mu_0 \nu_1 \nu_0 \sim \mu \nu \nu_0 \sim \mu. \]
Since \( m(\mu) < m(\nu_0) \) and \( \mu_0 \nu \nu_0 \sim \mu \), if follows that \( \mu_0 \neq 0 \). Hence \( m(\mu_0) \leq m(\mu) \leq m(\mu_0) \), that is, \( m(\mu_0) = m(\mu) \). Moreover, if \( 0 \neq \lambda \ll \mu_0 \) then \( \lambda \perp \nu_0 \), so \( m(\lambda) = m(\mu) = m(\mu_0) \).

Theorem 2.1.2: Given \( m \) and \( \mu \) as above, there exists a countable, singular family \( \{ \mu_n \} \subseteq \mathcal{M} \) such that each \( \mu_n \neq 0 \) has uniform multiplicity, and \( \mu \sim V_n \mu_n \).

Proof: Apply Zorn's Lemma to obtain a maximal singular family \( \{ \mu_n \} \) of nonzero measures \( \subseteq \mathcal{M} \) such that each \( \mu_n \) has uniform multiplicity and is absolutely continuous with respect to \( \mu \). (Such families exist by theorem 2.1.1.) If \( V_n \mu_n \sim \mu \) apply theorem 1.4.4 to obtain \( 0 \neq \mu_0 \perp V_n \mu_n \) such that \( \mu_0 \ll \mu \). Because of theorem 2.1.1 there exists \( \lambda \neq 0 \) such that \( \lambda \ll \mu_0 \) and has uniform multiplicity, contradicting the maximality of \( \{ \mu_n \} \), since \( \lambda \ll \mu \) and \( \lambda \perp V_n \mu_n \).

Theorem 2.1.3: Given \( m \) as above, there exists a singular family \( \{ \nu_k \} \) of nonzero measures \( \subseteq \mathcal{M} \) such that every \( \nu_k \) has uniform multiplicity and \( \mu \sim V_k (\mu \wedge \nu_k) \), for all \( \mu \in \mathcal{M} \).
Proof: Obtain a maximal singular family \( \{ \mu_j \} \) of nonzero measures \( \subseteq \mathcal{M} \) using Zorn's Lemma, and apply theorem 2.1.2 to every \( \mu_j \). The union of all of the families obtained in this way is a maximal singular family \( \{ \nu_k \} \) of nonzero measures \( \subseteq \mathcal{M} \), such that every \( \nu_k \) has uniform multiplicity. Again, if \( V_k(\nu_k \wedge \mu) \sim \mu \), for some \( \mu \in \mathcal{M} \), then by theorem 1.4.4 there exists \( \mu_0 \neq 0 \in \mathcal{M} \) such that \( \mu_0 \perp V_k(\nu_k \wedge \mu) \), and \( \mu_0 \ll \mu \). Hence \( \mu_0 \wedge \nu_k \sim \mu_0 \wedge \mu \wedge \nu_k \ll \mu_0 \wedge (V_k(\mu \wedge \mu_k)) = 0 \), that is \( \mu_0 \perp \nu_k \), for all \( k \), contradicting the maximality of \( \{ \nu_k \} \).

Theorem 2.1.4: Given \( m \) and \( \{ \nu_k \} \) as in theorem 2.1.3 and \( \mu \in \mathcal{M} \),

\[
m(\mu) = \min_k \{ m(\nu_k) \mid \nu_k \wedge \mu \neq 0 \}.
\]

Proof: Since \( m \) is a multiplicity function and \( \mu \ll V_k(\nu_k \wedge \mu) \ll \mu \),

\[
m(\mu) \geq m(V_k(\nu_k \wedge \mu)) \geq m(\mu), \quad \text{that is,}
\]

\[
m(\mu) = m(V_k(\nu_k \wedge \mu)) = \min_k \{ m(\nu_k \wedge \mu) \mid \nu_k \wedge \mu \neq 0 \}.
\]

However, every \( \nu_k \) has uniform multiplicity and \( \nu_k \wedge \mu \ll \nu_k \), hence if \( \nu_k \wedge \mu \neq 0 \) then

\[
m(\nu_k \wedge \mu) = m(\nu_k).
\]

Now that we know how multiplicity functions behave, we may formulate the objective:

To attach to every normal operator \( N \) a multiplicity function \( m_{H}(N) \) with the property that, given any singular family \( \{ \nu_k \} \) as in theorem 2.1.3, the spectral measure \( E_{N} \) of \( N \)

is unitarily equivalent to the standard spectral measure on \( K = \sum_k m_{H}(N)(\nu_k) \oplus ( \sum_{i=1}^{k} \oplus L_{2}(\nu_{k})) \). It
will follow that two normal operators \( N \) and \( N' \) are unitarily equivalent if and only if \( m_H(N) = m_H(N') \).

Section 2: Commutators and Double Commutators

We saw in Chapter 1, Section 3 that given a spectral measure \( E \) on a Hilbert space \( \mathcal{H} \) there is a useful subset of the set of all projections on \( \mathcal{H} \), namely the set of projections which commute with \( E \). Also, we know that \( \int f dE \) doubly commutes with \( E \), so long as \( f \) is a measurable function \( \in L_\omega(E) \). So let us study the projections which commute or doubly commute with \( E \).

Given a set \( A \) of projections on \( \mathcal{H} \), let \( A' \) denote the set of projections which commute with (every element of) \( A \). \( A' \) is called the commutator of \( A \). Note that \( A \subseteq A'' \), the double commutator of \( A \). Also, if \( B \subseteq A \) then \( A' \subseteq B' \).

**Theorem 2.2.1:** \( A' = A'' \).

**Proof:** Since \( A \subseteq A'' \), \( A'' \subseteq A' \subseteq A''' \).

**Theorem 2.2.2:** \( A \subseteq A' \) if and only if \( A'' \subseteq A''' \).

**Proof:** \( A \subseteq A' \) implies \( A'' \subseteq A' = A'' \). Conversely, \( A'' \subseteq A''' \) implies \( A \subseteq A'' \subseteq A' \).
If $V_iP_i$ denotes the smallest projection $\geq$ every $P_i$ and $\Lambda_iP_i$ denotes the largest projection $\leq$ every $P_i$, then $A'$ is closed under $V$ and $\Lambda$. Also, of course, $0, 1 \in A'$.

Section 3: Rows and Columns

Consider in particular the set $\xi = \{E(M) \mid M \in S\}$, the range of $E$. Here $\xi'$, the commutator of $\xi$, is the set of projections which commute with $E$. Since $\xi \subseteq \xi'$, $\xi'' \subseteq \xi'$ by theorem 2.2.2.

Given $P \in \xi'$, the column generated by $P$, denoted by $C(P)$, is defined as the smallest element of $\xi''$ which contains $P$, that is

$$C(P) = \Lambda \{F \in \xi'' \mid P \leq F\}.$$ 

Theorem 2.3.1: Given a family of projections $\{P_i\} \subseteq \xi'$, $C(V_iP_i) = V_iC(P_i)$.

Proof: Since every $P_i \leq C(P_i)$, $V_iP_i \leq V_iC(P_i) \in \xi''$.

Hence $C(V_iP_i) \leq C(V_iC(P_i)) = V_iC(P_i)$. Also, every $P_j \leq V_iP_i \leq C(V_iP_i)$, so every $C(P_j) \leq C(V_iP_i)$, and so $V_jC(P_j) \leq C(V_iP_i)$. 

Theorem 2.3.2: Given $P \in \xi'$ and $F \in \xi''$, $C(FP) = FC(P)$.

Proof: $FP \leq F$, so $C(FF) \leq C(F) = F$. Similarly, $C(FP) \leq C(P)$.

Hence $C(FP) \leq FC(P)$.

Also, $FC(P) = FC((1 - F)P + FP) \leq FC((1 - F) + FP) = FC(FP) \leq C(FP)$. 

$\Box$
Theorem 2.3.3: Given $P, F$ as above, if $0 \neq F \leq C(P)$ then $FP \neq 0$.

Proof: If $FP = 0$ then $C(FP) = 0 = FC(P) = F$, contradicting $F \neq 0$.

A row is a projection $R \in \xi'$ such that, for all $P \in \xi'$, if $R \geq P$ then $RC(P) = P$.

Theorem 2.3.4: Given $R, S \in \xi'$ such that $R \geq S$, if $R$ is a row then $S$ is a row.

Proof: Given $P \in \xi'$ such that $P \leq S \leq R$,

$P = SP = SRC(P) = SC(P), \text{ if } R \text{ is a row.}$

Theorem 2.3.5: Given a row $R \geq P, Q \in \xi'$, $C(P)C(Q) = C(PQ)$.

Proof: $C(PQ) = C(RC(P)RC(Q)) = C(C(P)C(Q)R) = C(P)C(Q)C(R)$ by theorem 2.3.2.

But $C(R) \geq C(P), C(Q) \in \xi''$, so $C(P)C(Q)C(R) = C(P)C(Q)$.

Note that given $P, Q$ and $R$ as above, $PQ = RC(P)RC(Q) = RC(Q)RC(P) = QP$.

Theorem 2.3.6: Given $R \in \xi'$, $R$ is a row if and only if whenever $R \geq P, Q \in \xi'$ such that $PQ = 0$, $C(P)C(Q) = 0$.

Proof: If $R$ is a row and $R \geq P, Q \in \xi'$ such that $PQ = 0$, then $0 = C(PQ) = C(P)C(Q)$ by theorem 2.3.5.
Conversely, if \( C(P)C(Q) = 0 \) whenever \( R \geq P \), \( Q \in \xi' \) such that \( PQ = 0 \) then in particular this is true when \( Q = RC(P) - P \leq C(P) \), \( P \) arbitrary \( \in \xi' \). But then \( C(Q) \leq C(P) \), so \( C(P)C(Q) = 0 = C(Q) \), so \( Q = 0 \), that is \( P = RC(P) \) whenever \( R \geq P \in \xi' \).

Because of results like the last one we introduce a bit of terminology: Given

\[ \text{if } C(P)C(Q) = 0 \text{ then we say that } P \text{ and } Q \text{ are } \text{very singular} \text{ (or } \text{very orthogonal}). \]

\section*{Section 4: Cyclic Projections Revisited}

Recall (see Chapter I, Section 3) that given \( X \in \mathcal{H} \), the cyclic projection \( Z_x \) is the smallest projection \( Q \in \xi' \) such that \( Qx = x \). Hence if \( P \in \xi' \) such that \( Px = x \), \( Z_x \leq P \).

Replacing \( P \) by \( 1-P \), we may conclude that if \( P \in \xi' \) such that \( Px = 0 \), \( PZ_x = 0 \).

Also note that if \( P \in \xi' \) and \( P \leq Z_x \), then \( P = PZ_x = Z_{p_x} \), since \( PE(M)x = E(M)Px \), for all \( M \in \mathcal{S} \).

\textbf{Theorem 2.4.1:} Given \( x \in \mathcal{H} \) and \( F \in \xi'' \), \( FZ_x = Z_{F_x} \).

\textbf{Proof:} Since \( Z_x \geq FZ_x \in \xi' \), \( FZ_x = Z_{FZ_x} = Z_{F_x} \), since \( Z_xx = x \).

Let us abbreviate \( C(Z_x) \) by \( C_x \). Since \( x = 0 \) if and only if \( Z_x = 0 \) and \( P = 0 \) if and only if \( C(P) = 0 \), \( x = 0 \) if and only if \( C_x = 0 \). In fact, we may show a little bit more:
Theorem 2.4.2: Given $x \in \mathcal{H}$ and $F \in \xi''$ such that $0 \neq F \leq C_x$, $Fx \neq 0$.

Proof: If $Fx = 0$ then $Z_{Fx} = 0 = FZ_x$, so that $F = FC_x = C(FZ_x) = 0$.

Cyclic projections are very useful for characterizing rows, as the following theorems demonstrate:

Theorem 2.4.3: Given $P \in \xi'$ there exists a singular family $\{Z_x\}$ of cyclic projections such that $P = V_j Z_x$.

Proof: Obtain a maximal family $\{x_j\}$ of non-zero vectors $\mathcal{M} \subset \mathcal{F}_P$ such that $\{Z_{x_j}\}$ is a singular family (we may assume $P \neq 0$). If there exists a non-zero vector $y \in (P - V_j Z_{x_j}) \mathcal{H}$ then $V_j Z_{x_j} y = 0$, so that $Z_{x_j} y = 0$, for all $j$, and so $Z_{x_j} Z_y = 0$, for all $j$, that is $y \in \mathcal{F}_P$ such that $\{Z_{x_j}\} \cup \{Z_y\}$ is a singular family, contradicting the maximality of $\{Z_{x_j}\}$.

Theorem 2.4.3 allows us to improve upon theorem 2.3.6:

Theorem 2.4.4: Given $R \in \xi'$, $R$ is a row if and only if whenever $R \geq Z_x, Z_y$ cyclic projections such that $Z_x Z_y = 0$, $C_x C_y = 0$.

Proof: If $R$ is a row and $Z_x, Z_y$ are singular, $\leq R$, then since $\xi'$ contains all cyclic projections, $C_x C_y = 0$ by theorem 2.3.6.
Conversely, if \( R \geq P, \ Q \in \xi' \) such that \( PQ = 0 \), then since \( P = V_iZ_{x_i} \) and \( Q = V_jZ_{y_j} \) by theorem 2.4.3, we may conclude that \( Z_{x_i}Z_{y_j} = 0 \), for all \( i \) and \( j \).

By hypothesis, \( C_{x_i}C_{y_j} = 0 \), for all \( i \) and \( j \), hence \( C(P)C(Q) = V_{i,j}C_{x_i}C_{y_j} = 0 \). By theorem 2.3.6, \( R \) is a row.

We know (theorem 2.4.3) that rows are built of cyclic projections. However, cyclic projections are themselves rows:

**Theorem 2.4.5:** Given \( x \in \mathcal{H} \), \( Z_x \) is a row.

**Proof:** Suppose \( Z_x \geq P \in \xi' \).

By theorem 1.3.2 there exists an isomorphism \( U : L_2(\mu_x) \rightarrow Z_x \) such that \( U^{-1}E(M)U(f) = \chi_M \cdot f \), for all \( M \in \mathcal{S} \), \( f \in L_2(\mu_x) \).

Hence \( U^{-1}PU \) is a projection on \( L_2(\mu_x) \), and

\[
U^{-1}PU(1) \cdot \chi_M = \chi_M \cdot U^{-1}PU(1) = U^{-1}E(M)UU^{-1}PU(1) = U^{-1}E(M)PU(1) = U^{-1}PE(M)U(1)
\]

\[
= U^{-1}PUU^{-1}E(M)U(1) = U^{-1}PU(\chi_M \cdot 1) = U^{-1}PU(\chi_M), \text{ for all } M \in \mathcal{S}. \text{ Hence (as in theorem 1.2.3) } U^{-1}PU \text{ is multiplication by } U^{-1}PU(1) \text{ on all of } L_2(\mu_x). \text{ Since }
\]

\( U^{-1}PU = U^{-1}PUU^{-1}PU \) (since \( P = P^2 \)) there exists \( N \in \mathcal{S} \) such that \( U^{-1}PU(1) = \chi_N \).

Now given \( y \in Z_x \), there exists \( f \in L_2(\mu_x) \) such that \( y = U(f) \). Hence

\[
P_y = PU(f) = UU^{-1}PU(f) = U(\chi_N \cdot f) = E(N)U(f) = E(N)y. \text{ Hence } P = E(N)Z_x, \text{ and so }
\]

\( C(P) = E(N)C_x \). Hence \( C(P)Z_x = E(N)C_xZ_x = E(N)Z_x = P \), since \( Z_x \leq C_x \).

We now know that rows exist. They also have nice properties:
Theorem 2.4.6: Given a very singular family \( \{Z_{x_j}\} \) of cyclic projections, \( V_jZ_{x_j} \) is a row.

Proof: Because of theorem 2.4.4 it suffices to show that if \( Z_w, Z_y \leq V_jZ_{x_j} \) such that

\[ Z_wZ_y = 0 \] then \( C_wC_y = 0 \). For all \( j \), let \( w_j = C_yC_{x_j}w \) and \( y_j = C_wC_{x_j}y \). Theorem 2.4.1 implies that \( Z_{w_j} = C_yC_{x_j}Z_w \leq Z_w \) and \( Z_{y_j} = C_wC_{x_j}Z_y \leq Z_y \), for all \( j \). Hence \( Z_{w_j}Z_{y_j} = 0 \), for all \( j \). However, every \( y_j \in C_{x_j}H \), and the \( Z_{x_j} \) are very singular, so every

\[ y_j \perp C_{x_k}H, \text{ whenever } k \neq j. \] Hence every \( y_j \perp Z_{x_k} \), whenever \( k \neq j \). But every

\[ y_j \in Z_{y_j} \subseteq Z_y, \text{ and } Z_y \subseteq V_jZ_{x_j}H, \text{ so every } y_j \in Z_{x_j}. \] Similarly, every \( w_j \in Z_{x_j} \). Hence

\[ Z_{y_j}, Z_{w_j} \leq Z_{x_j}, \text{ for all } j. \] However, every \( Z_{x_j} \) is a row, by theorem 2.4.5, so it follows

that \( Z_{y_j} = C(Z_{y_j})Z_{x_j} = C(C_wC_{x_j}Z_y)Z_{x_j} = C_wC_xZ_{x_j}, \) and similarly \( Z_{w_j} = C(Z_{w_j})Z_{x_j} = C_wC_xZ_{x_j}, \) for all \( j \). Hence \( Z_{y_j} = Z_{y_j}Z_{y_j} = Z_{y_j}Z_{w_j} = 0, \) for all \( j \). But \( C_y \leq C(V_jZ_{x_j}), \) since \( Z_y \leq V_jZ_{x_j}, \) so

\[ C_wC_y = C_wC_xC(V_jZ_{x_j}) = V_jC_wC_xC_{x_j} = V_jC(C_wC_xZ_{x_j}) = V_jC(0) = 0. \]

Theorem 2.4.7: Given a very singular family \( \{R_j\} \) of rows, \( V_jR_j \) is a row.

Proof: Theorem 2.4.3 implies that \( R_j = V_{k_j}Z_{j,k_j} \), where \( \{Z_{j,k_j}\} \) is a singular family of cyclic projections, for all \( j \). In fact, since every \( R_j \) is a row, theorem 2.4.4 implies that

\( \{Z_{j,k_j}\} \) is very singular, for all \( j \). So since \( \{R_j\} \) is very singular, \( \cup_j k_j \{Z_{j,k_j}\} \) is very singular. Hence \( V_jR_j = V_{j,k_j}Z_{j,k_j} \) is a row, by theorem 2.4.6.

Theorem 2.4.8: Given \( P \in \xi' \), there exists a row \( R \in \xi' \) such that \( R \leq P \) and \( C(P) \)
= C(R).

Proof: Without loss of generality we may assume \( P \neq 0 \). If \( x \) is a nonzero vector \( \in P H \) then \( Zx \) is a nonzero row (theorem 2.4.5) \( \leq P \). So let \( \{ R_j \} \) be a maximal very singular family of nonzero rows which are \( \leq P \), and let \( R \) be the row \( \vee R_j \leq P \). If
\[
P \leq C(V_j R_j) \leq V_j C(R_j)
\]
then there exists a nonzero \( x \in P H \) such that \( C(R_j)x = 0 \), for all \( j \). Hence \( C(R_j)x = C(C(R_j)Z_x) = C(ZC(R_j)x) = C(C(R_j)x) = 0 \), for all \( j \). But
\[
P x = x, \text{ so } Zx \text{ is a row } \leq P, \text{ contradicting the maximality of the very singular family } \{ R_j \}. \text{ Hence } P \leq V_j C(R_j) = C(R), \text{ so that } C(P) \leq C(C(R)) = C(R). \text{ But } R \leq P, \text{ so } C(R) \leq C(P). \text{ Hence the result.}
\]

Section 5: Separable Projections

Given \( F \in \xi'' \), if every singular family \( \{ F_j \} \subseteq \xi'' \) such that \( 0 \neq F_j \leq F \), for all \( j \), is countable then we say that \( F \) is separable. As we shall see, columns of the form
\[
C_x, x \in H,
\]
are characterized by the notion of separability.

Theorem 2.5.1: Given a countable singular family \( \{ F_j \} \) of separable projections \( \subseteq \xi'' \), \( \bigvee_j F_j \) is separable.

Proof: Given a singular family \( \{ H_k \}_k \) of projections \( \subseteq \xi'' \) such that \( 0 \neq H_k \leq \bigvee_j F_j \), for all \( k \), observe that for all \( j \) \( \{ F_j H_k \}_k \) is a singular family \( \subseteq \xi'' \) such that \( F_j H_k \leq F_j \), for all \( k \). Hence for all \( j \) only countably many \( F_j H_k \neq 0 \), since \( F_j \) is separable. So since
every $H_k = H_k \forall F_j = V_j F_j H_k$, and \{F_j\} is countable, it follows that \{H_k\} is countable.

Theorem 2.5.2: Given $x \in \mathcal{H}$, $C_x$ is separable.

Proof: Given a singular family $\{F_j\} \subseteq \xi''$ such that $0 \neq F_j \leq C_x$, for all $j$, $V_j F_j \leq C_x$, and so $\sum_j \|F_j x\|^2_{\mathcal{H}} = \|\sum_j F_j x\|^2_{\mathcal{H}} = \|V_j F_j x\|^2_{\mathcal{H}} \leq \|C_x x\|^2_{\mathcal{H}} = \|x\|^2_{\mathcal{H}}$.

But by theorem 2.4.2, every $F_j x \neq 0$, so $\sum_j \|F_j x\|^2_{\mathcal{H}}$ is a countable sum, that is \{F_j\} is a countable family.

Theorem 2.5.3: Given $P, Q \in \xi'$ such that $P \leq C(Q)$ which is separable, there exists $x \in \mathcal{PH}$ such that $C(P) = C_x$.

Proof: Obtain a maximal family $\{x_j\}$ of nonzero vectors $\subseteq \mathcal{PH}$ such that the family $\{Z_j\}$ is very singular, that is, such that $\{C_{x_j}\}$ is singular. Since every $C_{x_j} \leq C(Q)$, and $C(Q)$ is separable, \{C_{x_j}\} is countable, so that $\{x_j\}$ is countable. Hence without loss of generality we may assume, taking $x = \sum_j x_j$ that $\|x\|^2_{\mathcal{H}} < \infty$.

Since $x \in \mathcal{PH}$, $C_x \leq C(P)$. If $C(P) - C_x \neq 0$ then by theorem 2.3.3

$(C(P) - C_x)P \neq 0$, so that there exists nonzero $y \in (C(P) - C_x)\mathcal{PH}$. Hence $y \in \mathcal{PH}$, so that $y \in C(P)\mathcal{H}$. Since $y \in (C(P) - C_x)\mathcal{H}$, $C_x y = 0$. Hence $C_x y = C(C_x Z_y) = C(Z_c y) = C_{c_y} = 0$. But every $C_{x_j} x = \sum_k C_{x_j} x_k = x_j$, since $C_{x_j} x_k = 0$.
unless \( j = k \). Hence \( C_{x_j} C_x = C(C_{x_j} Z_{x_j}) = C(Z_{c_{x_j} x}) = C_{c_{x_j} x} = C_{x_j} \), for all \( j \). Hence

\[ C_{x_j} C_y = C_{x_j} C_x C_y = 0, \]

for all \( j \), contradicting the maximality of \( \{ x_j \} \). Therefore

\[ C(P) = C_x, \]

as required.

The point of theorem 2.5.3 is that (recall \( \xi'' \subseteq \xi' \)) if \( F \) is a separable projection \( \in \xi'' \) then \( F \leq C(F) = F \), so that there exists \( x \in \mathcal{H} \) such that \( F = C(F) = C_x \). Hence, as promised,

\[ \{ F \in \xi'' \mid F \text{ is separable} \} = \{ C_x \mid x \in \mathcal{H} \}. \]

Section 6: Measures and Columns of Cyclic Projections

In this section we develop connections between finite measures of the form \( \mu_x \) and columns of the form \( C_x \):

Theorem 2.6.1: Given \( x, y \in \mathcal{H} \), \( \mu_x \perp \mu_y \) if and only if \( C_x C_y = 0 \).

Proof: If \( \mu_x \perp \mu_y \) then there exists \( M \in \mathcal{S} \) such that \( \mu_x(\mathbb{M}M) = 0 = \mu_y(M) \). Hence

\[ E(M)y = 0 = E(\mathbb{M}M)x, \]

since \( \mu_z(N) = (E(N)z, z)_\mathcal{H} = \|E(N)z\|_\mathcal{H}^2 \) for all \( N \in \mathcal{S}, z \in \mathcal{H} \), and so \( E(M)x = x \) and \( E(\mathbb{M}M)y = y \). Hence

\[ Z_y \leq E(\mathbb{M}M) \text{ and } Z_x \leq E(M), \]

so

\[ C_y \leq C(E(\mathbb{M}M)) = E(\mathbb{M}M) \text{ and } C_x \leq C(E(M)) = E(M). \]

Hence

\[ C_x C_y \leq E(\mathbb{M}M)E(M) = E(\mathbb{M}M \cap M) = 0. \]

Conversely, suppose \( C_x C_y = 0 \). Since \( C_x, C_y \) are separable \( C_x V C_y = C(Z_x V Z_y) \) is separable by theorem 2.5.1, so that \( C_x V C_y = C_z \), for some \( z \in (Z_x V Z_y)\mathcal{H} \), by theorem 2.5.3. By theorem 1.3.2 there exists an isomorphism \( U: L_2(\mu) \to Z_z \) such that
$U^1E(M)U$ is multiplication by $\chi_M$, for all $M \in S$. Since $x, y \in \mathbb{Z}$ and

$C_x C_y = 0 = Z_x Z_y$, letting $f = U^1 x$ and $g = U^1 y$ we have

$$\int_M fg^*d\mu_z = \int_M f \chi_M f^*d\mu_z = (\chi_M f, g)_{L_2(\mu_z)} = (U\chi_M f, U\mu)_{E(M)} = (E(M)x, y)_{E(M)} = 0,$$

for all $M \in S$. Hence $f \cdot g^* = 0$ a.e. $(\mu_z)$, so there exists $N \in S$ such that $f = 0$ almost everywhere in $N$ and $g = 0$ almost everywhere in $MN$. Hence

$$0 = \int_N 0^2 d\mu_z = (\chi_N \cdot f, f)_{L_2(\mu_N)} = (U\chi_N \cdot f, U\mu)_{E(N)} = (E(N)x, x)_{E(N)} = x(N),$$

and similarly

$$\mu_y(M \setminus N) = 0.$$ Hence $\mu_x \perp \mu_y$.

**Theorem 2.6.2:** Given $x, y \in \mathcal{H}$, $\mu_x \ll \mu_y$ if and only if $C_x \leq C_y$.

**Proof:** If $C_y x = w$, say, then $w = C_y w$, so $Z_w \leq C_y$, so $C_w \leq C_y$. Also,

$$C_y C_x \cdot w = C(C_y Z_x \cdot w) = C(Z_y(x \cdot w)) = C(Z_0) = C_0 = 0.$$ Since

$$Z_w Z_x \cdot w \leq C_w C_x \cdot w \leq C_y C_x \cdot w = 0,$$

$\mu_x = \mu_w + \mu_x \cdot w$. Hence if $\mu_x \ll \mu_y$ then

$$\mu_x \cdot w \ll \mu_y.$$ But theorem 2.6.1 says that $\mu_x \perp \mu_y$, since $C_y C_x \cdot w = 0$. Hence $\mu_x \cdot w = 0$, so $x = w$, so $C_x = C_w \leq C_y$.

Conversely, if $C_x \leq C_y$ and $\mu_y(M) = 0 = \|E(M)y\|^2_{E(M)}$, then $E(M)y = 0$, so

$$C_{E(M)y} = E(M)C_y = 0,$$ and so $E(M)C_x = 0 = C_{E(M)x}$. Hence $E(M)x = 0$, so that

$$\mu_x(M) = 0.$$ So $\mu_x \ll \mu_y$.

**Theorem 2.6.3:** Given $x \in \mathcal{H}$ and $v \in \{v_0 \in \mathcal{M} : v_0 \ll \mu_x\}$, there exists $y \in \mathbb{Z}$ such that $\mu = \mu_y$. Moreover, if $v \sim \mu_x$ then $Z_y = Z_x$. 


Proof: If \( u << \mu_x \) then let \( v(.) = \int g \, d\mu_x \) for some nonnegative \( g \in L_1(\mu_x) \). Setting \( f \) to be the nonnegative square root of \( g \), and applying theorem 1.3.2 to obtain an isomorphism \( U: L_2(\mu_x) \to Z_x \) such that \( U^{-1}E(M)U \) is multiplication by \( \chi_M \), for all \( M \in S \), we have

\[
\nu(M) = \int_M f^* d\mu_x = ||\chi_M \cdot f||^2_{L_2(\mu_x)} = ||U\chi_M \cdot f||^2_{H^2} = ||E(M)Uf||^2_{H} = \mu_{Uf}(M), \text{ for all } M \in S.
\]

So let \( y = Uf \in Z_x \).

Moreover, if \( u \sim \mu_x \), then theorem 2.6.2 implies that \( C_x = C_y \). Hence (since \( Z_x \) is a row and \( Z_y \leq Z_x \)) we may conclude that \( Z_y = C_yZ_x = C_xZ_x = Z_x \).

Now we are in a position to introduce a projection \( \in \xi'' \) which will play a very important role in the definition of the multiplicity function of Halmos (and in the definition of the multiplicity function of Brown): Given \( \mu \in \mathcal{M} \), let \( F_\mu \) \( (= \mathcal{C}(\mu)) \) denote the projection onto the subspace \( \{ x \in \mathcal{H} \mid \mu_x << \mu \} \) of \( \mathcal{H} \). \( \{ x \in \mathcal{H} \mid \mu_x << \mu \} \) is a subspace since \( ||E(M)(\alpha x + \beta y)||_{\mathcal{H}} \leq ||\alpha|| \cdot ||E(M)x||_{\mathcal{H}} + ||\beta|| \cdot ||E(M)y||_{\mathcal{H}} \) implies \( \mu_{\alpha x + \beta y} << \mu_x \vee \mu_y \) and \( x_n \to x \) implies \( ||E(M)x_n||_{\mathcal{H}} \to ||E(M)x||_{\mathcal{H}} \), which implies \( \mu_x << \vee_n \mu_{x_n} \). Here are some useful properties of this type of projection:

Theorem 2.6.4: Given \( \mu \in \mathcal{M} \), \( F_\mu \in \xi'' \).

Proof: Given \( x \in \mathcal{H} \) and \( P \in \xi' \), \( \mu_{px} << \mu_x \), since \( ||E(M)Px||_{\mathcal{H}} = ||PE(M)x||_{\mathcal{H}} \leq ||E(M)x||_{\mathcal{H}} \), for all \( M \in S \).
Hence $\mu_x \ll \mu$ implies $\mu_{p_x} \ll \mu$, and $\mu_x \perp \mu$ implies $\mu_{p_x} \perp \mu$. Hence $P$ commutes with $F_\mu$, as required.

Theorem 2.6.5: Given $\mu \in M$, $F_\mu$ is separable, and hence is the column of some cyclic projection. Moreover, if $\mu = \mu_x$ then $C_x = F_\mu$.

Proof: Since $F_\mu \in \xi'' \subseteq \xi'$, theorem 2.4.8 implies that there exists a row $R \in \xi'$ such that $F_\mu = C(F_\mu) = C(R)$. By theorem 2.4.3 there exists a singular family $\{Z_j\}$ of cyclic projections such that $R = \bigvee_j Z_j$, and by theorem 2.4.4 $\{C_j\}$ is a singular family of columns. Hence $\{\mu_j\}$ is a singular family $\subseteq M$ by theorem 2.6.1. But every $x_j \in F_\mu M = \bigvee_j C_j M$, that is every $\mu_j \ll \mu$, so $\mu_j = 0$ except for countably many $j$. Hence $x_j = 0$ except for countably many $j$ (see the remark following theorem 1.4.3).

Hence by theorems 2.5.1 and 2.5.2 $F_\mu$ is separable.

Moreover, if $\mu = \mu_x$ and $y \in F_\mu M$ then by theorem 2.6.2 $C_y \leq C_x$, and so $y \in C_x M$. Hence $F_\mu \leq C_x$. But also $y \in C_x M$ implies $C_y \leq C_x$, which implies $\mu_y \ll \mu_x = \mu$, that is $y \in F_\mu M$. So $C_x \leq F_\mu$.

Theorem 2.6.6: Given $\mu, \nu \in M$, $F_{\mu \wedge \nu} = F_\mu F_\nu$, and so $\mu \ll \nu$ implies $F_\mu \leq F_\nu$.

Proof: If $x \in F_{\mu \wedge \nu} M$ then $\mu_x \ll \mu \wedge \nu \ll \mu, \nu$, so that $x \in F_\mu M, F_\nu M$.

Conversely, if $x \in F_\mu M, F_\nu M$ then $\mu_x \ll \mu, \nu$, so that (by definition of the infimum) $\mu_x \ll \mu \wedge \nu$, that is $x \in F_{\mu \wedge \nu} M$. Hence $F_{\mu \wedge \nu} = F_\mu F_\nu$.

If $\mu \ll \nu$ then $\mu \wedge \nu = \mu$, so that $F_\mu = F_{\mu \wedge \nu} = F_\mu F_\nu$, so $F_\mu \leq F_\nu$.

Theorem 2.6.7: Given $x \in M$ and $\mu \in M$ such that $F_\mu x = 0$, $\mu \perp \mu_x$. 
Proof: Since $\mu_x \land \mu \ll \mu$, theorem 2.6.3 implies that there exists $y \in Z_x$ such that

$$\mu_y = \mu_x \land \mu \ll \mu,$$

so that $y \in F_\mu H$. But also $F_\mu Z_x = Z_{F_\mu x} = Z_0 = 0$, so $F_\mu y = 0$.

Hence $y = 0$, so $\mu_y = 0 = \mu_x \land \mu$, so $\mu_x \perp \mu$.

Theorem 2.6.8: Given a bounded (countable) singular family $\{\mu_j\} \subseteq \mathcal{M}$, $F_\mu = \bigvee_j F_j$.

Proof: Theorem 2.6.6 implies that every $F_\mu_j \leq F_{\bigvee_j F_j}$, and so $\bigvee_j F_\mu_j \leq F_{\bigvee_j F_j}$.

Conversely, if $0 \neq x \in (F_{\bigvee_j F_j} - \bigvee_j F_\mu_j) H$ then $\mu_x \ll \bigvee_j F_\mu_j$, and since $F_\mu_j x = 0$ for all $j$, theorem 2.6.7 implies $\mu_x \perp \mu_j$ for all $j$. So $\mu_x \perp \bigvee_j F_\mu_j$. Hence

$$\mu_x = 0,$$

so $x = 0$, a contradiction.

Section 7: Singular (or Orthogonal) Systems

In order to use projections of the form $F_\mu$, $\mu \in \mathcal{M}$, to define Halmos' multiplicity function $m_H(N)$ we need one more fundamental concept:

Given $F \in \xi''$, a singular family $\{R_j\}$ of nonzero rows such that $C(R_j) = F$, for all $j$, is called a singular (or orthogonal) system of type $F$.

Theorem 2.7.1: Given a cardinal number $c$ and a singular family $\{F_j\}$ of nonzero projections $\subseteq \xi''$, if for every $j$ $\{R_{jk}\}_k$ is a singular system of type $F_j$ and of cardinality (or power) $c$, then $\{V_j R_{jk}\}$ is a singular system of type $\bigvee_j F_j$ and of cardinality $c$. 
Proof: Theorem 2.4.7 implies that \( V_j R_{jk} \) is a (nonzero) row, for all \( k \). Hence \( \{ V_j R_{jk} \} \) is a singular family of nonzero rows, and for all \( k \) \( C(V_j R_{jk}) = V_j C(R_{jk}) = V_j F_j \).

Note that since every element \( R_j \) of a singular system \( \{ R_j \} \) of type \( F \) is \( \leq C(R_j) = F \), we have \( V_j R_j \leq V_j C(R_j) = F \). If in fact \( V_j R_j = F \) then we say that \( \{ R_j \} \) is complete. Since every \( R_j \neq 0 \), any complete singular system is a maximal singular system.

Theorem 2.7.2: Given a singular system \( \{ R_j \} \) of type \( F \) and \( G \in \xi'' \) such that \( 0 \neq G \leq V_j R_j \) \( \{ GR_j \} \) is a singular system of type \( G \). Moreover, if \( \{ R_j \} \) is complete then \( \{ GR_j \} \) is complete.

Proof: Theorem 2.3.3 implies that every \( GR_j \neq 0 \), and theorem 2.3.4 implies that every \( GR_j \) is a row. Since every \( GR_j \leq R_j \) and \( C(GR_j) = GC(R_j) = GF = G \), for all \( j \), \( \{ GR_j \} \) is a singular system of type \( G \). Moreover, if \( V_j R_j = F \) then \( V_j GR_j = GV_j R_j = GF = G \).

Theorem 2.7.3: Given a singular system \( \{ R_j \} \) of type \( F \) and \( G \in \xi'' \) such that \( 0 \neq G \leq V_j R_j \) \( \{ GR_j \} \) is a complete singular system of type \( G \).

Proof: Since \( V_j R_j \leq F \) theorem 2.7.2 gives everything but completeness. However,
Theorem 2.7.4: Given a maximal singular system \( \{R_j\} \) of type \( F \), there exists nonzero \( x \in F \mathbb{H} \) such that \( \{C_xR_j\} \) is a complete singular system of type \( C_x \).

Proof: Clearly \( C(F - V_jR_j) \leq C(F) = F \). If in fact \( C(F - V_jR_j) = F \) then by theorem 2.4.8 there exists a row \( R \) such that \( C(R) = F \) and \( R \leq F - V_jR_j \), contradicting the maximality of \( \{R_j\} \).

So let \( G = F - C(F - V_jR_j) \neq 0 \). \( GC(F - V_jR_j) = 0 \), so \( G(F - V_jR_j) = 0 \). Hence \( G \leq V_jR_j \). Hence by theorem 2.7.3 \( \{GR_j\} \) is a complete singular system of type \( G \).

Since \( G \neq 0 \) there exists a nonzero \( x \in G \mathbb{H} \subseteq F \mathbb{H} \), and so \( 0 \neq C_x \leq G \). By theorem 2.7.2 \( \{C_xGR_j\} = \{C_xR_j\} \) is a complete singular system of type \( C_x \).

Halmos refers to the final theorem of this section as "the fundamental theorem of multiplicity theory."

Theorem 2.7.5: Given two maximal singular systems \( \{R_j\} \) and \( \{S_k\} \) of type \( F \), \( \{R_j\} \) and \( \{S_k\} \) have the same cardinality.

Proof: Suppose \( \{R_j\} \) has cardinality \( c \) and \( \{S_k\} \) has cardinality \( d \). By theorem 2.7.4 there exists a nonzero \( x \in F \mathbb{H} \) such that \( \{C_xR_j\} \) is a complete singular system of type \( C_x \neq 0 \). Since \( C_x \leq F \) (since \( Z_x \leq F \)), theorem 2.7.2 implies that \( \{C_xS_k\} \) is a singular system of type \( C_x \). Note that \( V_kC_xS_k \leq C_x = V_jC_xR_j \), and that \( \{C_xR_j\} \) and \( \{C_xS_k\} \)
have cardinality $c$ and $d$ respectively. Hence we may suppose without loss of generality that $\{R_j\}$ and $\{S_k\}$ are singular systems of type $C_x$ having cardinality $c$ and $d$ respectively, such that $V_k S_k \leq C_x = V J R_j$ (by replacing $F$, $\{R_j\}$ and $\{S_k\}$ by $C_x$, $\{C_x R_j\}$ and $\{C_x S_k\}$ respectively).

Now given $j$, $R_j \leq C(R_j) = C_x$ which is separable, so by theorem 2.5.3 there exists $x_j$ (say) $\in R_j \mathcal{H}$ such that $C(R_j) = C_{x_j}$. Since $R_j x_j = x_j$, $Z x_j \leq R_j$. If there exists nonzero $y \in (R_j - Z x_j) \mathcal{H}$ then $C_y \leq C(R_j) = C_{x_j}$, since $Z_y \leq R_j$ (since $R_j y = y$).

However, $Z x_j y = 0$, so $Z y Z x_j = 0$. So since $R_j$ is a row $\geq$ both $Z_y$ and $Z x_j$, theorem 2.4.4 implies that $C_y C x_j = 0$. But $C_y C x_j = C_y$, so $C_y = 0$. Hence $y = 0$, contradicting the hypothesis that $y \neq 0$. Hence $R_j = Z x_j$. Similarly, given $k$, there exists $y_k \in S_k \mathcal{H}$ such that $C(S_k) = C_{y_k}$ and $S_k = Z y_k$.

If $c \geq \aleph_0$ then let $K_j = \{k | Z y_k x_j \neq 0\}$, for all $j$. Since $\{S_k\}$ is a singular family, $K_j$ is countable, for all $j$. Now if there exists $k_0 \notin \bigcup_j K_j$, so that $Z y_{k_0} x_j = 0$ for all $j$, then $Z y_{k_0} Z x_j = 0$, for all $j$. Hence $Z y_{k_0} = S_{k_0} = S_{k_0} V J R_j = Z y_{k_0} V J Z x_j = 0$, which contradicts the fact that every $S_k$ is nonzero. Hence every $k \in \bigcup_j K_j$, so that $|\{S_k\}| = d \leq \aleph_0$. $|\{R_j\}| = \aleph_0 \cdot c = c$.

If, however, $c$ is finite, we apply theorem 1.3.2 to obtain, for all $j$, $U_j : L_2(\mu_{x_j}) \to Z x_j$ such that $U_j^\dagger E(M) U_j$ is multiplication by $\chi_M$, for all $M \in \mathcal{S}$. Putting all the $U_j$'s together we obtain an isomorphism $U$ from $K = \bigoplus_j L_2(\mu_{x_j})$ onto $V J Z x_j \mathcal{H}$ such that $U^\dagger E(M) U \{f_j\} = \{\chi_M \cdot f_j\}$, for all $M \in \mathcal{S}$ and all $\{f_j\} \in K$.

Since $C_x = C_{x_j}$, $\mu_{x_j} \sim \mu_x$ by theorem 2.6.2, for all $j$. Hence there exists a family $\{g_j\}$
of nonnegative functions \( \in \mathcal{L}_1(\mu_x) \) such that \( \mu_{x_j}^j(M) = \int g_j d\mu_x \), for all \( j \) and all \( M \in \mathcal{S} \).

Since every \( y_k \in \mathcal{V}_J \mathcal{Z}_x \mathcal{H} \) there exists \( \{f_{jk}\}_j \in \mathcal{K} \) such that \( U\{f_{jk}\}_j = y_k \), for all \( k \).

Hence for all \( M \in \mathcal{S} \),

\[
(E(M)y_{k_1}, y_{k_2})_{\mathcal{H}} = (E(M)U\{f_{jk_1}\}_j, U\{f_{jk_2}\}_j)_{\mathcal{H}} = (U\{X_M \cdot f_{jk_1}\}_j, U\{f_{jk_2}\}_j)_{\mathcal{H}}
\]

\[
= (\{X_M \cdot f_{jk_1}\}_j, \{f_{jk_2}\}_j)_{\mathcal{K}} = \sum_j \int X_M f_{jk_1} f_{jk_2}^* \, d\mu_x = \int \sum_j f_{jk_1} f_{jk_2}^* \, g_j \, d\mu_x,
\]

since the summation over \( j \) is finite. However, if \( \mu_x(M) \neq 0 \) then \( (E(M)y_{k_1}, y_{k_2}) = 0 \) if and only if \( k_1 \neq k_2 \), since \( \{Z_{x_k}\} \) is a singular family of nonzero projections. Hence given \( k_1, k_2 \in \text{a countable subset of } \{k\} \), there exists \( N \in \mathcal{S} \) such that \( \mu_x(N) = 0 \) and for all \( t \in \mathcal{M} \cap N \), \( \sum_j f_{jk_1}^* (t) f_{jk_2}^* (t) g_j (t) = 0 \) if and only if \( k_1 \neq k_2 \). But (given \( t \in \mathcal{M} \cap N \)) every \( \{f_{jk}(t)\}_j \) is a vector in a \( c \)-dimensional Hilbert space, so that \( d = |\{k\}| \leq c \). Hence \( d \leq c \) whether \( c \) is a finite or not. Similarly \( c \leq d \).

Section 8: The Multiplicity of a Projection Doubly Commuting with \( E \)

The reason that theorem 2.7.5 is so important is that we can use it to define a function \( u \) from \( \xi'' \), the double commutator of the range of \( E \), to cardinal numbers \( \leq \dim \mathcal{H} \) by letting \( u(F) \), the multiplicity of \( F \), be the cardinality of any maximal singular system of type \( F \), for all \( F \in \xi'' \). We give some useful properties of the function \( u \) below:

**Theorem 2.8.1:** Given \( F, G \in \xi'' \) such that \( 0 \neq F \preceq G \), \( u(G) \leq u(F) \).

**Proof:** Given a singular system \( \{R_j\} \) of type \( G \), theorem 2.7.2 implies that \( \{FR_j\} \) is a
singular system of type $F$. In particular this is true if $\{R_j\}$ is a maximal singular system of type $G$.

Theorem 2.8.2: Given a singular family $\{F_j\}$ of nonzero projections $\subseteq \xi^\prime$, $u(V_jF_j) = \min_j \{u(F_j)\}$.

Proof: Since every $F_i \leq V_jF_j$, theorem 2.8.1 implies that $u(V_jF_j) \leq$ every $u(F_i)$. Hence $u(V_jF_j) \leq \min_j \{u(F_j)\}$.

Conversely, since every $u(F_i) \geq \min_j \{u(F_j)\}$, there exists a singular system $\{R_{ik}\}_k$ of type $F_i$, having cardinality $\min_j \{u(F_j)\}$, for all $i$. Theorem 2.7.1 now implies that $\{V_jR_{jk}\}_k$ is a singular system of type $V_jF_j$, having cardinality $\min_j \{u(F_j)\}$. Hence $u(V_jF_j) \geq \min_j \{u(F_j)\}$.

The similarity of theorems 2.8.1 and 2.8.2 and properties ii) and iii) of multiplicity functions (see Chapter II, Section 1) is a hint of things to come. Note also that if $F = 0$ then $u(F) = 0$, since there are no nonzero rows $\leq F$ in this case. There is also a corresponding notion of uniform multiplicity: Given $F \in \xi^\prime$, if for every nonzero projection $G \in \xi^\prime$ such that $G \leq F$ $u(G) = u(F)$ then we say that $F$ has uniform multiplicity $u(F)$.

Theorem 2.8.3: Given a singular family $\{F_j\} \subseteq \xi^\prime$ such that every $F_j$ has uniform multiplicity $c$, $V_jF_j$ has uniform multiplicity $c$. 
Proof: Given $G \in \xi''$ such that

$$0 \neq G \leq V_{j}F_{j}, \quad G = GV_{j}F_{j} = V_{j}GF_{j} = V_{j}GF_{j}^{'},$$

so by theorem 2.8.2 $u(G) = \min \{ u(GF_{j}) \mid GF_{j} \neq 0 \}$. But since every $F_{j}$ has uniform multiplicity $c$, if $0 \neq GF_{j} \leq F_{j}$ then $u(GF_{j}) = c$. Hence $u(G) = c$.

The following theorem gives a valuable characterization of uniform multiplicity:

Theorem 2.8.4: Given nonzero $F \in \xi''$, $F$ has uniform multiplicity if and only if there exists a complete singular system of type $F$.

Proof: Since a complete singular system $\{R_{j}\}$ of type $F$ is necessarily maximal, if

$$0 \neq G \leq F \quad \text{and} \quad G \in \xi'' \quad \text{then} \quad \{GR_{j}\} \quad \text{is a maximal singular system of type} \quad G \quad \text{by theorem 2.7.2.}$$

Hence $u(G) = u(F)$.

Conversely, if $F$ has uniform multiplicity, let $\{F_{j}\}$ be a maximal singular family $\subseteq \xi''$ such that $0 \neq F_{j} \leq F$ and such that there exists a complete singular system of type $F_{j}$, for all $j$ (such an $F_{j}$ exists by theorem 2.7.4). If $F - V_{j}F_{j} \neq 0$ then we may apply theorem 2.7.4 to obtain $G \in \xi'$ such that $0 \neq G \leq F - V_{j}F_{j}$ and such that there exists a complete singular system of type $G$, contradicting the maximality of $\{F_{j}\}$. Hence $F = V_{j}F_{j}$.

By hypothesis every $u(F_{j}) = u(F)$, so let us denote the complete (hence maximal) singular system of type $F_{j}$ by $\{R_{jk}\}$, for all $j$. Theorem 2.7.1 implies that $\{V_{j}R_{jk}\}$ is
a singular system of type \( F \) and having cardinality \( u(F) \). Since
\[
V_k(V_jR_{jk}) = V_j(V_kR_{jk}) = V_jF_j = F, \quad \text{this singular system is complete.}
\]

Theorem 2.8.5: For every cardinal \( c \leq \dim \mathcal{H} \) let
\[
G_c = V\{F \in \xi'' | F \text{ has uniform multiplicity } c\}. \quad \text{Then } \{G_c\} \text{ is a singular family such that}
\]
\[
V_cG_c = 1, \quad \text{and for all } c, \ G_c \text{ either } 0 \text{ or has uniform multiplicity } c.
\]

Proof: Given \( c \leq \dim \mathcal{H} \), obtain a maximal singular family \( \{F_{cj}\}_j \subseteq \xi'' \) such that \( F_{cj} \) has uniform multiplicity \( c \), for all \( j \). (This can be done so long as \( G_c \neq 0 \).) Then
\[
G_c = V_cF_{cj}; \quad \text{Otherwise there exists } F \in \xi'' \text{ such that } F \text{ has uniform multiplicity } c \text{ and}
\]
\[
F(G_c - V_jF_{cj}) \neq 0, \quad \text{so that } F(G_c - V_jF_{cj}) \text{ has uniform multiplicity } c \text{ and is singular to every } F_{cj}', \text{ contradicting the maximality of } \{F_{cj}\}_j.
\]

Theorem 2.8.3 implies that \( G_c \) has uniform multiplicity \( c \), therefore, unless \( G_c = 0 \).

Hence \( \{G_c\} \) is a singular family, since if \( 0 \neq G_cG_d \leq G_c, G_d \), then \( u(G_cG_d) = c = d \).

Finally, if \( V_cG_c \neq 1 \), theorem 2.7.4 implies that there exists a nonzero \( x \in (1 - V_cG_c)\mathcal{H} \) and a complete singular system of type \( C_x \). Hence \( C_x \) has uniform multiplicity, by theorem 2.8.4. Hence \( C_x \leq V_cG_c \), so that \( x \in V_cG_c\mathcal{H} \), a contradiction.

Theorem 2.8.5 allows us to decompose \( \mathcal{H} \) more delicately than in the proof of

Theorem 1.3.3: \( \mathcal{H} = \sum_{c}^\dim \mathcal{H} \oplus G_c\mathcal{H}, \) and every \( G_c (\neq 0) \) has uniform multiplicity \( c \). Hence
\[
G_c \neq 0
\]
by theorem 2.8.4 there exists a complete singular system \( \{ R_{c_j} \}_j \) of type \( G_c \), for all \( c \) such that \( G_c \neq 0 \). Finally, every \( R_{c_j} \) is made of singular cyclic projections by theorem 2.4.3. All that remains now is to introduce the desired multiplicity function.

Section 9: The Multiplicity Function of a Normal Operator

Given a normal operator \( N \) with corresponding spectral measure \( \mathcal{E}_N \), the multiplicity function of \( N \) is the function \( m_H(N): \mathcal{M} \to \{ \text{cardinals} \leq \dim H \} \) defined by

\[
m_H(N)(\mu) = \min \{ \mu(F_v) \mid 0 \neq v << \mu \} \text{ for all nonzero } \mu \in \mathcal{M}, \text{ and } m_H(N)(0) = 0.
\]

Let us show that \( m_H(N) \) is actually a multiplicity function. Recall the three properties it must satisfy:

i) if \( \mu = 0 \) then \( m_H(N)(\mu) = 0 \),

ii) if \( 0 \neq u << \mu \) then \( m_H(N)(\mu) \leq m_H(N)(u) \), and

iii) if \( \{ \mu_n \} \) is a bounded (hence countable) singular family of nonzero measures \( \subseteq \mathcal{M} \),

\[
m_H(N)(V_n \mu_n) = \min \{ m_H(N)(\mu_n) \}.
\]

We see that the first property of multiplicity functions is satisfied by definition. The other properties are easily demonstrated:

Theorem 2.9.1: Given \( \mu, v \in \mathcal{M} \) such that \( 0 \neq v << \mu \), \( m_H(N)(v) \geq m_H(N)(\mu) \).

Proof: If \( 0 \neq v << \mu \) then \( \{ \lambda \mid 0 \neq \lambda << v \} \subseteq \{ \lambda \mid 0 \neq \lambda << \mu \} \).

Theorem 2.9.2: Given a countable singular family \( \{ \mu_n \} \) of nonzero measures \( \subseteq \mathcal{M} \),

\[
m_H(N)(V_n \mu_n) = \min \{ m_H(N)(\mu_n) \}.
\]
Proof: Given \( 0 \neq \psi < \psi_n \mu_n \), \( \psi = \psi_n (\psi \wedge \mu_n) = \psi_n (\psi \wedge \mu_n) \).

Theorem 2.6.8 implies that \( F_{\psi} = V_n F_{\psi \wedge \mu_n} \). Hence by theorem 2.8.2,

\[ u(F_{\psi}) = \min_n \{ u(F_{\psi \wedge \mu_n} \mid \psi \wedge \mu_n \neq 0) \} \geq \min \{ m_H(N)(\mu_n) \}. \]

Hence

\[ m_H(N)(V_n \mu_n) = \min_n \{ u(F_{\psi}) \mid 0 \neq \psi < \psi_n \mu_n \} \geq \min \{ m_H(N)(\mu_n) \}. \]

However, since \( 0 \neq \mu_m < \psi_n \mu_n \), for all \( m \), theorem 2.9.1 implies that every

\[ m_H(N)(\mu_n) \geq m_H(N)(V_n \mu_n). \]

Hence \( \min_n \{ m_H(N)(\mu_n) \} \geq m_H(N)(V_n \mu_n). \)

In order to prove that two normal operators having the same multiplicity function are unitarily equivalent we need the following theorem (which we shall have occasion to use in a later chapter also):

**Theorem 2.9.3:** Given \( x \in \mathcal{H} \), if \( C_x \) has uniform multiplicity (with respect to \( u \)) then \( \mu_x \) has uniform multiplicity (with respect to \( m_H(N) \)). Conversely, given nonzero \( \mu \in \mathcal{M} \) having uniform multiplicity and \( x \in \mathcal{H} \) such that \( F_\mu = C_x \) (see theorem 2.6.5), there exists \( y \in \mathcal{H} \) such that \( \mu = \mu_y \), \( Z_y = Z_x \), and \( C_y = F_\mu \) has uniform multiplicity.

Proof: If \( C_x \) has uniform multiplicity and \( 0 \neq \psi < \mu_x \) then there exists \( y \in \mathcal{Z}_x \) such that \( \psi = \mu_y \), by theorem 2.6.3. By theorems 2.6.6 and 2.6.5 \( 0 \neq F_\psi \leq F_{\mu_x} = C_x \). Hence \( u(F_\psi) = u(C_x) \) by hypothesis. Hence \( m_H(N)(\mu_x) = \min \{ u(F_\psi) \mid 0 \neq \psi < \mu_x \} = u(C_x) \). If also \( 0 \neq \lambda < \psi < \mu_x \) then similarly \( m_H(N)(\mu_\lambda) = u(F_\lambda) \). Hence
m_H(N)(\nu) = \min \{ u(F_\lambda) \mid 0 \neq \lambda \ll \nu \} = m_H(N)(\mu_x). Hence \mu_x has uniform multiplicity.

Conversely, given nonzero \mu \in \mathcal{M} having uniform multiplicity (necessarily nonzero) and \lambda \in \mathcal{H} such that F_\mu = C_\lambda, applying theorem 1.4.4 we obtain \mu_0 \in \mathcal{M} such that 

\mu_0 \perp \mu_x and \mu_0 \nu \mu_x \sim \mu \nu \mu_x (= \mu, since \mu_x \ll \mu). Now if \lambda \in F_{\mu_0} \mathcal{H} then \mu_\lambda \ll \mu_0.

Hence \mu_\lambda \perp \mu_x, so that C_\lambda C_\mu = 0 = C_\lambda F_\mu. But \mu_0 \ll \mu, so that \mu_z \ll \mu, so that 

C_z \leq F_\mu \ (\text{see theorems 2.6.1 and 2.6.2 for justification of the above}). Hence 

C_z F_\mu = C_z = 0, so \lambda = 0. Hence F_{\mu_0} = 0. Hence m_H(N)(\mu_0) = 0, since if 0 \neq \nu \ll \mu_0

then F_0 \leq F_{\mu_0} = 0 by theorem 2.6.6. So since \mu_0 \ll \mu which has uniform multiplicity 

\neq 0, \mu_0 = 0. Hence \mu_x \sim \mu. We apply theorem 2.6.3 to obtain \gamma \in \mathcal{Z}_x such that 

\mu_\gamma = \mu \ and \ Z_\gamma = Z_x.

Hence C(Z_\gamma) = C_\gamma = C_\lambda = C(Z_x) = F_\mu.

Now given \ F \in \mathcal{Z}_x such that 0 \neq \ F \leq F_\mu, \ F \ is separable (since \ F_\mu \ is separable by theorem 2.6.5). Hence \ F = C_\gamma = F_{\mu_\gamma}, for some \ \gamma \in \mathcal{H}, \ by theorem 2.6.5. Since 

F \neq 0, \ \gamma \neq 0, \ and \ so \ \mu_\gamma \neq 0. Hence m_H(N)(\mu_\gamma) = m_H(N)(\mu), since 0 \neq \mu_\gamma \ll \mu. But 

0 \neq \nu_0 \ll \mu_\gamma \ implies \ u(F_{\nu_0}) \geq u(F_{\mu_\gamma}). Hence m_H(N)(\mu_\gamma) \geq u(F_{\mu_\gamma}), and so 

m_H(N)(\mu) \geq u(F_{\mu_\gamma}). But also \ u(F_{\mu_\gamma}) \geq \min \{u(F_{\nu_0}) \mid 0 \neq \nu_0 \ll \mu\} = m_H(N)(\mu). Hence 

u(F_{\mu_\gamma}) = m_H(N)(\mu) = m_H(N)(\mu_\gamma). In particular, \ u(F_{\mu_\gamma}) = m_H(N)(\mu) = u(F_{\mu_\gamma}) = u(F), as required to show that \ F_\mu \ has uniform multiplicity.

Theorem 2.9.4: \ Given normal operators \ N \ and \ N' with spectral measures 

E_N \ and \ E_{N'}, respectively, \ m_H(N) = m_H(N') \ if and only if \ N \equiv N'.
Proof: Applying theorem 2.1.3 to the multiplicity function \( m_H(N) \) we obtain a singular family \( \{ u_j \} \) of nonzero measures \( \subseteq \mathcal{M} \) such that every \( u_j \) has uniform multiplicity (with respect to \( m_H(N) \)) and such that \( \mu \sim V_j(\mu \wedge u_j) \), for all \( \mu \in \mathcal{M} \).

Theorem 2.6.6 implies that \( \{ F_{u_j} \} \) is a singular family \( \subseteq \xi'' \). Now given

\[
x \in H, \quad \mu_x \sim V_j(\mu_x \wedge u_j), \quad \text{so that by theorems 2.6.8 and 2.6.6} \quad F_{\mu_x} = V_j F_{\mu_x \wedge u_j}
\]

\[
= V_j F_{\mu_x F_{u_j}} \lesssim V_j F_{u_j}. \quad \text{Hence} \quad x \in F_{\mu_x} H \subseteq V_j F_{u_j} H. \quad \text{Hence} \quad V_j F_{u_j} = 1.
\]

Theorem 2.9.3 implies that every \( F_{u_j} \) has uniform multiplicity (with respect to \( u = u_N \), say). Hence by theorem 2.8.4 there exists a complete singular system \( \{ R_{jk} \} \) of type \( F_{u_j} \), having cardinality \( u_N(F_{u_j}) = m_H(N)(u_j) \), for all \( j \). Theorem 2.6.5 implies that every \( F_{u_j} \) is separable, however. Hence by theorem 2.5.3 there exists \( y_{jk} \) such that

\[
F_{u_j} = C(R_{jk}) = C_{y_{jk}} \quad \text{and} \quad y_{jk} \in R_{jk} H, \quad \text{for all} \quad j \quad \text{and} \quad k. \quad \text{But then} \quad Z_{y_{jk}} \leq R_{jk}', \quad \text{which is a row, so that} \quad Z_{y_{jk}} = C_{y_{jk}} R_{jk} = R_{jk}, \quad \text{for all} \quad j \quad \text{and} \quad k. \quad \text{So given} \quad j, \quad \text{there exists a set of vectors} \quad \{ x_{jk} \} \subseteq H \quad \text{such that} \quad \mu_{x_{jk}} = u_j, \quad C_{x_{jk}} = C_{y_{jk}} = F_{u_j} \quad \text{which has uniform multiplicity, and} \quad Z_{x_{jk}} = Z_{y_{jk}} = R_{jk}, \quad \text{for all} \quad k, \quad \text{by theorem 2.9.3}.
\]

Applying theorem 1.3.2 we obtain unitary transformations of \( L_2(u_j) \) onto every \( Z_{x_{jk}} \) making the restriction of \( E_N \) on \( Z_{x_{jk}} \) unitarily equivalent to the standard spectral measure on \( L_2(u_j) \). Combining these transformations we obtain a unitary transformation of

\[
\sum_j \oplus \left( \sum_k L_2(u_j) \right) \quad \text{onto} \quad H \quad \text{making} \quad E_N \quad \text{unitarily equivalent to the standard spectral measure associated with} \quad \{ u_j \} \quad \text{and} \quad \{ m_H(N)(u_j) \} \quad \text{(see the end of Chapter I, Section 2)}.
\]

Hence if \( m_H(N) = m_H(N') \) then also \( E_N \) is unitarily equivalent to this standard spectral measure, and so \( E_N \equiv E_{N'} \). Hence by theorem 1.2.2 \( N \equiv N' \).
Conversely given normal operators \( N \) and \( N' \) on \( \mathcal{H} \) and \( \mathcal{K} \) respectively such that 
\[ N \equiv N', \text{ there exists a unitary transformation } U : \mathcal{H} \rightarrow \mathcal{K} \text{ such that } U E_N U^{-1} = E_N. \]

Define, for \( x \in \mathcal{H} \), \( \mu_x \) by 
\[ \mu_x(M) = (E_N(M)x, x)_{\mathcal{H}} \]
for all \( M \in \mathcal{S} \), and similarly define 
\[ \nu_y \text{ by } \nu_y(M) = (E_N(M)y, y)_{\mathcal{K}}, \]
for all \( M \in \mathcal{S} \).

Now given \( x \in \mathcal{H} \) and \( M \in \mathcal{S} \),
\[ \nu_{Ux}(M) = (E_N(M)Ux, Ux)_{\mathcal{K}} = (U^{-1}E_N(M)Ux, x)_{\mathcal{H}} = (E_N(M)x, x)_{\mathcal{H}} = \mu_x(M), \]
so given \( \mu \in \mathcal{M} \), \( \mu_x << \mu \) if and only if \( \nu_{Ux} << \mu \).

If \( F_\mu \) is the projection of \( \mathcal{H} \) onto \( \{ x \in \mathcal{H} | \mu_x << \mu \} \) and \( G_\mu \) is the projection of \( \mathcal{K} \) onto \( \{ y \in \mathcal{K} | \nu_y << \mu \} \) then by the above \( UF_\mu = G_\mu U \), that is \( UF_\mu U^{-1} = G_\mu \). Hence the multiplicity (with respect to \( u = u_N \)) of \( F_\mu \) is equal to the multiplicity (with respect to \( u_N \)) of \( G_\mu \). Hence \( m_N(N) = m_N(N') \).
CHAPTER III: BROWN'S VERSION OF MULTIPLICITY THEORY

Section 1: The $\sigma$-ideals of Finite Measures

Given $T \in \mathcal{P}(\mathcal{M})$, we say that $T$ is a $\sigma$-ideal of $\mathcal{M}$ if

i) given $\mu, \nu \in \mathcal{M}$ such that $\nu \ll \mu$ and $\mu \in T$, $\nu \in T$, and

ii) given a countable family $(\mu_n) \subseteq T$, $\forall \mu_n \in T$.

For example, given $\mu \in \mathcal{M}$, $(\mu) = \{\nu \in \mathcal{M} \mid \nu \ll \mu\}$ is the principal $\sigma$-ideal generated by $\mu$ (as always, interpreted up to equivalence of measures).

Before developing some properties of $\sigma$-ideals, it is worthwhile to recall that one reason Brown introduces them is that he wishes to define his multiplicity function $m_B(N)$ as a function taking cardinal numbers to certain $\sigma$-ideals of $\mathcal{M}$. This is a rather different notion of a multiplicity function from Halmos' (see Chapter II, Section 1), so let us return briefly to the special case where $\dim \mathcal{H}$ is finite, in order to understand this difference better.

In Chapter II, Section 1 we introduced the function $m(N): \mathbb{C} \to \mathbb{Z}^+$ defined by $m(N)(z) = \text{the multiplicity of } z \text{ as an eigenvalue of the normal operator } N$ (if $z$ is not an eigenvalue $m(N)(z) = 0$). If for all $i \in \mathbb{Z}^+$, $A_i = \{z \in \mathbb{C} \mid m(N)(z) = i\}$, define a function $\rho(N): \mathbb{Z}^+ \to \mathcal{P}(\mathbb{C})$ by $\rho(N)(i) = A_i$, $i \in \mathbb{Z}^+$.

Clearly $\rho(N) = \rho(N')$ if and only if $m(N) = m(N')$, and so $\rho(N) = \rho(N')$ if and only if $N \cong N'$. 
The parallel development in the case of infinite dimensional Hilbert space is to (again) replace \( \mathbb{Z}^+ \) by the set of cardinal numbers \( (\leq \dim \mathcal{H}) \) and \( \mathcal{C} \) by \( \mathcal{M} \), to find an appropriate function \( m_B(N) : \{ \text{nonzero cardinals } c \leq \dim \mathcal{H} \} \to \mathcal{P}(\mathcal{M}) \). As we shall see, the range \( m_B(N) \) will be restricted to \( \mathcal{A} \), the set of \( \sigma \)-ideals of \( \mathcal{M} \).

First, however, we give some results concerning \( \mathcal{A} \):

**Theorem 3.1.1:** Given \( \mathcal{J} \in \mathcal{A} \) and a bounded family \( \{ \mu_j \} \subseteq \mathcal{J} \), \( \forall j, \mu_j \in \mathcal{J} \).

**Proof:** By Theorem 1.4.3 there exists a countable subfamily \( \{ \mu_j \} \) of \( \{ \mu_j \} \) such that

\[
\forall j \mu_j = V_k \mu_j \in \mathcal{J}.
\]

**Theorem 3.1.2:** Given \( \mathcal{J} \in \mathcal{A} \) and \( \mu \in \mathcal{M} \) such that \( \mathcal{J} \subseteq (\mu) \), there exists \( \nu \in \mathcal{M} \) such that \( \mathcal{J} = (\nu) \).

**Proof:** Let \( \nu = V(\nu_0) \). Then \( \nu \in \mathcal{J} \) by theorem 3.1.1. Since every element of \( \mathcal{J} \) is absolutely continuous with respect to \( \nu \), \( \mathcal{J} \subseteq (\nu) \subseteq \mathcal{J} \).

Since the intersection of any nonempty family \( \{ \mathcal{J}_\alpha \} \subseteq \mathcal{A} \) is also \( \in \mathcal{A} \), the supremum and infimum of \( \{ \mathcal{J}_\alpha \} \) exist and are denoted by \( V_\alpha \mathcal{J}_\alpha \) and \( \Lambda_\alpha \mathcal{J}_\alpha \) respectively. For example, \( V_\alpha \mathcal{J}_\alpha \), the smallest \( \sigma \)-ideal containing every \( \mathcal{J}_\alpha \), is equal to

\[
\bigcap \{ \mathcal{J} \in \mathcal{A} \mid \mathcal{J} \supseteq \text{every } \mathcal{J}_\alpha \}.
\]

In fact, we have the following:

**Theorem 3.1.3:** Given a nonempty family \( \{ \mathcal{J}_\alpha \} \subseteq \mathcal{A} \),
i) \( \bigwedge \alpha \mathcal{T}_\alpha = \bigcap \mathcal{T}_\alpha \) and

ii) \( \bigvee \alpha \mathcal{T}_\alpha = \{ \mu \in \mathcal{M} \mid \mu = \bigvee \alpha \mu_\alpha, \{ \mu_\alpha \} \text{ a (bounded) family such that every } \mu_\alpha \in \mathcal{T}_\alpha \} \).

Proof:

i) Since \( \bigcap \mathcal{T}_\alpha \) is a \( \sigma \)-ideal of \( \mathcal{M} \), and is \( \subseteq \) every \( \mathcal{T}_\alpha \), this is clear.

ii) By theorem 3.1.1 \( \bigvee \alpha \mathcal{T}_\alpha \supseteq \{ \mu \in \mathcal{M} \mid \mu = \bigvee \alpha \mu_\alpha, \text{ every } \mu_\alpha \in \mathcal{T}_\alpha \} \supseteq \text{ every } \mathcal{T}_\alpha \).

Hence if \( \{ \bigvee \alpha \mu_\alpha \mid \{ \mu_\alpha \} \text{ a (bounded) family, every } \mu_\alpha \in \mathcal{T}_\alpha \} \) is a \( \sigma \)-ideal then it must be \( \bigvee \alpha \mathcal{T}_\alpha \), the smallest \( \sigma \)-ideal \( \supseteq \text{ every } \mathcal{T}_\alpha \).

If \( \nu \ll \mu = \bigvee \mathcal{M}_\alpha \), where \( \{ \mu_\alpha \} \) is a bounded family such that every \( \mu_\alpha \in \mathcal{T}_\alpha \), then

\[ \nu = \bigvee \alpha (\nu \land \mu_\alpha), \text{ and every } (\nu \land \mu_\alpha) \in \mathcal{T}_\alpha, \text{ so that by theorem 3.1.1 } \nu \in \bigvee \alpha \mathcal{T}_\alpha. \]

If \( \{ \nu_n \} \) is a countable family \( \subseteq \bigvee \alpha \mathcal{T}_\alpha \) every \( \nu_n = \bigvee \alpha \mu_\alpha_n \), where \( \{ \mu_\alpha_n \} \) is a bounded family such that every \( \mu_\alpha_n \in \mathcal{T}_\alpha \). Hence \( \nu_n \nu_n = \bigvee \alpha \nu_\alpha \mu_\alpha_n = \bigvee \alpha \nu_\alpha \nu_\alpha_n = \bigvee \alpha \mathcal{T}_\alpha \), because every \( \nu_\alpha \mu_\alpha_n \in \mathcal{T}_\alpha \) and \( \{ \bigvee \mathcal{T}_\alpha \} \) is bounded by \( \nu_\nu_n \).

Theorem 3.1.4: Given \( \mathcal{C} \subseteq \mathcal{P}(\mathcal{M}), C^\perp = \{ \mu \in \mathcal{M} \mid \mu \perp \text{ (every element of) } \mathcal{C} \} \) is a \( \sigma \)-ideal.

Proof: If \( \nu \in \mathcal{M} \) and \( \nu \ll \mu \in C^\perp \) then \( \nu \land \lambda \ll \mu \land \lambda = 0 \), for all \( \lambda \in \mathcal{C} \), so \( \nu \perp \mathcal{C} \), that is \( \nu \in C^\perp \).

If \( \{ \mu_n \} \) is a countable family \( \subseteq C^\perp \) then \( (\bigvee \mathcal{M}_n) \land \lambda = \bigvee \mathcal{N} (\mu_n \land \lambda) = 0 \), for all \( \lambda \in \mathcal{C} \), so that \( \bigvee \mathcal{N} \mu_n \in C^\perp \).
Theorem 3.1.5: (Lebesgue decomposition) Given $\mu \in \mathcal{M}$ and $\mathcal{T} \in \hat{\mathcal{M}}$, there exist $\mu_0 \in \mathcal{T}$ and $\mu_1 \in \mathcal{T}^\perp$ such that $\mu = \mu_0 \lor \mu_1$ (up to equivalence, of course).

Proof: Since $\mathcal{T} \cap (\mu) \subseteq (\mu)$, theorem 3.1.2 implies that there exists $\mu_0 \in \mathcal{M}$ such that $\mathcal{T} \cap (\mu) = (\mu_0)$. Decomposing $\mu$ into $\mu_0 \lor \mu_1$ where $\mu_1 \perp \mu_0$, we need only show that $\mu_1 \in \mathcal{T}^\perp$, since $\mu_0 \in (\mu_0) \subseteq \mathcal{T}$.

If $\nu \in \mathcal{T}$, $\mu_1 \lor \nu = (\mu_1 \lor \mu) \lor \nu = \mu_1 \lor (\mu \lor \nu) \ll \mu_1 \lor \mu_0 = 0$, since $\mu \lor \nu \ll \mu_0$.

Hence $\mu_1 \perp \mathcal{T}$, that is $\mu_1 \in \mathcal{T}^\perp$.

The consequence of theorem 3.1.5 is that given $\mathcal{T} \in \hat{\mathcal{M}}$, $\mathcal{T} \lor \mathcal{T}^\perp = \mathcal{M}$. Hence $\hat{\mathcal{M}}$ is (with the inclusion ordering) a complete, complemented lattice. It is also distributive:

Theorem 3.1.6: Given $\mathcal{T} \in \hat{\mathcal{M}}$ and $\{\mathcal{T}_\alpha\} \subseteq \hat{\mathcal{M}}$, $\mathcal{T} \land (\lor \mathcal{T}_\alpha) = \lor (\mathcal{T} \land \mathcal{T}_\alpha)$ and $\mathcal{T} \lor (\land \mathcal{T}_\alpha) = \land (\mathcal{T} \lor \mathcal{T}_\alpha)$.

Proof: Given $\mu \in \mathcal{M}$ such that $\mu = \lor \mathcal{T}_\alpha$, where every $\mu_\alpha \in \mathcal{T}_\alpha$, and such that $\mu \in \mathcal{T}$, every $\mu_\alpha \ll \mu$. So every $\mu_\alpha \in \mathcal{T}$, so that every $\mu_\alpha \in \mathcal{T}_\alpha \cap \mathcal{T}$. Hence $\mu = \lor \mathcal{T}_\alpha \in V_\alpha (\mathcal{T}_\alpha \cap \mathcal{T}) = V_\alpha (\mathcal{T} \land \mathcal{T}_\alpha)$. Hence $(\lor \mathcal{T}_\alpha) \land \mathcal{T} \subseteq V_\alpha (\mathcal{T} \land \mathcal{T}_\alpha)$. On the other hand, if $\mu = \lor \mathcal{T}_\alpha$, such that every $\mu_\alpha \in \mathcal{T} \land \mathcal{T}_\alpha$, every $\mu_\alpha \in \mathcal{T}$ and every $\mu_\alpha \in \mathcal{T}_\alpha$, so that $\mu = \lor \mathcal{T}_\alpha \in \mathcal{T}$ and $\in \lor V_\alpha \mathcal{T}_\alpha$.

Hence $V_\alpha (\mathcal{T} \land \mathcal{T}_\alpha) \subseteq \mathcal{T} \land (\lor \mathcal{T}_\alpha) \subseteq V_\alpha (\mathcal{T} \land \mathcal{T}_\alpha)$.

If $\mu \in \mathcal{T} \lor (\land \mathcal{T}_\alpha)$ then $\mu = \lor \mathcal{T}_\alpha V_\alpha \lambda$, where $V_\alpha \mathcal{T}_\alpha$ and $\lambda \in \land \mathcal{T}_\alpha = \land \mathcal{T}_\alpha$ so that
\[ \mu = \vee \vee \lambda \in \text{ every } \mathcal{T} \vee \mathcal{T}_\alpha, \text{ that is } \mu \in \wedge_\alpha (\mathcal{T} \vee \mathcal{T}_\alpha). \]

Hence \( \mathcal{T} \vee (\wedge_\alpha \mathcal{T}_\alpha) \subseteq \wedge_\alpha (\mathcal{T} \vee \mathcal{T}_\alpha). \)

On the other hand, if \( \mu \in \text{ every } \mathcal{T} \vee \mathcal{T}_\alpha, \) express \( \mu \) as \( \mu_0 \vee \mu_1, \) where \( \mu_0 \in \mathcal{T} \) and \( \mu_1 \in \mathcal{T}^\perp \) (using theorem 3.1.5), so that if \( \mu = \vee_\alpha \vee \mu_\alpha \) where \( \vee_\alpha \in \mathcal{T} \) and \( \mu_\alpha \in \mathcal{T}_\alpha \) then

\[ \mu_1 = \mu_1 \wedge \mu = \mu_1 \wedge (\vee_\alpha \vee \mu_\alpha) = \mu_1 \wedge \mu_\alpha \ll \mu_\alpha. \]

Then \( \mu_1 \in \text{ every } \mathcal{T}_\alpha, \) that is \( \mu_1 \in \cap_\alpha \mathcal{T}_\alpha = \wedge_\alpha \mathcal{T}_\alpha. \) Hence \( \mu \in \mathcal{T} \vee (\wedge_\alpha \mathcal{T}_\alpha). \)

Hence \( \wedge_\alpha (\mathcal{T} \vee \mathcal{T}_\alpha) \subseteq \mathcal{T} \vee (\wedge_\alpha \mathcal{T}_\alpha) \subseteq \wedge_\alpha (\mathcal{T} \vee \mathcal{T}_\alpha). \)

Theorem 3.1.7: The mapping \( \eta: \mu \to (\mu) \) maps \( \mathcal{M} \) (with the ordering \( \ll \)) isomorphically into \( \hat{\mathcal{M}}. \)

Proof: Certainly \( \eta \) is 1 - 1. given a nonempty bounded family \( \{ \mu_j \} \subseteq \mathcal{M}, \) if

\[ \mu \in \wedge_j \{ \eta(\mu_j) \} = \bigcap_j (\mu_j) \text{ then } \mu \ll \text{ every } \mu_j, \] so that \( \mu \ll \wedge_j \mu_j, \) that is \( \mu \in (\wedge_j \mu_j) = \eta(\wedge_j \mu_j). \)

Conversely, if \( \mu \in (\wedge_j \mu_j) \) then \( \mu \ll \text{ every } \mu_j, \) so \( \mu \in \bigcap_j (\mu_j). \)

Hence \( \eta(\wedge_j \mu_j) = \wedge_j \eta(\mu_j). \)

If \( \mu \in \bigvee_j \eta(\mu_j) = \bigvee_j (\mu_j) \text{ then } \mu = \bigvee_j \vee_j \) where every \( \vee_j \ll \mu_j, \) so that \( \mu \ll \bigvee_j \mu_j, \) that is

\[ \mu \in \eta(\bigvee_j \mu_j). \]

Conversely, \( \mu \ll \bigvee_j \mu_j \) implies that \( \mu = \mu \wedge (\bigvee_j \mu_j) = \bigvee_j (\mu \wedge \mu_j), \) where every \( \mu \wedge \mu_j \ll \mu_j, \) so that every \( \mu \wedge \mu_j \in (\mu_j). \)

Hence \( \mu \in \bigvee_j (\mu_j) = \bigvee_j \eta(\mu_j). \) Hence \( \eta(\bigvee_j \mu_j) = \bigvee_j \eta(\mu_j). \)

The usefulness of \( \sigma \)-ideals will become apparent in the next two sections, as we incorporate \( \sigma \)-ideals into the theory developed already concerning cyclic subspaces of \( \mathcal{H}. \)
Section 2: Cyclic Projections and \( \sigma \)-ideals

Given \( \mu \in \mathcal{M} \) and \( f \in L_2(\mu) \), let \( \mu_f \) be the measure on \( (M, \mathcal{S}) \) defined by

\[
\mu_f(M) = \langle \chi_M f, f \rangle_{L_2(\mu)} = \int_M |f|^2 \, d\mu.
\]

Theorem 3.2.1: Given \( \mu \in \mathcal{M} \), \((\mu) = \{\mu_f \mid f \in L_2(\mu)\}\).

Proof: Given \( u \in (\mu) \), there exists \( f \in L_2(\mu) \) such that \( u(M) = \int_M |f|^2 \, d\mu \), for all \( M \in \mathcal{S} \), by Radon-Nikodym. Hence \( u = \mu_f \). Conversely, given \( f \in L_2(\mu) \), \( \mu_f \ll \mu \), so \( \mu_f \in (\mu) \).

Theorem 3.2.2: Given a spectral measure \( E \) on \( \mathcal{H} \) and \( x \in \mathcal{H} \), \((\mu_x) = \{\mu_y \mid y \in Z_x\}\).

Proof: Given \( u \in (\mu_x) \), there exists \( y \in Z_x \) such that \( u = \mu_y \) by theorem 2.6.3. Conversely, if \( y \in Z_x \) then \( Z_y = y \), so \( Z_y \leq Z_x \), so \( C_y \leq C_x \).

Hence \( \mu_y \ll \mu_x \) by theorem 2.6.2, so \( \mu_y \in (\mu_x) \).

The next theorem is very important in Brown's treatment of multiplicity theory:

Theorem 3.2.3: Given a spectral measure \( E \) on \( \mathcal{H} \) and a very singular countable family \( \{Z_n\} \) of cyclic projections on \( \mathcal{H} \), \( \vee_n Z_n \) is a cyclic projection.
Proof: By theorem 2.4.6 $V_n Z_{x_n}$ is a row, and $C(V_n Z_{x_n}) = V_n C_{x_n}$ is separable, by theorem 2.5.1 (since every $C_{x_n}$ is separable). Hence theorem 2.5.3 implies that there exists $x \in (V_n Z_{x_n})^H$ such that $C(V_n Z_{x_n}) = C_x$.

Since $V_n Z_{x_n} x = x$, $Z_x \subseteq V_n Z_{x_n}$ which is a row, and so $Z_x = C_x V_n Z_{x_n} = C(V_n Z_{x_n}) V_n Z_{x_n} = V_n Z_{x_n}$, as required.

Note that in the above proof, $\mu_x = V_n \mu_{x_n}$. Hence the following:

**Theorem 3.2.4:** Given a spectral measure $E$ on $\mathcal{H}$, $\{\mu_x \mid x \in \mathcal{H}\} \in \mathcal{M}$.

Proof: Theorem 2.6.3 gives the first requirement that $\{\mu_x \mid x \in \mathcal{H}\}$ be a $\sigma$-ideal of $\mathcal{M}$. If $\{x_n\}$ is a countable subset of $\mathcal{H}$ then there exists a singular family $\{u_n\} \subseteq \mathcal{M}$ such that $u_n << \mu_{x_n}$ and $V_n u_n = V_n \mu_{x_n}$ by Lebesgue decomposition. Theorem 2.6.3 implies that every $u_n = \mu_{y_n}$, for some $y_n \in \mathcal{H}$, and by theorem 3.2.3 there exists $x \in \mathcal{H}$ such that $\mu_x = V_n \mu_{y_n} = V_n u_n = V_n \mu_{x_n}$, so that $V_n \mu_{x_n} \in \{\mu_x \mid x \in \mathcal{H}\}$.

The $\sigma$-ideal $\{\mu_x \mid x \in \mathcal{H}\}$ is called the $\sigma$-ideal of $E$, and is denoted by $\mathcal{T}_E$. Note that it follows from theorem 3.2.4 that if $K$ is a subspace of $\mathcal{H}$ invariant under $E$ then $\{\mu_x \mid x \in K\}$ is the $\sigma$-ideal of $E$ restricted to $K$.

**Section 3: Subspaces from $\sigma$-ideals**

Given a spectral measure $E$ on $\mathcal{H}$ and $\mu \in \mathcal{M}$, $\{\mu_x \mid x \in \mathcal{H} \mid \mu_x \in (\mu)\}$ is a subspace of
Theorem 3.3.1: Given a spectral measure \( \mathcal{E} \) on \( \mathcal{H} \) and \( T \in \mathcal{H}^\prime \), \( \{ x \in \mathcal{H} \mid \mu, \nu \in T \} \) is a subspace of \( \mathcal{H} \).

Proof: As noted in the special case where \( T = (\mu) \), \( \mu_x + \beta_y \ll \mu_x + \nu_y \) and if

\[ x_n \to x \text{ then } \mu_x \ll \nu_{\|x\|}. \]

We denote the projection of \( \mathcal{H} \) onto \( \{ x \in \mathcal{H} \mid \mu_x \in T \} \) by \( F_T \), except that we continue to write \( F_\mu \) for \( F_{(\mu)} \). Note that given \( T \in \mathcal{H}^\prime \), \( T \cap T \in \mathcal{H}^\prime \) also, and \( F_T = F_{T \cap T} \), since \( \mu_x \in T \) for all \( x \in \mathcal{H} \).

Theorem 3.3.2: Given a family \( \{ Z_x \} \) of cyclic subspaces spanning \( \mathcal{H} \), the family \( \{ \mu_x \} \) generates \( T_E \).

Proof: If \( \{ \mu_x \} \) is the \( \sigma \)-ideal generated by \( \{ \mu_x \} \) then, since every \( Z_x \leq F_{(\mu_x)} \),

\[ \mathcal{H} = \sum_j Z_x \subseteq F_{(\mu_x)} \mathcal{H} \text{, so } F_{(\mu_x)} \mathcal{H} = \mathcal{H} \text{. Hence } \mu_x \in (\mu_x)_j \text{, for all } x \in \mathcal{H} \text{, so that } T_E \subseteq (\mu_x)_j \subseteq T_E. \]

In fact (see the remark at the end of the previous section), given any family \( \{ Z_x \} \) of
cyclic subspaces of $\mathcal{H}$, $(\mu_{x_j})_j = \{\mu_x \mid x \in \mathbb{V}_j \mathcal{Z}_x\}$, since $\mathbb{V}_j \mathcal{Z}_x$ is the smallest projection $\varepsilon \xi'$ containing every $x_j$.

Theorem 3.3.3: Given $T \in \hat{\mathcal{H}}$, $\mathcal{H} = F_T \mathcal{H} \oplus F_{T^\perp} \mathcal{H}$.

Proof: Given $x \in F_T \mathcal{H}$ and $y \in F_{T^\perp} \mathcal{H}$, $\mu_x \in T$ and $\mu_y \in T^\perp$, so $\mu_x \perp \mu_y$.

Hence $C_x C_y = 0$ by theorem 2.6.1, so $x \perp y$. Hence $F_T F_{T^\perp} = 0$. Given $x \in \mathcal{H}$, apply theorem 3.1.5 to obtain $\mu_0 \in T$ and $\mu_1 \in T^\perp$ such that $\mu_x = \mu_0 + \mu_1$. There exists $M \in \mathcal{S}$ such that $\mu_0 \sim \mu_x |_M$ and $\mu_1 \sim \mu_x |_{M \setminus M}$ (see proof of theorem 1.4.2). For all $N \in \mathcal{S}$, however, $\mu_{x_M}(N) = (\chi_N \cdot \chi_M)_{L^2(\mathcal{H})} = \int_{\chi_N \cdot \chi_M} d\mu_x = \mu_x (N \cap M) = \mu_x |_M (N)$ and similarly $\mu_{x_{M \setminus M}}(N) = \mu_x |_{M \setminus M} (N)$, so $\mu_x \in T$ and $\mu_{x_{M \setminus M}} \in T^\perp$.

Applying theorem 1.3.2 we obtain a unitary transformation $U: L_2(\mu_x) \to \mathcal{Z}_x$ such that $U^{-1}EU$ is the standard spectral measure on $L_2(\mu_x)$, and $U(1) = x$. Since

$$1 = \chi_M + \chi_{M \setminus M}, \quad x = U(\chi_M) + U(\chi_{M \setminus M}).$$

But for all $N \in \mathcal{S}$,

$$\mu_U(\chi_M)(N) = (E(N)U(\chi_M), U(\chi_M))_{\mathcal{H}} = (U^{-1}E(N)U(\chi_M), \chi_M |_{L^2(\mu_x)}) = (\chi_N \cdot \chi_M, \chi_M |_{L^2(\mu_x)})$$

$= \mu_{\chi_M}(N)$ and similarly $\mu_U(\chi_{M \setminus M})(N) = \mu_{\chi_{M \setminus M}}(N)$, so $\mu_U(\chi_M) \in T$ and $\mu_U(\chi_{M \setminus M}) \in T^\perp$, so $U(\chi_M) \in F_T \mathcal{H}$ and $U(\chi_{M \setminus M}) \in F_{T^\perp} \mathcal{H}$.

Hence $\mathcal{H} = F_T \mathcal{H} + F_{T^\perp} \mathcal{H} = F_T \mathcal{H} \oplus F_{T^\perp} \mathcal{H}$.

Theorem 3.3.4: Given a spectral measure $E$ on $\mathcal{H}$, $\{F_T \mid T \in \hat{\mathcal{H}}\} = \xi''$. 0
Proof: Given $T \in \xi'$ and $x \in \mathcal{H}$, $\mu_{Tx}(M) = \|E(M)Tx\|^2_\mathcal{H} = \|TE(M)x\|^2_\mathcal{H}$, for all $M \in \mathcal{S}$, so $\mu_{Tx} \ll \mu_x$.

Hence for all $T \in \mathcal{H}$, if $\mu_x \in T$ then $\mu_{Tx} \in T$, and if $\mu_x \in T^\perp$ (a $\sigma$-ideal) then $\mu_{Tx} \in T^\perp$. Hence $TF \leq F_T$ and $TF^\perp \leq F_T^\perp$, so $TF_T = F_T T$. Hence $F_T \in \xi''$.

Conversely, suppose $F \in \xi'' \subseteq \xi'$. We know (by the remark following theorem 3.2.4) that $T = \{\mu_x \mid x \in \mathcal{H}\}$ is a $\sigma$-ideal.

If $x \in \mathcal{PH}$ then $\mu_x \in T$, so that $x \in \{z \in \mathcal{H} \mid \mu_z \in T\} = F_T \mathcal{H}$. So $F \leq F_T$. Suppose that $y \in F_T \mathcal{H}$, on the other hand, so that $\mu_y \in T$.

Since $F \in \xi'$, $\mu_{(1-F)y}(M) = (E(M)(1-F)y, (1-F)y)_\mathcal{H} = ((1-F)E(M)y, (1-F)y)_\mathcal{H}$, for all $M \in \mathcal{S}$. Hence $\mu_{(1-F)y} \ll \mu_y$, so that $\mu_{(1-F)y} \in T$.

Hence there exists $z \in \mathcal{PH}$ such that $\mu_{(1-F)y} = \mu_z$, and so $C_{(1-F)y} = C_z$ by theorem 2.6.2. But $Fz = z$, so $Z_z \leq F$, so $C_z \leq C(F) = F$, and also $F(1-F)y = 0$, so $C_z = FC_z = FC_{(1-F)y} = C_{F(1-F)y} = 0$, so $z = 0$.

Hence $\mu_z = 0 = \mu_{(1-F)y}$, so $(1-F)y = 0$, that is $y = Fy$. Hence $y \in \mathcal{PH}$. Hence $F_T \leq F \leq F_T$, so $F = F_T$.

Section 4: Stacks and Multiplicity

Before providing the remaining tools for the solution of our problem, we give a fundamental lemma:
Theorem 3.4.1: Given nonzero $\mu \in \mathcal{M}$ and $n \in \mathbb{N}$, let $H = \sum L_2(\mu)$ and $E$ be the standard spectral measure on $H$. If $\{Z_x\}_{x=1}^m$ is a singular family of cyclic projections on $H$ such that every $\mu_x \sim \mu$ then $m \leq n$.

Proof: By theorem 2.6.3 there exists a family $\{y_i\}_{i=1}^m \in H$ such that for all $i$,

$\mu = \mu_{y_i}$ and $Z_{y_i} = Z_{x_i}$. For all $i$, let $y_i = \{f_{i,1}, \ldots, f_{i,n}\}$. Since

$\mu(M) = \mu_{y_i}(M) = \sum_{j=1}^n \int_M f_{i,j}^2 \, d\mu = \sum_{j=1}^n \int_M f_{i,j}^2 \, d\mu$, for all $M \in S$ and all $i$,

$\sum_{j=1}^n \int_M f_{i,j}^2$ equals 1 a.e. ($\mu$), for all $i$. Also, $(E(M)y_i, y_k) = 0$ whenever $i \neq k$, for all $M \in S$, so that $\sum_{j=1}^n \int_M f_{i,j}^* f_{k,j} \, d\mu = \sum_{j=1}^n \int_M f_{i,j}^* f_{k,j} \, d\mu = 0$, whenever $i \neq k$, for all $M \in S$.

Hence $\sum_{j=1}^n f_{i,j}^* f_{k,j} = 0$ a.e. ($\mu$), whenever $i \neq k$.

Hence the $y_i(z)$, $1 \leq i \leq m$, form an orthonormal set in $\mathbb{C}^n$, for almost all $z \in M$.

Since $\mu(M) > 0$, $\{y_i(z)\}_{i=1}^m$ is an orthonormal set in $\mathbb{C}^n$ for some $z \in M$.

Hence $m \leq \dim (\mathbb{C}^n) = n$.

Given $\mu \in \mathcal{M}$ and a singular family $\{Z_x\}$ of cyclic subspaces of $H$ such that every $\mu_x \sim \mu$, $\sum Z_x$ is called a stack with measure $\mu$. If $|\{Z_x\}| = c$, say, then we say that the stack has multiplicity $c$, or is $c$-fold.
Theorem 3.4.2: Given a spectral measure $E$ on $\mathcal{H}$, nonzero $\mu \in \mathcal{M}$, and stacks $S$ and $T$ with measure $\mu$ and having multiplicity $c$ and $d$ respectively, if $S \subseteq T$ then $c \leq d$.

Proof: If $c$ is finite and $d$ is infinite then $c < d$. If $c$ is finite and $d$ is finite, apply theorem 1.3.2 to every cyclic subspace making up the stack $T$ to obtain a unitary transformation: $T \rightarrow \Sigma \oplus L_2(\mu)$ making $E$ restricted to $T$ unitarily equivalent to the standard spectral measure on $\Sigma \oplus L_2(\mu)$. Then $c \leq d$ by theorem 3.4.1.

If $c = \aleph_0$ then we may apply the previous case to every $m < c$ to conclude that $d \geq m$. Hence $d \geq c$. If $c > \aleph_0$ then $c = c \cdot \aleph_0 = \dim S \leq \dim T = d \cdot \aleph_0 = d$. O

Note that the previous theorem has an important corollary:

Theorem 3.4.3: Given a stack $\Sigma \oplus \mathcal{Z}_x$ having multiplicity $c$ and a stack $\Sigma \oplus \mathcal{Z}_y$ having multiplicity $d$, if $\Sigma \oplus \mathcal{Z}_x = \Sigma \oplus \mathcal{Z}_y$ then $c = d$.

Proof: If $\Sigma \oplus \mathcal{Z}_x$ and $\Sigma \oplus \mathcal{Z}_y$ have measure $\mu$ and $\nu$ respectively, then

$$(\mu) = (\mu_x)_j = \{\mu_x | x \in \Sigma \oplus \mathcal{Z}_x\} \quad \text{(see the remark following theorem 3.3.2)}$$

$$= \{\mu_x | x \in \Sigma \oplus \mathcal{Z}_y\} = (\mu_y)_k = (\nu), \quad \text{so } \mu \sim \nu. \quad \text{The result follows from theorem 3.4.2. O}$$

The previous theorem is useful for defining $m_B(N)$ in much the same way that
Theorem 2.7.5 is useful for defining $m_{\mathcal{H}}(N)$, but first here is another useful theorem:

Theorem 3.4.4: Given a $c$-fold stack $S = \sum \oplus Z_{x_j}$ with measure $\mu$ and $0 \neq \nu \ll \mu$, $S \cap F_{\nu} \mathcal{H}$ is a $c$-fold stack with measure $\nu$.

Proof: For all $j$, there is a unique cyclic subspace $Z_{y_j} \subseteq Z_{x_j}$ such that $\mu_{y_j} = \nu$ (see theorem 2.6.3), and so $\sum \oplus Z_{y_j}$ is a $c$-fold stack with measure $\nu$. But in fact every $Z_{y_j} = Z_{x_j} \cap F_{\nu} \mathcal{H}$, that is $Z_{y_j} = Z_{x_j} F_{\nu} = Z_{x_j} C_{y_j}$, since $Z_{y_j} \subseteq Z_{x_j}$ which is a row, and since $F_{\nu} = C_{y_j}$ (by theorem 2.6.5).

Hence $\sum \oplus Z_{y_j} = (\sum \oplus Z_{y_j}) \cap F_{\nu} \mathcal{H} = S \cap F_{\nu} \mathcal{H}$.  

Theorem 3.4.5: Given a spectral measure $E$ on $\mathcal{H}$, there exists a singular family $\{\mu_j\}$ of nonzero measures $\subseteq \mathcal{M}$ and a family $\{S_j\}$ of subspaces of $\mathcal{H}$ such that every $S_j$ is a stack with measure $\mu_j$ and such that $\mathcal{H} = \sum \oplus S_j$.

Moreover, if there exists another singular family $\{\nu_k\}$ of nonzero measures $\subseteq \mathcal{M}$ and another family $\{T_k\}$ of stacks such that every $T_k$ has measure $\nu_k$ and such that $\mathcal{H} = \sum \oplus T_k$ then $(\mu_j \mid S_j \text{ is } c\text{-fold}) (= \sigma\text{-ideal generated by } \{\mu_j \mid S_j \text{ is } c\text{-fold}\}) = (\nu_k \mid T_k \text{ is } c\text{-fold})$, for all $c > 0$.

Proof: Given nonzero $x \in \mathcal{H}$, $\mu_x \neq 0$, $\mu_x \in T_E$. Apply Zorn's lemma to obtain a maximal singular family $\{Z_{x_j}\}$ such that every $\mu_{x_j} \sim \mu_x$, and let $S = \sum \oplus Z_{x_j}$. If $F_{\mu_x} \mathcal{H} \cap S \neq \{0\}$
then \( \{ \mu_y \mid y \in F_{\mu_x} \} \) is a nonzero ideal \( \subseteq (\mu_x) \), so that by theorem 3.1.2 there exists a nonzero \( v \ll \mu_x \) such that \( \{ \mu_y \mid y \in F_{\mu_x} \} = (v) \). If \( v \sim \mu_x \) then there exists a cyclic subspace in \( F_{\mu_x} \) with measure \( \mu_x \), contradicting the maximality of \( \{ Z_{j'} \} \).

Hence there exists nonzero \( v' \perp v \) such that \( v'v \sim \mu_x \). By theorem 3.4.4

\( S' = S \cap F_0 \) is a stack with measure \( v' \). Also, if \( t \in F_0 \) then \( \mu_t \ll v' \), so that \( \mu_t \ll \mu_x \) and \( \mu_t \perp v \), so that \( t \in S \), and hence \( F_0 \subseteq S \). Hence \( S' = F_0 \). We conclude that if \( H \neq \{0\} \) then there exists a nonzero stack \( S \subseteq H \) with nonzero measure \( \mu \) such that \( S = F_0 \).

Now apply Zorn's lemma again to obtain a singular family \( \{ S_j \} \) of stacks which is maximal with respect to the property that if \( S_j \) has measure \( \mu_j \) then \( S_j = F_{\mu_j} \). Letting \( T = (\mu_j)_j \) the \( \sigma \)-ideal generated by \( \{ \mu_j \} \), \( F_T = \Sigma \Theta S_j \). Now suppose that

\( H \Omega \Sigma \Theta S_j = HOF_T \subseteq F_T H \neq \{0\} \). Then repeating the above argument with \( E \)

restricted to \( F_T \), \( F_T H \) contains a nonzero stack \( S \) with nonzero measure \( \mu \) such that \( S = F_{\mu} F_T H \). But \( S \subseteq F_T H \) implies that there exists \( x \in F_T H \) such that \( \mu = \mu_x \), so that \( \mu \in T \), so that \( F_{\mu} \leq F_T \). Hence \( S = F_{\mu} F_T H = F_{\mu} H \), and \( S \perp S_j \), contradicting the maximality of \( \{ S_j \} \). Hence \( S = F_T H = \Sigma \Theta S_j \). Of course, since \( F_{\mu_i} F_{\mu_j} = 0 \) for \( i \neq j \), \( \mu_i \perp \mu_j \) for \( i \neq j \).

Now suppose that there exists a singular family \( \{ T_k \} \) of stacks and a singular family \( \{ \nu_k \} \) of nonzero measures \( \subseteq \mathcal{M} \) such that every \( T_k \) has measure \( \nu_k \) and such that \( \mathcal{H} = \Sigma \Theta T_k \). Given a cardinal number \( c > 0 \), if \( (\mu_j \mid S_j \) is \( c \)-fold) is zero then \( (\mu_j \mid S_j \) is \( c \)-fold) \( \subseteq (\nu_k \mid T_k \) is \( c \)-fold). Otherwise, there exists a nonzero \( \mu_j \) such that \( S_j \)
is c-fold. Given k such that \( u = \mu_j \wedge u_k \neq 0 \), \( u \perp \mu_j \) whenever \( j \neq j' \). Hence

\( \mathcal{F}_v \mathcal{H} \subseteq S_j \), and similarly \( \mathcal{F}_v \mathcal{H} \subseteq T_k \). Hence by theorem 3.4.4 \( \mathcal{F}_v \mathcal{H} = S_j \cap \mathcal{F}_v \mathcal{H} \) is a c-fold stack with measure \( u \), and also \( \mathcal{F}_v \mathcal{H} = T_k \cap \mathcal{F}_v \mathcal{H} \) is a stack with measure \( u \) and multiplicity equal to the multiplicity of \( T_k \). Theorem 3.4.3 implies that \( T_k \) is c-fold.

Hence given k such that \( T_k \) is not c-fold, \( \mu_j \wedge u_k = 0 \), that is \( \mu_j \perp u_k \). But

\( \mu_j \in \mathcal{T}_E = (u_k)_k \), the \( \sigma \)-ideal generated by \( \{u_k\} \), so \( \mu_j \in (u_k \mid T_k \text{ is c-fold}) \).

Hence \( (\mu_j \mid S_j \text{ is c-fold}) \subseteq (u_k \mid T_k \text{ is c-fold}) \). Similarly, we have that

\( (u_k \mid T_k \text{ is c-fold}) \subseteq (\mu_j \mid S_j \text{ is c-fold}) \), and so we have, for all nonzero \( c \), that

\( (\mu_j \mid S_j \text{ is c-fold}) = (u_k \mid T_k \text{ is c-fold}) \).

Section 5: Brown's Multiplicity Function

Given a normal operator \( N \) on \( \mathcal{H} \) with spectral measure \( E_N \), the multiplicity function of \( N \) \( m_B(N) : \{ \text{nonzero cardinals } c \leq \dim \mathcal{H} \} \rightarrow \mathbb{N} \) is defined by

\( m_B(N)(c) = \mathcal{T}_c = (\mu_j \mid S_j \text{ is c-fold}) \), where \( \{\mu_j\} \) and \( \{S_j\} \) are families of measures and stacks as in theorem 3.4.5. Of course, theorem 3.4.5 ensures that the function \( m_B(N) \) is well-defined.

Theorem 3.5.1: Given normal operators \( N \) and \( N' \) on \( \mathcal{H} \) and \( \mathcal{K} \) and with spectral measures \( E_N \) and \( E_{N'} \), respectively, \( m_B(N) = m_B(N') \) if and only if \( N \equiv N' \).

Proof: Suppose \( N \equiv N' \). Then there exists a unitary transformation \( U : \mathcal{K} \rightarrow \mathcal{H} \) such that

\( U^{-1}E_N U = E_{N'} \). Hence \( UZ_x = Z_{Ux} \), for all \( x \in \mathcal{K} \). Hence if \( S \) is a c-fold stack with
respect to $E_N$, having measure $\mu$, such that $S = F_{\mu}K$, then $US$ is a $c$-fold stack with respect to $E_N$ having measure $\mu$, such that $US = F_{\mu}K$. So if $K = \sum_j S_j$ is a decomposition according to theorem 3.4.5 with respect to $E_N$, then $U\sum_j US_j$ is a decomposition according to theorem 3.4.5 with respect to $E_N$, and so $m_B(N) = m_B(N')$.

Conversely, suppose $m_B(N) = m_B(N')$. Given a cardinal number $c > 0$ and $T \in \hat{\mathfrak{M}}$, suppose that $0 \neq T = (\mu_\alpha)_{\alpha} = (\nu_\beta)_\beta$, where $\{\mu_\alpha\}$ is a singular family of nonzero measures $\leq \mathfrak{M}$, and so is $\{\nu_\beta\}$. Then if $E$ and $F$ are the standard spectral measures on $\sum_\alpha L_2(\mu_\alpha)$ and $\sum_\beta L_2(\nu_\beta)$ respectively then $E \equiv F$: For the standard spectral measure on each $L_2(\mu_\alpha)$ may be identified with the standard spectral measure on $\sum_\beta L_2(\mu_\alpha \land \nu_\beta)$, since $\mu_\alpha = V_\beta(\mu_\alpha \land \nu_\beta)$, for all $\alpha$. Hence $E$ is unitarily equivalent to the standard spectral measure on $\sum_{\alpha, \beta} L_2(\mu_\alpha \land \nu_\beta)$, and so is $F$ by a similar argument. Hence $E \equiv F$.

Hence the standard spectral measure on $c$ copies of $\sum_\alpha L_2(\mu_\alpha)$ is unitarily equivalent to the standard spectral measure on $c$ copies of $\sum_\beta L_2(\nu_\beta)$.

Now suppose that $\mathfrak{H} = \sum_j S_j$ and $K = \sum_k T_k$ are decompositions (with respect to $E_N$ and $E_{N'}$ respectively) according to theorem 3.4.5. If every $S_j = \sum_a Z_{j,a}$ and every $T_k = \sum_w Z_{k,w}$, with measures $\mu_j$ and $\nu_k$ respectively, then $E_N$ restricted to $F_{T_c} \mathfrak{H}$ is unitarily equivalent to $E_{N'}$ restricted to $F_{T_c} \mathfrak{K}$, for all $c$: Simply apply theorem 1.3.2 to every $Z_{j,a}$ such that $\mu_j \in T_c$ and to every $Z_{k,w}$ such that $\nu_k \in T_c$, so that $E_N$ restricted to $F_{T_c} \mathfrak{H}$ is unitarily equivalent to the standard spectral measure on $c$ copies of $\sum_{\mu_j \in T_c} L_2(\mu_j)$ and $E_{N'}$ restricted to $F_{T_c} \mathfrak{K}$ is unitarily equivalent to the standard spectral measure on $c$ copies of $\sum_{\nu_k \in T_c} L_2(\nu_k)$. 
Since $\mathcal{H} = \sum_{c \geq 1} \oplus F_c \mathcal{H}$ and $\mathcal{K} = \sum_{c \geq 1} \oplus F_c \mathcal{K}$, $E_N \equiv E_{N'}$.

Section 6: The separable case

It is useful to examine how Brown's version of multiplicity theory is simplified when the Hilbert space $\mathcal{H}$ is assumed to be separable. In this case we have the following results:

Theorem 3.6.1: Given a normal operator $N$ on $\mathcal{H}$ and a nonzero cardinal $c \leq \dim \mathcal{H}$, $m_B(N)(c)$ is principal.

Proof: Given a (countable) basis $\{x_n\}$ for $\mathcal{H}$, the family $\{Z_{x_n}\}$ spans $\mathcal{H}$. Hence by theorem 3.3.2 the family $\{\mu_{x_n}\}$ generates $T_{E_N}$, that is $T_{E_N} = (\mu_{x_n})_n = V_n(\mu_{x_n})$. However, given $\nu \in V_n(\mu_{x_n})$, theorem 3.1.3 implies that $\nu = V_n \nu_n$ where every $\nu_n \ll \mu_{x_n}$. Since every $\nu_n \ll V_n \mu_{x_n}$ we conclude that $\nu = V_n \nu_n \ll V_n \mu_{x_n}$. So $T_{E_N} \subseteq (V_n \mu_{x_n})$. On the other hand every $\mu_{x_n} \in T_{E_N}$, so $V_n \mu_{x_n} \in T_{E_N}$. Hence $(V_n \mu_{x_n}) \subseteq T_{E_N}$. Hence $T_{E_N} = (V_n \mu_{x_n})$, a principal ideal. Since $c > 0$, $m_B(N)(c) \subseteq T_{E_N}$. Theorem 3.1.2 implies that $m_B(N)(c)$ is principal also.

The above theorem implies that there exists a family $\{\mu_c\}_{c=1}^{\kappa_0} \subseteq T_{E_N}$ such that

$m_B(N)(c) = (\mu_c)$ for all $c$. Since $(\mu) = (\nu)$ if and only if $\mu \sim \nu$, theorem 3.4.5 reduces to the following:
Theorem 3.6.2: Given a spectral measure $E$ on $\mathcal{H}$, there exists a countable singular family $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ and a family $\{S_n\}_{n=1}^{\infty}$ of subspaces of $\mathcal{H}$ such that every $S_n$ is an $n$-fold stack with measure $\mu_n$ and such that $\mathcal{H} = \sum_{n} \oplus S_n$. Moreover, if there exists another singular family $\{\nu_n\} \subseteq \mathcal{M}$ and another family $\{T_n\}$ of subspaces such that every $T_n$ is an $n$-fold stack with measure $\nu_n$ and such that $\mathcal{H} = \sum_{n} \oplus T_n$, then $\nu_n \sim \mu_n$ and $T_n = S_n$, for all $n$.

Proof: The only thing requiring proof is that $T_n = S_n$, for all $n$.

Given $n$, suppose $T_n = \sum_{j=1}^{n} \oplus Z_{x_j} \mathcal{H}$, where every $\mu_{x_j} \sim \nu_n$. If $y \in T_n$ then $Z_y \leq V_{x_j}$. Hence $C_y \leq C(V_{x_j}) = V_{x_j} C_{x_j} = V_{x_j} F_{\nu_n} = F_{\nu_n}$. Hence $y \in C_y \mathcal{H} \subseteq F_{\nu_n} \mathcal{H}$. Hence $T_n \subseteq F_{\nu_n} \mathcal{H}$. However, $\{F_{\nu_n}\}$ is a singular family since $\{\nu_n\}$ is a singular family (apply theorems 2.6.1 and 2.6.5). Hence $\sum_{n} \oplus T_n \subseteq \sum_{n} \oplus F_{\nu_n} \mathcal{H} \subseteq \mathcal{H} = \sum_{n} \oplus T_n$, so $\sum_{n} \oplus F_{\nu_n} \mathcal{H} = \sum_{n} \oplus T_n = \mathcal{H}$. Since every $T_n \subseteq F_{\nu_n} \mathcal{H}$, every $T_n = F_{\nu_n} \mathcal{H}$. Similarly every $S_n = F_{\mu_n} \mathcal{H}$. But since $(\nu_n) = (\mu_n)$ by theorem 3.4.5, $\nu_n \sim \mu_n$ for all $n$. Hence $F_{\nu_n} = F_{\mu_n}$ for all $n$. Hence $T_n = F_{\nu_n} \mathcal{H} = F_{\mu_n} \mathcal{H} = S_n$, for all $n$. 

Theorem 3.6.2 can be used to derive the version of spectral multiplicity theory due to Hellinger:

Theorem 3.6.3: Given a spectral measure $E$ on a (separable) Hilbert space $\mathcal{H}$, there exists a countable family $\{\nu_n\}$ of nonzero measures $\subseteq \mathcal{M}$ and a family $\{Z_{x_n}\}$ of cyclic
subspaces of \( \mathcal{H} \) such that

i) \( \psi_{n+1} \ll \psi_n \), for all \( n \),

ii) \( \mathcal{H} = \bigoplus_{n} \mathcal{Z}_{x_n} \), and

iii) \( \mu_{x_n} \sim \psi_n \), for all \( n \).

Moreover, every \( \psi_n \) is unique up to equivalence.

Proof: Apply theorem 3.6.2 to obtain \( \{\mu_n\} \subseteq \mathcal{M} \) and \( \{S_n\} \) with the appropriate properties. For all \( k \), let \( \psi_k = \bigoplus_{n=k}^{\infty} \mu_n \), keeping only those \( \psi_k \neq 0 \). For all \( n \), there exists \( y_{1,n} \in S_n \) such that \( \mu_n \sim \mu_{y_{1,n}} \). Since the \( \mu_n \) are pairwise singular, so are the \( C_{y_{1,n}} \). Hence by theorem 3.2.3, \( \bigoplus_{n}^{\infty} \mathcal{Z}_{y_{1,n}} \) is a cyclic projection, say \( \mathcal{Z}_{x_1} \). Hence \( \mu_{x_1} \sim \psi_1 \), since the \( \mu_{y_{1,n}} \) are pairwise singular. Continue by choosing \( y_{2,n} \in S_n \bigoplus \mathcal{Z}_{y_{1,n}} \), for all \( n \geq 2 \), and letting \( \mathcal{Z}_{x_2} = \bigoplus_{n>1} \mathcal{Z}_{y_{2,n}} \). Since every \( \mu_n \sim \mu_{y_{2,n}} \), \( \mu_{x_2} \sim \psi_2 \). Note that

\[ S_1 = \mathcal{Z}_{y_{1,1}} \text{ and } S_2 = \mathcal{Z}_{y_{1,2}} \oplus \mathcal{Z}_{y_{2,2}} \].

At the \( m \)th stage, choose

\[ y_{m,n} \in S_n \bigoplus (\mathcal{Z}_{y_{1,n}} \oplus \mathcal{Z}_{y_{2,n}} \oplus \ldots \oplus \mathcal{Z}_{y_{m-1,n}}) \] for all \( n \geq m \), and let \( \mathcal{Z}_{x_m} = \bigoplus_{n \geq m} \mathcal{Z}_{y_{m,n}} \).

Then every \( \mu_n \sim \mu_{y_{m,n}} \) and \( \psi_m \sim \mu_{x_m} \). Continue by induction on \( m \).

Therefore there exists a family \( \{\mathcal{Z}_{x_n}\} \) such that every \( \mu_{x_n} \sim \psi_n \). Moreover, for all \( n \),

\[ S_n = \sum_{m=1}^{n} \mathcal{Z}_{y_{m,n}} \]. Hence

\[ \mathcal{H} = \bigoplus_{n \geq 1} S_n = \bigoplus_{n \geq 1} \bigoplus_{m=1}^{n} \mathcal{Z}_{y_{m,n}} = \bigoplus_{m \geq 1} \bigoplus_{n \geq m} \mathcal{Z}_{y_{m,n}} = \bigoplus_{m \geq 1} \mathcal{Z}_{x_m} \].

Next, it is obvious that \( \psi_{n+1} \ll \psi_n \), for all \( n \), since
\[ \nu_n = \frac{\mathbb{1}_n}{m=1} \mu_m = \mu_{n+1} \nu_{n+1}, \text{ for all } n. \]

Finally, given any family \( \{\nu_n\} \) as above, setting every \( S_n = F_{\nu_n} \cap F_{\nu_{n+1}} \) we obtain a family of stacks as in theorem 3.6.2. Since the \( S_n \) are unique and their measures \( \mu_n \) are unique up to equivalence, and every \( \nu_n = \frac{\mathbb{1}_n}{m=1} \mu_m \), we conclude that every \( \nu_n \) is unique up to equivalence also.

There is one more simplification in the separable case:

**Theorem 3.6.4:** Given a spectral measure \( E \) on \( \mathcal{H} \), \( \xi = \xi'' \).

**Proof:** According to the proof of theorem 3.6.1 \( T_E \) is principal. Suppose \( T_E = (\mu) \).

Given a \( \sigma \)-ideal \( T \subseteq T_E \), theorem 3.1.2 implies that there exists \( \nu \in \mathcal{M} \) such that \( T = (\nu) \). Since \( \nu \ll \mu \), theorem 1.4.2 implies that there exists \( M \in \mathcal{S} \) such that \( \nu \sim \mu | M \).

Hence \( T = (\mu | M) \). If we let \( (M) \) denote the \( \sigma \)-ideal \( \{ \lambda \in \mathcal{M} \mid \lambda \text{ is supported on } M \} \), then \( T = (\mu) \cap (M) \). Hence \( F_T = F_{(\mu) \cap (M)} = F_{T_E \cap (M)} = F_M \).

In fact, \( F_M = E(M) \): If \( x \in F_M \mathcal{H} \) then \( \mu_x(M \mathcal{M}) = 0 = \|E(M \mathcal{M})x\|_\mathcal{H}^2 \). Hence \( E(M \mathcal{M})x = 0 \), so that \( E(M)x = E(M)x + E(M \mathcal{M})x = E(M)x \). Hence \( x \in E(M) \mathcal{H} \).

Conversely, if \( x \in E(M) \mathcal{H} \) then \( E(M)x = x \). Hence \( \mu_x(M \mathcal{M}) = \|E(M \mathcal{M})x\|_\mathcal{H}^2 = 0 \), so \( \mu_x \in (M) \). Hence \( x \in F_M \mathcal{H} = \{ y \in \mathcal{H} \mid \mu_y \in (M) \} \).

Therefore given a \( \sigma \)-ideal \( T \subseteq T_E \), \( F_T = E(M) \) for some \( M \in \mathcal{S} \). Given any \( T \in \mathcal{H}, T \cap T_E \) is a \( \sigma \)-ideal \( \subseteq T_E \). Hence by the preceding remark \( F_T = F_{T \cap T_E} = E(M) \).
for some $M \in \mathcal{S}$, that is $\{F_T \mid T \in \hat{\mathcal{M}}\} \subseteq \xi$. However, according to theorem 3.3.4

\[(F_T \mid T \in \hat{\mathcal{M}}) = \xi'', \text{ so } \xi'' \subseteq \xi. \text{ But it is trivial that } \xi \subseteq \xi'', \text{ and so we have equality.}\]

It should be noted that the proof of theorem 3.6.4 uses only the assumption that $\mathcal{T}_E$ is principal. Hence $\xi = \xi''$ not just in the separable case, but whenever $\mathcal{T}_E$ is principal.

Section 7: The Relationship between $m_H(N)$ and $m_B(N)$

It follows from theorems 2.9.4 and 3.5.1 that given normal operators $N$ and $N'$,

$m_B(N) = m_B(N')$ if and only if $m_H(N) = m_H(N')$. In order to show this result more directly we require some results which will increase our understanding of Halmos' multiplicity function.

Theorem 3.7.1: Given a spectral measure $E$ and $\nu \in \mathcal{M}$, there exists $\lambda \in \mathcal{T}_E$ such that $F_\nu = F_\lambda$ and $\lambda \ll \nu$.

Proof: Theorem 3.1.3 implies that $(\nu) \cap \mathcal{T}_E$ is a $\sigma$-ideal. By theorem 3.1.2 there exists $\lambda \in \mathcal{M}$ such that $(\lambda) = (\nu) \cap \mathcal{T}_E$. Hence $\lambda \ll \nu$ and $\lambda \in \mathcal{T}_E$. The remark preceding theorem 3.3.2 implies that $F_\nu = F_{(\nu) \cap \mathcal{T}_E} = F_\lambda$.

Theorem 3.7.2: Given a normal operator $N$ with spectral measure $E_N$ and $\mu \in \mathcal{M}$, there exists $\mu_0 \in \mathcal{T}_{E_N}$ such that $m_H(N)(\mu_0) = m_H(N)(\mu)$ and $\mu_0 \ll \mu$. 

Proof: If $m_H(N)(\mu) = 0$ then we may take $\mu_0 = 0 \in \mathcal{T}_{E_N}$. 

Otherwise $0 < m_H(N)(\mu) = \min \{u(F_{\nu_0}) | 0 \neq \nu_0 << \mu\}$. Hence $u(F_{\nu_0}) > 0$ whenever $0 \neq \nu_0 << \mu$. For all $\nu_0$ such that $0 \neq \nu_0 << \mu$, theorem 3.7.1 implies that there exists $\lambda \in \mathcal{T}_{E_N}$ such that $\lambda << \nu_0 << \mu$ and $F_{\nu_0} = F_{\lambda}$.

Hence $u(F_{\lambda}) = u(F_{\nu_0}) > 0$, so $\lambda \neq 0$.

Hence $m_H(N)(\mu) = \min \{u(F_{\lambda}) | 0 \neq \lambda << \mu, \lambda \in \mathcal{T}_{E_N}\}$.

Now apply theorem 3.1.5 to obtain $\mu_0 \in \mathcal{T}_{E_N}$ and $\mu_1 \in \mathcal{T}_{E_N}^\perp$ such that $\mu \sim \mu_0 \vee \mu_1$.

Since $\mathcal{T}_{E_N}$ is a $\sigma$-ideal, if $\nu_0 << \mu_0 \in \mathcal{T}_{E_N}$ then $\nu_0 \in \mathcal{T}_{E_N}$. Hence

$m_H(N)(\mu_0) = \min \{u(F_{\nu_0}) | 0 \neq \nu_0 << \mu_0\} = \min \{u(F_{\nu_0}) | 0 \neq \nu_0 << \mu_0, \nu_0 \in \mathcal{T}_{E_N}\}$.

Therefore in order to show that $m_H(N)(\mu) = m_H(N)(\mu_0)$ we need only show that

$\{\nu_0 \in \mathcal{T}_{E_N} | 0 \neq \nu_0 << \mu_0\} = \{\lambda \in \mathcal{T}_{E_N} | 0 \neq \lambda << \mu\}$. Since $\mu_0 << \mu$, if $\nu_0 << \mu_0$ then $\nu_0 << \mu$. So $\{\nu_0 \in \mathcal{T}_{E_N} | 0 \neq \nu_0 << \mu_0\} \subseteq \{\lambda \in \mathcal{T}_{E_N} | 0 \neq \lambda << \mu\}$. Now suppose $\lambda \in \mathcal{T}_{E_N}$ such that $0 \neq \lambda << \mu$. We wish to show that $\lambda << \mu_0$. Since $\mu_1 \in \mathcal{T}_{E_N}^\perp$,

$\lambda \wedge \mu_1 = 0$. Hence $\lambda \sim \lambda \wedge \mu \sim (\mu_0 \vee \mu_1) = (\lambda \wedge \mu_0) \vee (\lambda \wedge \mu_1) = (\lambda \wedge \mu_0) << \mu_0$.

Hence $\{\lambda \in \mathcal{T}_{E_N} | 0 \neq \lambda << \mu\} \subseteq \{\nu_0 \in \mathcal{T}_{E_N} | 0 \neq \nu_0 << \mu_0\}$.

Theorem 3.7.3: Given normal operators $N$ and $N'$ on $\mathcal{H}$ and $\mathcal{K}$ with spectral measures $E_N$ and $E_{N'}$, respectively, $m_H(N) = m_H(N')$ if and only if $m_B(N) = m_B(N')$.

Proof: Consider the multiplicity function $m_H(N)$, and let $\{\nu_j\}$ be a singular family of nonzero measures $\subseteq \mathcal{M}$ as in theorem 2.1.3. According to the proof of theorem 2.9.4
\( V_j F_{u_j} = 1 \) and every \( F_{u_j} = V_k Z_{\mu_j k} \), where \( \{ Z_{\mu_j k} \}_{k=1}^{m_{H(N)}(u_j)} \) is a singular family of cyclic projections such that \( \mu_{\mu_j k} \sim u_j \) for all \( k \). Hence every \( F_{u_j} \) is the projection onto a 

\[ m_{H(N)}(u_j) \text{-fold stack with measure } u_j, \text{ and } \mathcal{H} = \sum F_{u_j} \mathcal{H} \text{ (as in theorem 3.4.5).} \]

Hence for every nonzero cardinal \( c \) \( m_B(N)(c) \) is the \( \sigma \)-ideal generated by every \( u_j \) such that \( m_{H(N)}(u_j) = c \). If \( m_{H(N)}(N) = m_{H(N')} \), then (with the same family \( \{ u_j \} \)) every \( F_{u_j} \mathcal{K} \) is a \( m_{H(N')} (u_j) \)-fold stack with measure \( u_j \), and \( \mathcal{K} = \sum F_{u_j} \mathcal{K} \). Hence for every 

\[ c > 0 \ m_B(N')(c) \text{ is the } \sigma \text{-ideal generated by every } u_j \text{ such that} \]

\[ m_{H(N')}(u_j) = m_{H(N)}(u_j) = c. \text{ Hence } m_B(N')(c) = m_B(N)(c) \text{ for every } c, \text{ that is} \]

\[ m_B(N') = m_B(N). \]

Conversely, suppose that \( m_B(N) = m_B(N') \). Given \( \mu \in \mathcal{M} \), suppose that 

\[ m_{H(N)}(\mu) = c. \text{ If } \mu = 0 \text{ then } m_{H(N)}(\mu) = 0 = m_{H(N')}(\mu). \text{ Otherwise, by theorem 3.7.2 there exists } \mu_0 \in \mathcal{T}_{E_N} \text{ such that } \mu_0 \ll \mu \text{ and } m_{H(N)}(\mu_0) = c. \text{ Assume first that } c > 0. \]

Theorem 2.1.1 implies that there exists a nonzero \( \upsilon \in \mathcal{M} \) such that \( \upsilon \ll \mu_0 \) and \( \upsilon \) has uniform multiplicity \( c \) with respect to \( m_{H(N)} \). Note that \( \upsilon \in \mathcal{T}_{E_N} \) (a \( \sigma \)-ideal). Hence \( F_{\upsilon} \) has uniform multiplicity by theorem 2.9.3. Hence (as in the proof of theorem 2.9.4, since 

\( F_{\upsilon} \) is separable) \( F_{\upsilon} \mathcal{H} \) is a \( c \)-fold stack with measure \( \upsilon \).

Hence \( \upsilon \in m_B(N)(c) = m_B(N')(c) \).

Apply theorem 3.4.5 to obtain a singular family \( \{ \mu_j \} \subseteq \mathcal{M} \) and a family \( \{ S_j \} \) of subspaces of \( \mathcal{K} \) such that every \( S_j \) is a stack with measure \( \mu_j \), every \( S_j = F_{\mu_j} \mathcal{K} \), and 

\[ \mathcal{K} = \sum S_j. \text{ Since } m_B(N')(c) = (\mu_j \mid S_j \text{ is } c \text{-fold}), \upsilon = V_j (\upsilon \wedge \mu_j \mid S_j \text{ is } c \text{-fold}). \]

Let 

\[ \{ u_k \} = (\mu_j \mid S_j \text{ is } c \text{-fold}), \text{ so that } \upsilon = V_k (\upsilon \wedge u_k). \text{ Since every } F_{u_k} \mathcal{K} \text{ is a } c \text{-fold stack,} \]

\[ \text{(continue reading...)} \]
Theorem 2.8.4 implies that every $F_{u_k}$ has uniform multiplicity $c$. Hence by theorem 2.9.3 every $u_k$ has uniform multiplicity $c$ with respect to $m_H(N')$. Hence

$$m_{H}(N')(u) = m_{H}(N')(V_k(u \wedge u_k)) = \min_k (m_{H}(N')(u \wedge u_k) \mid u \wedge u_k \neq 0) = c.$$ Since $0 \neq u << m_0 << \mu$, $m_{H}(N') (\mu) \leq m_{H}(N')(u) = c = m_{H}(N)(\mu)$.

Finally, if $0 = c = m_{H}(N)(\mu)$ then since $\mu \neq 0$, theorem 2.1.1 implies that there exists a nonzero $u << \mu$ such that $u$ has uniform multiplicity $0$ with respect to $m_{H}(N)$. Hence $F_{u}$ has uniform multiplicity $0$, so that $u(F_{u}) = 0$. But by theorem 2.6.5 $F_{u}$ is separable. Hence theorem 2.5.3 implies that $F_{u} = C_x$ for some $x \in \mathcal{H}$. Since $Z_x$ is a row such that $Z_x \leq F_{u}$ and $C(Z_x) = C_x = F_{u}$, $Z_x = 0$, (else $u(F_{u}) \geq 1 > 0$). Hence $F_{u} = C_x = 0$. So $u \in T_{E_N}$. But since $m_{B}(N) = m_{B}(N')$, $T_{E_N} = V_{c > 0} m_{B}(N)(c) = V_{c > 0} m_{B}(N')(c) = T_{E_N}$.

Hence $u \in T_{E_N}$. Hence $F_{u} = \{ x \in \mathcal{K} \mid x << u \} = 0$, so $F_{u} = 0$ (with respect to $N'$ now). Hence $u(F_{u}) = 0$. Hence $m_{H}(N')(u) = \min \{ u(F_{u}) \mid 0 \neq u << u \} = 0$. However, $0 \neq u << \mu$, so $m_{H}(N')(\mu) \leq m_{H}(N')(u) = 0 = m_{H}(N)(\mu)$.

Hence in every case $m_{H}(N')(\mu) \leq m_{H}(N)(\mu)$. Similarly, we may show the reverse inequality. Hence $m_{H}(N')(\mu) = m_{H}(N)(\mu)$, for all $\mu \in \mathcal{M}$, that is $m_{H}(N') = m_{H}(N)$.

In fact, we may give an explicit formula for $m_{B}(N)$ using the proof of theorem 2.9.4:

Theorem 3.7.4: Given a normal operator $N$ on $\mathcal{H}$ with spectral measure $E_{N}$ and a nonzero cardinal $c$, $m_{B}(N)(c) = \{ \mu \in \mathcal{M} \mid \mu = 0 \text{ or } \mu \text{ has uniform multiplicity } c \}$.

Proof: Apply theorem 2.1.3 to obtain a singular family $\{ \mu_j \}$ of nonzero measures $\subseteq \mathcal{M}$
such that every $\mu_j$ has uniform multiplicity and $\mu \sim V_j(\mu \land \mu_j)$, for all $\mu \in \mathcal{M}$. Let

$\{\nu_k\} = \{\mu \mid m_H(N)(\mu) > 0\}$. We claim that the new family $\{\nu_k\}$ along with $\{F_{\nu_k}\}$ satisfies the conditions of theorem 3.4.5. Indeed, every $F_{\nu_k}$ is separable (theorem 2.6.5) and has uniform multiplicity $m_H(N)(\nu_k) > 0$ (theorem 2.9.3). Hence every $F_{\nu_k}$ is a $m_H(N)(\nu_k)$-fold stack with measure $\nu_k$, as in the proof of theorem 2.9.4. Moreover, given $x \in \mathcal{H}$, $\mu_x \in \mathcal{T}_{E_N}$. If $\mu_x \sim V_k(\mu_x \land \nu_k)$ then there exists a $\mu_j$ with uniform multiplicity 0 such that $\mu_x \land \mu_j \neq 0$, since $\mu_x \sim V_j(\mu_x \land \mu_j)$. But then $m_H(N)(\mu_x \land \mu_j) = 0$, so $m_H(N)(\mu_x) = \min \{m_H(N)(\mu_j) \mid \mu_x \land \mu_j \neq 0\} = 0$.

However, we saw in the last part of the proof of theorem 3.7.3 that if $\nu = \mu_x \land \mu_j \neq 0$ and has uniform multiplicity 0 (since $\mu_j$ has uniform multiplicity 0), then $\nu \in \mathcal{T}_{E_N}$. But $\nu \ll \mu_x \in \mathcal{T}_{E_N}$, so $\nu \in \mathcal{T}_{E_N}$. Hence $\nu = 0$, a contradiction. Hence $\mu_x \sim V_k(\mu_x \land \nu_k)$.

Hence $C_x = F_{\mu_x} = V_k F_{\mu_x \land \nu_k} \leq V_k F_{\nu_k}$, again as in the proof of theorem 2.9.4. Since $x \in C_x \mathcal{H}$, we conclude that $x \in V_k F_{\nu_k} \mathcal{H} = \bigoplus_k F_{\nu_k} \mathcal{H}$. Since $x$ was arbitrary, $\mathcal{H} = \bigoplus_k F_{\nu_k} \mathcal{H}$. This proves the claim.

Therefore $m_B(N)(c) = (\nu_k \mid F_{\nu_k} \mathcal{H} \text{ is c-fold}) = (\nu_k \mid m_H(N)(\nu_k) = c) = V(\nu_k \mid m_H(N)(\nu_k) = c) = \{\mu \in \mathcal{M} \mid \mu = V(\mu \land \nu_k \mid m_H(N)(\nu_k) = c)\}$, by theorem 3.1.3. Suppose $\mu \in \mathcal{M}$ such that $0 \neq \nu = V(\mu \land \nu_k \mid m_H(N)(\nu_k) = c)$. Then if $\nu \in \mathcal{M}$ such that $0 \neq \nu \ll \mu$ then $\nu = 0$. Hence $m_H(N)(\nu) = \min \{m_H(N)(\nu_k) \mid \nu \land \nu_k \neq 0\} = c$.

Therefore $\mu$ has uniform multiplicity $c$. Conversely, if $\mu \in \mathcal{M}$ such that $\mu \neq 0$ and $\mu$ has uniform multiplicity $c$ then for all $k$ such that $0 \neq \mu \land \nu_k$ we have
\[ m_H(N)(u_k) = m_H(N)(u_k \land \mu) = m_H(N)(\mu) = c, \text{ since } u_k \text{ has uniform multiplicity also.} \]

Hence \[ \mu = V_j(\mu \land \mu_j) = V(\mu \land u_k \mid m_H(N)(u_k) = c). \]

Since also \[ 0 = V(0 \land u_k \mid m_H(N)(u_k) = c), \]
we conclude that

\[ m_B(N)(c) = \{ \mu \in \mathcal{M} \mid \mu = 0 \text{ or } \mu \text{ has uniform multiplicity } c \} = \{ \mu \in \mathcal{M} \mid \mu = V(\mu \land u_k \mid m_H(N)(u_k) = c) \}, \] as required.
REFERENCES


