The Dynamics of Hanging Tubular Cantilevers in Axial Flow: An Experimental and Theoretical Investigation

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ABSTRACT

This thesis describes an investigation into the dynamics of hanging, heavy, flexible cantilever tubes. More precisely, the flexible tubes are concentric within a wider, shorter, rigid tube; thus the outer tube forms an annulus over part of the length of the inner tube. The flexible cantilevers both convey and are immersed in water at all times.

Experiments were performed to determine the comparative stability of the following flow configurations: (i) discharging tube and no flow in the annulus, (ii) aspirating tube and no flow in the annulus, (iii) discharging tube with an aspirating annulus, (iv) aspirating tube with a discharging annulus. The flow configurations listed in order of increasing stability are as follows: configuration (iv)→(iii)→(i)→(ii); as was determined by the experiments.

Camera-based motion capture was used to record the three-dimensional motions of the cantilevered tubes. It was found that, for a sufficiently high flow velocity, a discharging tube with or without aspirating flow in the annulus leads to second-mode quasi-planar limit-cycle oscillations in a precessing plane. In addition, an aspirating pipe with a discharging annulus leads to first-mode flutter, and in some cases first-mode and second-mode flutter oscillations of random orientation. Experiments with aspirating pipes and no flow in the annulus revealed extremely small movements and were thus inconclusive about the nature of the motion.
In configurations (i), (iii), and (iv), further increase of the flow velocity resulted in impacting and/or rubbing of the flexible tube with the rigid outer tube, introducing randomness and multiple frequencies into the system behaviour.

Linear theoretical investigations of configurations (b) and (c) were also undertaken to predict the evolution of critical flow velocity and associated frequency with annulus length. Agreement between theory and experiments was moderate to good.

Finally, numerical results based on the developed models were obtained for very long tubes on the order of one kilometre, such as those used in certain industrial applications. In both cases, increasing the cantilever length revealed an asymptotic behaviour, where the critical flow velocity and associated frequency reached limiting values.

For (i), a long discharging cantilever, flutter at ever increasing modes was predicted, but the critical flow velocity remained essentially constant after a certain critical length was surpassed. For (iii), a long discharging cantilever with an aspirating annulus, upon increasing the length first flutter at ever increasing modes and finally static divergence (buckling) was predicted, with the critical flow velocity again reaching a limiting value.
SOMMAIRE

La dynamique de tuyaux lourds encastrés-libres suspendus (le bout en haut encastré, le bout en bas libre) et soumis à des écoulements divers a été étudiée expérimentalement et théoriquement. Plus précisément, un tuyau vertical, dont la partie supérieure est entourée par un tube rigide, tel qu’un espace annulaire est formé, est soumis à un écoulement interne ou un écoulement axial annulaire, ou bien aux deux écoulements simultanément. Le système entier est immergé dans un réservoir d’eau.

Des expériences ont été effectuées pour déterminer le comportement dynamique et la stabilité du système dans les configurations suivantes: (i) le tuyau déchargeant, sans écoulement annulaire; (ii) le tuyau aspirant, sans écoulement annulaire; (iii) le tuyau déchargeant et l’espace annulaire aspirant; (iv) le tuyau aspirant et l’espace annulaire déchargeant. Les configurations d’écoulement énumérés dans l’ordre de stabilité croissant sont les suivantes: configuration (iv) → (iii) → (i) → (ii); comme a été déterminée par les expériences.

Le mouvement du tuyau a été mesuré cinématographiquement, en utilisant des marqueurs passifs afin d’enregistrer le mouvement en trois dimensions. Il a été observé qu’avec un écoulement déchargeant assez élevé dans le tuyau, avec ou sans écoulement externe, des oscillations planaires en second mode de poutre sont déclenchées, avec précession du plan d’oscillations. Par contre, un tuyau aspirant,
avec l’espace annulaire déchargeant, déclenche un flottement au premier mode avec une orientation aléatoire. Des expériences avec le tuyau aspirant sans écoulement annulaire donnent lieu à de mouvements très faibles dont la nature est difficile à déterminer.

Dans les configurations (i),(ii) et (iv), avec une vitesse d’écoulement assez élevée, le tuyau vibrant impacte la paroi annulaire et cause un comportement aléatoire avec de fréquences multiples.

Une étude théorique linéaire a été entreprise dans les cas (i) et (iii). Les solutions aux modèles théoriques ont donné la vitesse d’écoulement critique et la fréquence associée en fonction de la longueur le l’espace annulaire. Il y a un bon accord entre la théorie et les expériences.

De plus, des résultats numériques ont été obtenus à partir des modèles théoriques développés pour des tuyaux de grande longueur, tels que ceux utilisés dans certaines applications industrielles. Pour les deux cas (i) et (iii), en augmentant progressivement la longueur du tuyau, un comportement asymptotique a été révélé, où la vitesse d’écoulement critique et la fréquence associée atteignent des valeurs limites.

Pour le cas (i), dont un tuyau long déchargeant, un flottement aux modes croissants a été obtenu, mais la vitesse d’écoulement critique reste constante une fois qu’une longueur critique a été dépassée. Pour le cas (iii), un tuyau encastré-libre avec l’espace annulaire aspirant, avec l’augmentation de la longueur un flottement se déclenche dans de modes croissants avec la vitesse d’écoulement et finalement donne lieu à une instabilité statique (flambage), encore avec de valeurs critiques asymptotiques.
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CHAPTER 1
Introduction

1.1 Background and motivation

1.1.1 Fluid structure interactions and applications

Despite its imposing name, “Fluid-structure interactions” is simply the study of the interactions of fluids and structures. These interactions are all around us. Everyone is familiar with the incessant motions of a garden hose turned on perhaps a little too much and left unattended, or the waving of a flag in the wind. These phenomena belong to this field of study, in fact they are classical cases of an instability known as flutter, or fluttering.

The fluid-structure interactions involving slender, tubular, or cylindrical structures and axial flows are of intense interest in both the academic world, and in countless industries. Given the extensive use of such geometries, this is hardly surprising.

The case of a fluid-conveying pipe has been described as a paradigm of dynamics (Paidoussis and Li, 1993). Any modification of the flow configuration around the structures comes with its own set of challenges. Case in point, although the dynamics of a cantilevered tube carrying fluid from the clamped end to the free end are well established, a reversal of the flow direction leads to behaviour that is still a matter of active discussion (Giacobbi et al., 2012; Paidoussis, 2014). Furthermore, these systems sometimes exhibit non-linear behaviour such as chaotic dynamics. In
other cases the system behaviour, although far from exotic, is just counter-intuitive, such as the phenomena of (i) negative stiffness, and (ii) destabilization by increased dissipation (Paidoussis, 2014).

Real-world applications of fluid-structure interactions of slender structures in axial flow are numerous. They range from the small, such as: the self-cleaning filters of a Dyson bag-less vacuum cleaner (Dyson Ltd, 2014), Coriolis type flow-meters Ghayesh et al. (2012), and alternative marine propulsion applications for robots and small vessels (Paidoussis, 1998, 2014; Streffing et al., 2012); to the very large, such as: pipelines conveying liquid hydrocarbons, heat exchanger tubes, oil-carrying barges (Hawthorne, 1961), nuclear fuel rods (Paidoussis, 1973) and fuel rod bundles (Paidoussis, 1981), as well as ocean mining (Paidoussis and Luu, 1985).

More recently, fluid-elastic instability has been recognized as a possible culprit of the failure of the very long pipe-strings used in solution mining and hydrocarbon storage applications (Ratigan, 2008). The strings may bend or even break, requiring replacement with associated costs in both pipe material, as well as down-time. Investigating the potential of fluid-elastic instability in these systems is the main practical motivation of the research described herein.

1.1.2 Solution mining and hydrocarbon storage

Solution mining is a technique by which very long tubes, on the order of one kilometre, are lowered into the ground until they reach a water soluble deposit, such as sodium or potassium salts. Water is then pumped down the tube, dissolving the salt and exiting the borehole as saturated brine. The product is then extracted from
the brine by evaporation. This method of extraction leaves behind massive underground caverns, as large as one million cubic meters, that due to the impermeability of the surrounding rock salt are suitable for use as hydrocarbon storage reservoirs.

In order to use the aforementioned cavern as a reservoir, a very long pipe is lowered into the cavern, concentric with an impermeable larger outer pipe, which serves to support the sides of the borehole. The long pipe, also known as the string, is then clamped at surface level and hangs nearly to the bottom of the cavern, as shown in figure 7–1. The liquid hydrocarbon product floats on top of the brine and can be extracted when needed by displacement with additional brine. If a gaseous product is stored, then extraction occurs by allowing the product to expand through the annular region created by the inner and outer pipes.

1.1.3 The system under investigation, and organization of this thesis

The research described in this thesis, was performed at McGill University and funded by the Solution Mining Research Institute (SMRI). Apart from purely academic interest, it is applicable in investigating the stability of brine-strings, as a natural continuation of the previous study of brine-string vibration performed at McGill (SMRI, 2010; Jamin, 2010).

There are two parts to this work: an experimental component and a theoretical component. Four flow configurations were investigated experimentally:

- Configuration (i), the tube discharging in the tank and no flow in the annulus; figure 7–5(a).
- Configuration (ii), the tube aspirating and no flow in the annulus; figure 7–5(b).
• Configuration (iii), the tube discharging in the tank and the annulus aspirating fluid; figure 7–5(c).
• Configuration (iv), the tube aspirating fluid and the annulus discharging in the tank; figure 7–5(d).

These configurations represent the different modes of operation of the salt caverns during solution mining and hydrocarbon storage.

The experiments were performed in the purpose-built, SMRI apparatus at the Fluid Structure Interactions Laboratory of McGill University.

Of the above, configurations (i) and (iii) were also investigated theoretically, and the results were compared to the experiments.

Lastly, the dynamics of a full sized brine-string were also investigated theoretically, by asymptotic analysis.

1.2 Literature review and basic theory

The following, selective rather than exhaustive, literature review aims to provide both a historical background as well as to elucidate some basic theory behind the analysis of the dynamics of tubular cantilevers. For a complete literature review refer to Paidoussis (2004, 2014).

We are concerned with tubular cantilevers that are both conveying fluid and are immersed in axially flowing fluid. The complete problem involves the combination of the effects of internal and external flow, which although deceptively simple to implement, results in systems with very diverse and complex dynamical behaviour.
Historically, internal and external flows have been investigated separately, and later combined to describe systems with both internal and external flows. It is therefore logical that we follow suit and first consider the dynamics of pipes conveying fluid (internal flow) where the fluid flows from the clamped end to the free end; then internal flow where the fluid flows from the free end to the clamped end. Subsequently, we briefly consider the dynamics of cantilevered cylinders subjected to external axial flow. Finally we consider the case of tubular cantilevers subjected to simultaneous internal and external flows.

For a complete review of the subject, refer to Paidoussis (2004, 2014).

1.2.1 Discharging cantilever

Benjamin (1961a,b) systematically studied the problem of the free motions of an articulated fluid-conveying cantilever pipe, composed of \( N \) rigid pipe segments that are flexibly interconnected (7–2(a)). This is a system of \( N \) degrees of freedom (dof); as \( N \to \infty \), it becomes an approximation to the continuously flexible cantilever.

Using a Lagrangian approach, Benjamin arrived at an accurate statement of Hamilton’s principle for a system with infinite energy, supplied here by the stream of flowing fluid. Using this approach, he derived the following expression of the work done on the articulated pipe by the fluid forces over a period of oscillation \( t_1 \),

\[
\Delta W = \int_{0}^{t_1} -MU \left( \dot{R}^2 + U \tau \cdot \dot{R} \right) \, dt ,
\]

where \( \mathbf{R} \) denotes the position vector of the end of the last pipe segment from which fluid is discharged, \( \tau \) is a unit vector in the direction along the last pipe segment, \( U \) is the internal flow velocity, \( M \) is the mass per unit length of the fluid stream, and the
over-dot denotes differentiation with respect to time. From equation (1.1) Benjamin showed that for a sufficiently high flow velocity, $\Delta W > 0$; i.e. energy transfers from the fluid to the pipe, provided that the free end of the articulated pipe: “performs a ‘dragging’ sort of motion, like the end of a flexible wand waved back and forth in a viscous medium”.

Using a plug flow assumption, Benjamin derived the following linear equation for small lateral motions of the continuously flexible cantilever:

$$EI \frac{\partial^4 w}{\partial x^4} + MU^2 \frac{\partial^2 w}{\partial x^2} + 2MU \frac{\partial^2 w}{\partial x \partial t} + (M + m) \frac{\partial^2 w}{\partial t^2} = 0$$

(1.2)

where $m$ is the tube mass per unit length, $x$ and $w$ are the longitudinal and lateral coordinates respectively. He obtained solutions for a two degree-of-freedom ($N = 2$) system which results in a pair of second-order, coupled, ordinary differential equations. The stability of the $N = 2$ system was then determined by classical eigenvalue analysis.

Benjamin predicted theoretically, and confirmed experimentally, the existence of amplified oscillations for the articulated cantilever at sufficiently high flow velocities. He also observed the occurrence of buckling, also known as divergence, in the case of a hanging articulated cantilever conveying water, as opposed to air, when gravity was the predominant restoring force. It is noteworthy that in equation (1.2) there is absence of terms related to friction due to the internally flowing fluid. As stressed in Paidoussis (2014), the internal fluid is not taken to be inviscid, but rather, and as was rigorously shown by Benjamin (1961a), the fluid-frictional effects of the internally flowing fluid are offset by the associated pressure loss.
Following Benjamin, Gregory and Paidoussis (1966a) were concerned specifically with the problem of the flexible cantilever confined in a horizontal plane (figure 7–2(b)). They arrived at equation (1.2) using a Newtonian approach and obtained solutions in the following two ways:

I. They obtained exact solutions by letting $w(x,t) = \text{Re} \left[ \sum_{j=1}^{4} A_j \exp(i\Lambda_j x/L) \times \exp(i\Omega_j t) \right]$ and applying the four boundary conditions (4.43). The existence of non trivial solutions requires that the determinant of the $A_j$’s must vanish. The difficulty of this task necessitated the use of elaborate numerical schemes.

II. They obtained approximate solutions by discretizing (1.2) in terms of the eigenfunctions of a cantilevered Euler-Bernoulli beam, using a Galerkin method.

The exact as well as approximate solutions to the theoretical model predicted the onset of amplified oscillations.

Gregory and Paidoussis also performed experiments using metal and rubber cantilevered pipes (Gregory and Paidoussis, 1966b), which were in good agreement with the theory. Predictions were further improved when taking into account the internal (due to the pipe material) and external (due to the surrounding air) dissipation. Furthermore, they discovered that in some cases small amounts of dissipation can actually destabilize the system.

Paidoussis (1970) built upon the work for horizontal cantilevers and taking into account the effect of gravity as well as internal and external dissipation, he arrived
at the following equation for small lateral motions:

\[
EI \left( 1 + \frac{\mu}{\Omega} \frac{\partial}{\partial t} \right) \frac{\partial^4 w}{\partial x^4} + \left[ MU^2 - (M + m)(L - x)g \right] \frac{\partial^2 w}{\partial x^2} \\
+ 2MU \frac{\partial^2 w}{\partial x \partial t} + (M + m)g \frac{\partial w}{\partial x} + k \frac{\partial w}{\partial t} + (M + m) \frac{\partial^2 w}{\partial t^2} = 0,
\]

valid for \( \text{Im}(\Omega) \ll \text{Re}(\Omega) \), where \( g \) is the gravitational acceleration, and \( \mu \) and \( k \) are the coefficients of hysteretic and viscous damping respectively.

Most importantly, Paidoussis (1970) provided an expression for the work (\( \Delta W \)) done by the fluid forces on the cantilever, over a period of oscillation \( t_1 \), namely

\[
\Delta W = -MU \int_0^{t_1} \left[ \left( \frac{\partial w}{\partial t} \right)^2 + U \left( \frac{\partial w}{\partial t} \right) \left( \frac{\partial w}{\partial x} \right) \right] |_{L} dt.
\]  

This is analogous to (1.1), and it can be seen from (1.4), that if the downstream end of the tube is also supported, then \( \partial w/\partial t |_{L} = 0 \) and consequently \( \Delta W = 0 \); thus the system becomes conservative\(^1\). For small values of \( U \), \( \Delta W < 0 \) and energy flows from the tube to the fluid. Any motions of the system are hence damped. On the other hand, for \( U \) sufficiently large, \( \Delta W > 0 \) and energy can flow from the fluid to the cantilever provided that \( \int_0^{t_1} (\partial w/\partial t) (\partial w/\partial x) |_{L} dt < 0 \); that is: the end of the tube performs a “dragging, lagging motion”. This was experimentally verified, and is also in line with Benjamin’s observations.

Theoretical and experimental results were obtained for the case of hanging, and standing (where the free end is above the clamped end) cantilevers. For hanging

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\(^1\) More specifically the system is “gyroscopic conservative” (Paidoussis and Issid, 1974) and as corrected by Weaver (1974) it is actually non-conservative.
cantilevers, theoretical results predicted only flutter (amplified oscillations). This was also the only instability experimentally observed, with fairly good agreement. For standing cantilevers already buckling under their own weight, it was found that in some cases the internal flow could re-stabilize the system, before it lost stability again by flutter at even higher flow velocities.

At odds with Benjamin’s (1996a,b) work, buckling was neither predicted nor observed. However, the possibility of buckling could not be theoretically refuted. This paradoxical behaviour was later resolved by Paidoussis and Deksnis (1970), when they showed that as $N$ was increased, the critical flow velocity for buckling tends to infinity.

A more complete equation of motion for small lateral motions, including the effects of externally imposed tensioning $\bar{T}$ and pressurization $\bar{p}$, is given by Paidoussis and Issid (1974):

$$E^* I \frac{\partial^5 w}{\partial x^5 \partial t} + E I \frac{\partial^4 w}{\partial x^4 \partial t} + \left\{ M U^2 - \bar{T} + \bar{p} A (1 - 2\nu\delta) - \left[ (M + m)g - M \left( \frac{dU}{dt} \right) \right] (L - x) \right\} \frac{\partial^2 w}{\partial x^2} = 0 \; .$$

In (1.5) $E^*$ is the coefficient of Kelvin-Voigt type internal dissipation, $A$ is the internal cross-sectional area of the tube, $\nu$ is the Poisson ratio of the pipe material, and $\delta$ is a parameter equal to 0 if axial sliding of one end is permitted (such as in the case of a cantilever), and 1 if it is not permitted (such as for a simply supported tube).
Doaré and de Langre (2002) examined the dynamics of long hanging cantilevered pipes conveying fluid both theoretically and experimentally, by considering the propagation of bending waves along their length, i.e. using a travelling-wave rather than a standing wave approach.

They made use of a simplified form of equation (1.3), by ignoring the effects of dissipation ($\mu \to 0, k \to 0$). In addition, they systematically investigated the effect of length $L$ on the system behaviour, by introducing the following scaling length:

$$\tilde{L} = \left[ \frac{EI}{(M + m)g} \right]^{\frac{1}{3}},$$

which took into account both the effect of stiffness and gravity acting on the cantilever.

The stability of the system is an interplay between the internal flow which tends to de-stabilize the system by creating an effective compressive force, and gravity which serves to stabilize the system by inducing a tension due to the cantilever’s weight.

Doaré and de Langre, discovered that the system exhibits asymptotic behaviour with increasing length, and that beyond a certain point the critical flow velocity for flutter remains essentially the same. Furthermore, they ascertained that for a sufficiently long system ($L > L_c$), the flutter instability occurs only in a region of length $L_c$ from the downstream end of the cantilever, for which they also provide an expression. The length of the cantilever above the unstable region, i.e. $0 \leq x \leq L_c$, is essentially “rigidified” by the action of gravity. Experimental results were found to be in very good agreement with the theoretical model.
1.2.2 Aspirating cantilever

A very special place in the dynamics of fluid conveying cantilevers is occupied by the case where the flow is directed from the free end to the clamped end, also known as the aspirating cantilever. Historically, the path towards understanding the dynamics of the aspirating system has been a difficult one “...plagued by missteps and reversals...” (Paidoussis, 2014).

Fundamental scientific interest sparked an initial experimental investigation by Paidoussis in the mid-1960s at the Chalk River Nuclear Laboratories, involving flexible aspirating cantilevers immersed in water. No flutter oscillations were observed\(^2\), and the matter was left to rest for almost two decades.

Motivated by developments in ocean mining, Paidoussis and Luu (1985) realized that an aspirating cantilevered pipe with an end-mass was essentially an idealization of an ocean mining system (figure 7–4), and undertook a theoretical investigation of the dynamics of the system. They arrived at the following equation of motion:

\[
EI \frac{\partial^4 w}{\partial x^4} + MUU_j \frac{\partial^2 w}{\partial x^2} - 2MU \frac{\partial^2 w}{\partial x \partial t} + (M + m + Ma) \frac{\partial^2 w}{\partial t^2} \\
- \left\{ (M + m)g(L - x) + Mg - F_b - p_{oA} \left( \frac{L/\alpha^2 - x}{L} \right) \right\} \frac{\partial^2 w}{\partial x^2} \\
+ [(M + m)g - p_{oA} L] \frac{\partial w}{\partial x} + c \frac{\partial w}{\partial x} = 0
\]

\(^2\) No flutter was observed in the classical sense, i.e. similar to flutter instability of discharging pipes. However, Paidoussis did notice some amplified oscillation relating to the intermittent ingestion of air by the pipe when the lever of the water was sufficiently low, a parametric instability.
where $M$ is mass of internal fluid per unit length, $U_j$ is velocity of the entering jet, $m$ is the mass of the pipe per unit length, $M_a$ is added fluid mass per unit length, $\bar{M}$ is the end-mass, $\bar{F}_b$ is the buoyancy force acting on $\bar{M}$, and $\alpha^2 = A_o/A_j = U_j/U$ relates the entering jet cross-section and jet velocity to the external cross-sectional area of the pipe.

At $x = 0$, the classical clamped boundary conditions apply, and for the free end ($x = L$) we have the following boundary conditions

\[
EI \frac{\partial^3 w}{\partial x^3} - \bar{M} \bar{d} \frac{\partial^3 w}{\partial x \partial t^2} - (\bar{M} g - \bar{F}_b) \frac{\partial w}{\partial x} - (\bar{M} + \bar{M}_a) \frac{\partial^2 w}{\partial t^2} - c \frac{\partial w}{\partial t} = 0,
\]

\[
EI \frac{\partial^2 w}{\partial x^2} + (\bar{M} g - \bar{F}_b) \bar{d} \frac{\partial w}{\partial x} + \bar{M} \bar{d} \frac{\partial^2 w}{\partial t^2} + (\bar{J} + \bar{M} \bar{d}^2) \frac{\partial^3 w}{\partial x \partial t^2} = 0,
\]

where $\bar{d}$ is the distance of the end-mass centre of mass from the tube free end, $\bar{J}$ is the end-mass moment of inertia, and the other quantities with an over-bar are the same as those in equation (1.7) but pertaining to the end-mass.

The system was predicted to lose stability by flutter at very low flow velocities, with the end-mass having a very small effect on the system behaviour. Furthermore, Paidoussis notes that, in the absence of dissipation, the system lost stability at infinitesimally small flow velocities $U^+$ ($U \approx 0.2 \text{ ms}^{-1}$ for the ocean mining analogue).

The predicted system behaviour in the 1985 study appeared wildly unrealistic, and with the construction of a brand-new apparatus in 1986 Paidoussis embarked on an ill-fated series of experiments, which resulted in severe damages to the experimental equipment, along with flooding of the laboratory (Paidoussis, 2014), without producing any results.
Further work (Paidoussis, 1998, 2014), showed the aforementioned result, i.e. that aspirating pipes flutter at infinitesimally small flow velocities, to be completely wrong. Revisiting the problem, Paaidoussis (1999) in a short note undertook a new theoretical and experimental investigation. He reasoned that the flow entering the free end of the pipe undergoes a de-pressurization (or negative pressurisation) equal to

$$\bar{p} = -\rho U^2 \equiv -MU^2/A.$$  \hspace{0.6cm} (1.9)

Which given the simplest form of equation (1.3)

$$EI \frac{\partial^4 w}{\partial x^4} + (\bar{p}A + MU^2) \frac{\partial^2 w}{\partial x^2} + 2MU \frac{\partial^2 w}{\partial x \partial t} + (M + m) \frac{\partial^2 w}{\partial t^2} = 0, \hspace{0.6cm} (1.10)$$

results in cancellation of the centrifugal term $$(\bar{p}A + MU^2) \frac{\partial^2 w}{\partial x^2}$$, and thus (and in view of (1.4)) flutter cannot occur. It is noted that in equation (1.10) $U > 0$ and $\bar{p} = 0$ correspond to a discharging pipe, and $U < 0$ corresponds to an aspirating pipe. A rather simple experiment performed in water, seemed to support the findings, with no instability being observed.

Kuiper and Metrikine (2005), argued that the inlet de-pressurization depends strongly on the inlet geometry, and that the correct value for the negative pressure developed at the pipe free-end lies somewhere between

$$-\rho_f U^2 < \bar{p} < -\rho_f U^2/2.$$  \hspace{0.6cm} (1.11)

Furthermore, they found that the value of the pressurization plays but a small role in the value of the critical velocity, giving a variation on the order of 2% for the calculated critical velocities; they postulated that dissipation to the surrounding fluid is responsible for the experimentally observed behaviour.
Paidoussis et al. (2005), in a special brief communication address the critique of Kuiper and Metrikine (2005), and propose a model which takes into account the flow field in the inlet in a more detailed manner. In particular, they propose that there is a mean velocity \(-v\), a fraction of the mean internal flow velocity \(U\), that has a particular orientation. They explore the possibilities of \(-v\) (i) being tangent to the un-deflected axis of the pipe or (ii) being tangent to the pipe inlet. The work done in Paidoussis et al. (2005) was incorporated in a more recent model proposed in Giacobbi et al. (2012) (in addition refer to Paidoussis (2014)) to be addressed in more detail further down.

New sets of experiments of aspirating pipes in water performed by Kuiper and Metrikine (2008), and experiments in air performed by Rinaldi (2009), lent support to the fact that flutter does in fact occur, beyond a critical, non-zero flow velocity. However, it is of a weak, feeble variety.

In addition, the dynamics of aspirating cantilevers was studied numerically, using coupled structural and fluid mechanical models, as described in Giacobbi et al. (2008), Giacobbi (2010), and Giacobbi et al. (2012), which despite large computational difficulties they did reveal the emergence of low amplitude oscillations that were comparable to the experimental results, and occurred at similar flow velocities.
The most recent linear analytical model is given in Giacobbi et al. (2012), see also Paidoussis (2014). The equation of small lateral motions is

\[
EI \left[ 1 + \left( \bar{\alpha} + \frac{\bar{\nu}^*}{\Omega} \right) \frac{\partial}{\partial t} \right] \frac{\partial^4 w}{\partial x^4} + \left\{ [1 - \left( \frac{3}{2} - \alpha \right)(1 + \gamma)]MU^2 - (M + m - \rho_f A_o) g (L - x) \right\} \frac{\partial^2 w}{\partial x^2} - 2MU \frac{\partial^2 w}{\partial x \partial t} + (M + m - \rho_f A_o) g \frac{\partial w}{\partial x} + c \frac{\partial w}{\partial t} + \left( M + m + M_a \right) \frac{\partial^2 w}{\partial t^2} + MU \left[ \frac{\partial w}{\partial t} - \alpha U (1 - \psi) \frac{\partial w}{\partial x} \right] \delta_D(x - L) = 0
\]

where \( \alpha = |v|/|U| \), \( v \) is a mean velocity at the pipe inlet directed at an angle \( \theta \) from the non-deflected centreline of the tube, the parameter \( \psi \) is defined from \( \theta = \psi \chi \) where \( \chi \approx \left[ \frac{(\partial w)/(\partial x)}{|w|} \right]_{x=L} \) is the slope of the tube free end approximated by the first derivative for small motions, \( \tilde{\gamma} = [p_{\text{lip}}/\bar{p}][\vec{A}/A_o - A/A] \) relates the pressure acting on the pipe lip to the ambient pressure at the pipe free end, and \( \delta_D(x - L) \) is the Dirac delta function.

The stability of the system, i.e. whether flutter arises or not, depends on the particular values of the parameters \( \alpha, \tilde{\gamma}, \) and \( \psi \). This is further illustrated by the expression for the work done on the aspirating cantilever given in Paidoussis (2014),

\[
\Delta W = MU \int_0^{t_1} \left\{ -[1 - \left( \frac{3}{2} - \alpha \right)(1 + \gamma)] + [\alpha(1 - \psi)] \right\} \left. \left( \frac{\partial w}{\partial t} \right) \right|_L \left. \left( \frac{\partial w}{\partial x} \right) \right|_L \, dt , \quad (1.13)
\]

where these parameters, in addition to the term \( [(\partial w/\partial t)(\partial w/\partial x)]_L \) determine the sign of \( \Delta W \).

As detailed in Giacobbi et al. (2012), the values for the parameters \( 0.7 \leq \alpha \leq 0.9, \tilde{\gamma} \approx 0.35, 0.9 \leq \psi \leq 1 \) were determined through CFD simulations, and analytical results were obtained based thereupon. For the cantilevers considered, the range of
the critical velocities for flutter obtained with equation (1.12) were relatively close to the ranges determined by simulations and experiments.

1.2.3 Cantilevered flexible cylinders subjected to external flow

In one of the first studies of cylinders in axial flow, Hawthorne (1961) studied the stability of the Dracone barge, a flexible streamlined cylindrical container towed behind a vessel and designed to carry liquids less dense than water (figure 7–3(a)). Through both a theoretical and experimental investigation, Hawthorne concluded that, above a critical towing velocity, the Dracone barge was “...found to move laterally like an enraged sea serpent...”. This motion is illustrated in figure 7–3(b). Furthermore, he noticed that replacement of the streamlined trailing edge with a totally blunt flat disc, eliminated the “snaking” motions, at the expense of increased drag.

Based on Hawthorne’s 1961 formulation, Paidoussis (1966a,b) investigated both theoretically and experimentally, horizontal cylinders in axial flow, including cantilevered cylinders clamped at the upstream end and free at the downstream end, fitted with a tapering end-piece. He derived the following equation of small lateral motions:

\[
EI \frac{\partial^4 w}{\partial x^4} + M \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 w - \frac{\partial}{\partial x} \left\{ \frac{1}{2} \frac{4C_T}{\pi} \frac{MU^2}{D} \times \right. \\
\left. \left[ \left( 1 - \frac{1}{2} \delta \right) L - x \right] \frac{\partial w}{\partial x} \right\} - \left[ \delta T + \frac{1}{2} (1 - \delta) \frac{4C_b}{\pi} MU^2 \right] \frac{\partial^2 w}{\partial x^2} \\
+ \frac{1}{2} \frac{4C_N}{\pi} \frac{MU}{D} \left( \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} \right) + m \frac{\partial^2 w}{\partial t^2} = 0 ,
\]

given here in slightly modified form for consistency with more recent derivations. In (1.14), \( D \) is the cylinder diameter, \( U \) is the mean external flow velocity, \( C_T \) and \( C_N \)
are the skin friction coefficients associated with the longitudinal and circumferential
directions along the cylinder surface, $C_b$ is a form drag coefficient referred to as
base drag, $\bar{T}$ is an externally applied tension, and $\delta$ is parameter equal to 1 if the
downstream end is supported and 0 if it is not supported. $M$ is the added (or virtual)
mass of the fluid per unit length and it is equal to $M = \rho A = \rho \pi D^2 / 4$ for a circular
cylinder in unconfined flow, provided that the wavelength of the motion is large in
relation to the cylinder diameter.

The effects of the tapering end-piece of length $l$ were included in the downstream
boundary conditions (provided that $(l/L) \ll 1$), leading to the following boundary
conditions for the free end:

$$\frac{\partial^2 w}{\partial x^2} = 0 \ ,$$  \hspace{1cm} \text{(1.15)}

and

$$EI \frac{\partial^3 w}{\partial x^3} + fMU \left( \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} \right) - (m + fM)x_e \frac{\partial^2 w}{\partial t^2} = 0 \ ,$$  \hspace{1cm} \text{(1.16)}

where $0 \leq f \leq 1$ is a parameter related to the three-dimensional nature of the
flow around the end-piece and ranges from 0 for a blunt end to 1 for a perfectly
streamlined end; in (1.16),

$$x_e = \frac{1}{A} \int_{L-l}^L A(x) \, dx \ .$$  \hspace{1cm} \text{(1.17)}

The theoretical predictions and experimental results were in very good agree-
ment. It was found that small flow velocities stabilized the system, while once the
flow velocity was high enough, buckling in the first mode occurred. A further in-
crease in $U$ led to flutter in the second mode and subsequently in the third mode.
The parameter $f$ had a strong influence: the system lost stability at higher flow ve-
locities when the downstream was less streamlined. All instabilities vanished when
the downstream end was made sufficiently blunt, i.e. when \( f \to 0 \). This is similar to Hawthorne’s observation regarding the Dracone barge.

Unfortunately, the original equation of small lateral motions in Paidoussis (1966a), equation (1.14) here, was in error due to the omission of the term \( F_N (\partial w)/(\partial x) \) in the force balance in the \( w \) (lateral) direction, and this mistake found its way into the work of other researchers as well, for example (Pao, 1970; Chen and Wambsganss, 1972). Luckily it had minimal effect on general behaviour of the cantilevered cylinder (Paidoussis, 1973) where the inviscid hydrodynamic forces are dominant.

A more general linear equation of motion was provided by Paidoussis (1973), taking in account internal dissipation, gravity (acting in the positive \( x \)-direction), externally imposed tension, as well as the effect of confinement, i.e. the proximity of the cylinder to the surrounding flow channel. In addition, the error in equation (1.14) was corrected

\[
E^*I \frac{\partial^5 w}{\partial t \partial x^4} + EI \frac{\partial^4 w}{\partial x^4} + \chi \rho A \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right)^2 w - \left\{ \delta \left[ \bar{T} + (1 - 2\nu) \bar{p}A \right] \right. \\
+ \left[ \frac{1}{2} \rho D U^2 C_f \left( 1 + \frac{D}{D_h} \right) + (m - \rho A)g \right] \left[ (1 - \frac{1}{2}\delta) L - x \right] \\
+ \frac{1}{2} \rho D^2 U^2 (1 - \delta) C_b \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \rho D U C_f \left( \frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} \right) \\
+ \frac{1}{2} \rho D C_D \frac{\partial w}{\partial t} + \left[ (m - \rho A)g + \frac{1}{2} \rho D U^2 C_f \left( \frac{D}{D_h} \right)^2 \right] \frac{\partial w}{\partial x} + m \frac{\partial^2 w}{\partial t^2} = 0.
\]

18
In (1.18), $C_N = C_T \equiv C_f$ has been used, although in general $C_N \neq C_T^3$. The diameter of the flow channel is $D_{ch}$, and $D_h = 4A_{ch}/S_{tot}$ is the hydraulic diameter of the annular region between the flow channel and the cylinder, where $A_{ch}$ and $S_{tot}$ are the cross-sectional area and the total wetted perimeter of the annular region. The parameter $\chi = [(D_{ch}^2 + D^2)/(D_{ch}^2 - D^2)] \geq 1$, gives a measure of the confinement, with $\chi = 1$ for an unconfined cylinder. Increased confinement results in an increased added mass, given by $M = \chi \rho A$.

If $\chi \to 1$ and $D/D_h \to 0$, equation (1.18) is reduced to the case of the unconfined cylinder, also provided in (Paidoussis, 1973).

The results for the cantilevered cylinder, obtained with this more general theory, were qualitatively the same with those obtained with (1.14), with perhaps a small improvement in the quantitative agreement between experiment and theory, as per Paidoussis (1973).

It was also found, that increasing the confinement of the cylinder led to essentially the same sequence of instabilities, but at a much lower flow velocity. This is attributed to the increased added mass $M$, which leads to higher inviscid hydrodynamic forces; these forces are dominant in the case of the cantilevered cylinder.

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3 The equation of small lateral motions, where $C_N \neq C_T$ can be found in Paidoussis (2004), equation 8.16.

4 $A_{ch} = \pi (D_{ch}^2 - D_o^2)/4$.

5 For the case of the pinned-pinned or otherwise doubly supported cylinder, although not of interest here, the correction of (1.14) had a profound effect on the predicted instabilities.
In the configurations seen here so far, the external flow is directed from the clamped end to the free end. An interesting study was performed by Rinaldi (2009), refer to Rinaldi and Paidoussis (2012), involving a cantilevered cylinder fitted with an end piece, and with the flow directed from the free end to the clamped end, referred to as a free-clamped cylinder. The equation of motion is reproduced here

\[ EI \left[ 1 + \left( \bar{\alpha} + \frac{\bar{\mu}^*}{\Omega} \right) \frac{\partial}{\partial t} \right] \frac{\partial^4 w}{\partial x^4} - \left[ \frac{1}{2} \rho DU^2 C_T \left( \frac{D}{D_h} \right) \right. \]

\[- (m - \rho A) g \frac{\partial w}{\partial x} + \left\{ \frac{1}{2} \rho D^2 U^2 C_b \right\} \frac{\partial^2 w}{\partial x^2} + \chi \rho A \left( \frac{\partial^2 w}{\partial t^2} - 2U \frac{\partial^2 w}{\partial x \partial t} + U^2 \frac{\partial^2 w}{\partial x^2} \right) \]

\[ + \frac{1}{2} \rho D U C_N \left( \frac{\partial w}{\partial t} - U \frac{\partial w}{\partial x} \right) + m \frac{\partial^2 w}{\partial t^2} = 0 . \]

Although following an involved derivation, equation (1.19) amounts to taking equation (1.18) with \( U \rightarrow -U \) to account for reverse flow. The coefficient \( C_b \) no longer represents base-drag but rather a frontal pressure acting on the free end. Internal dissipation is presumed to be a mixture of hysteretic and viscoelastic damping (Rinaldi, 2009), and regular viscous dissipation is not accounted for independently \( (C_D \rightarrow 0) \) since the effects of viscosity due to the surrounding fluid (in this case air) cannot be separated from the experimentally determined coefficients \( \bar{\alpha} \) and \( \bar{\mu}^* \).

Depending on the parameters \( f \) and \( C_b \), the theoretical model for low flow velocities predicts very low damping (almost zero) or flutter in the first mode. As \( U \) is increased further, divergence in the first mode is predicted. The theoretical predictions were at least qualitatively confirmed by a set of experiments conducted in air.
For small flow velocities, very low amplitude oscillations were observed. In accordance with the theoretical predictions, these could be flutter or vibrations induced by secondary flows impinging on the weakly damped system.

A further increase of the flow velocity led to diminution of the frequency of oscillation until divergence in the first mode developed. As opposed to the case of the clamped-free cylinder, the shape of the end-piece did not appear to have a significant effect on the system dynamics. Despite good qualitative agreement, the quantitative agreement between theory and experiment was rather poor.

Finally, de Langre et al. (2007) studied the effect of length for very long, horizontal, unconfined, clamped-free cylinders. They used the following form of the equation of motion for small lateral motions:

\[
EI \frac{d^4w}{dx^4} - \frac{d}{dx} \left( \Theta \frac{dw}{dx} \right) + \frac{1}{2} \rho U^2 D(C_N - C_T) \frac{dw}{dx} \\
+ \frac{1}{2} \rho U D C_N \frac{dw}{dt} + 2 \rho AU \frac{\partial^2 w}{\partial x \partial t} + (m + \rho A) \frac{\partial^2 w}{\partial t^2} = 0,
\]

(1.20)

where \( \Theta(x) \) is the local tension in the cylinder given by

\[
\Theta(x) = -\rho U^2 \left( A - \frac{1}{2} D^2 C_b + \frac{1}{2} DC_T x \right).
\]

(1.21)

The tension \( \Theta(x) \) includes the drag forces, which tend to stabilize the system by friction, as well as the inviscid hydrodynamic force (Lighthill, 1960), which acts as an effective compressive load of magnitude \( \rho AU^2 \) and tends to destabilize the system. They found a point on the \( x \)-axis, a distance

\[
L_c = D(\pi - 2C_b)/(2C_T)
\]

(1.22)

from the free end of the cylinder, at which the centrifugal and frictional forces cancel out, leading to zero tension \( (\Theta = 0) \) at that point.
Using $L_c$ in place of the cylinder length $L$ for scaling the equation of motion, de Langre et al. (2007) found that both divergence and flutter exist in the case of long cylinders, and in general succeed each other as the flow velocity increases, subject of course to the variation of other parameters of the system such as $f$ (defined previously). More importantly, however, the authors concluded that as the length of the cylinder increases, both divergence and flutter tend to be restricted to a downstream region of length $L = L_c$, while the upstream portion of the cylinder is made rigid through friction induced tension. This conclusion is similar to the results obtained in Doaré and de Langre (2002) in the case of hanging cantilevered tubes.

1.2.4 Cantilevers subjected to both internal and external flows

Hannoyer and Paidoussis (1978) examined the dynamics of clamped-free and clamped-clamped unconfined tubular beams, subjected to both internal and external concurrent axial flow. The cantilevered (clamped-free) tubes were terminated by an externally tapered end-piece of length $l$. In addition, the effect of the developing boundary layer was taken into consideration in a simplified manner. They derived the following equation for small lateral motions:

$$
E^* I \frac{\partial^5 w}{\partial t \partial x^4} + EI \frac{\partial^4 w}{\partial x^4} + \rho_i A_i \left( \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \right)^2 w \\
+ \rho_e A_e \left( \frac{\partial}{\partial t} + U_e \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + U_e \frac{\partial}{\partial x} \right) - (\rho_e A_e - \rho_i A_i - m)g \frac{\partial w}{\partial x} \\
- \left\{ T(L) - \int_x^L \left[ (\rho_e A_e - \rho_i A_i - m)g - \frac{1}{2} C_{f e} \rho_e D_e U_e^2 \right] \, dx \right\} \frac{\partial^2 w}{\partial x^2} \\
+ \frac{1}{2} C_{f n} \rho_e D_e U_e \left( \frac{\partial}{\partial t} + U_e \frac{\partial}{\partial x} \right) + \frac{1}{2} \mu_e C_D \frac{\partial w}{\partial t} + m \frac{\partial^2 w}{\partial t^2} = 0 ,
$$

(1.23)
where $A_i$ and $A_e$ are the internal and external cross-sectional areas, $\rho_i$ and $\rho_e$ the internal and external fluid densities, and $U_i$ and $U_e$ the internal and external flow velocities, and $T(L)$ is the tension at $x = L$. Furthermore, $U_e^*$ (in the fourth term) is the reduced velocity and it is related to the boundary layer growth, and $\mu_e$ in the external fluid dynamic viscosity. For the cantilevered tube free end, the merging internal and external flows create the following rather complicated boundary conditions:

$$\frac{\partial^2 w}{\partial x^2} = 0,$$

(1.24)

and

$$[(\rho + f\rho_e)\bar{A}_e + (\rho_i - \rho)A_i] l \left[ \frac{\partial^2 w}{\partial t^2} \right]_{L-l} - f\rho_e(A_e - A_i)U_e^* \left[ \frac{\partial w}{\partial t} \right]_{L-l}$$

$$+ [f\rho_e(A_iU_e^* + \bar{A}_eU_e) + 2\rho_iA_iU_i] l \left[ \frac{\partial^2 w}{\partial x \partial t} \right]_{L-l}$$

$$- \left( E + E' \frac{\partial}{\partial t} \right) \left[ I \frac{\partial^3 w}{\partial x^3} \right]_{L-l} - \{ f\rho_e(A_e - A_i)U_eU_e^* +$$

$$[(\rho_e - \rho)\bar{A}_e + (\rho - \rho_i)A_i] \bar{g}l \} \left[ \frac{\partial w}{\partial x} \right]_{L-l} = 0,$$

(1.25)

where $\rho$ is the density of the tube material, $0 \leq f \leq 1$ is the familiar parameter related to the end piece streamlining and $\bar{A}_e$ is the end-piece average external area defined by

$$\bar{A}_e = \frac{1}{l} \int_{L-l}^L A_e(x)dx.$$

(1.26)

For the sake of simplicity, the end-piece was presumed to be conical, and the following approximations were used

$$\bar{A}_e \approx \frac{1}{3}(A_e^{3/2} - A_i^{3/2})/(A_e^{1/2} - A_i^{1/2}), \quad f \approx 4l^2/[4l^2 + (D_e - D_i^2)].$$

(1.27)
When the tubular cantilever was fitted with a blunt end-piece \( f \approx 0 \), the system dynamics was dominated by the internal flow, with flutter being precipitated for sufficiently high \( U_i \). The external flow velocity \( U_e \) had a stabilizing effect on the system, such that a tube undergoing flutter could be stabilized by an increase in \( U_e \). Similar to the case of the cantilevered cylinder, no divergence was possible with a blunt end.

For a very well streamlined end-piece, the system dynamics was much more complex, with both divergence and flutter possible, as well as domains of divergence with superimposed flutter. Agreement between theory and experiment was reasonably good.

A further theoretical study in the dynamics of hanging tubular cantilevers subjected to simultaneous flows was performed by Paidoussis et al. (2008). This time the flows were counter-current, where the internal flow was directed from the clamped to the free end, and upon exiting the tube was directed upward as confined external axial flow. In this case the cantilever was blunt, with no end-piece fitted. This system serves as a model for the dynamics of a drill-string with a floating, fluid-powered,
drill-bit. The following linear equation of motion was obtained:

\[
EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} + \rho_f A_f \left( \frac{\partial^2 w}{\partial t^2} + 2U_i \frac{\partial^2 w}{\partial x \partial t} + U_i^2 \frac{\partial^2 w}{\partial x^2} \right) \\
+ \chi \rho_f A_o \left( \frac{\partial^2 w}{\partial t^2} - 2U_o \frac{\partial^2 w}{\partial x \partial t} + U_o^2 \frac{\partial^2 w}{\partial x^2} \right) - \left[ (T - A_f p_i + A_o p_o) \right]_L \\
+ (m + \rho_f A_f - \rho_f A_o) g (L - x) - \frac{1}{2} C_f \rho_f D_o U_o^2 \left( 1 + \frac{D_o}{D_h} \right) (L - x) \frac{\partial^2 w}{\partial x^2} \\
+ \left[ (m + \rho_f A_f - \rho_f A_o) g - \frac{1}{2} C_f \rho_f D_o U_o^2 \left( 1 + \frac{D_o}{D_h} \right) \right] \frac{\partial w}{\partial x} \\
+ \frac{1}{2} C_f \rho_f D_o U_o \frac{\partial w}{\partial t} + k \frac{\partial w}{\partial t} = 0 ,
\]

where \( \rho_f \) is the fluid density, \( D_i \) and \( D_o \) are the internal and external tube diameter, \( A_f \) and \( A_o \) are the cross-sectional areas based on \( D_i \) and \( D_o \) respectively, and \( C_T = C_N \equiv C_f \). The other parameters have been defined in section 1.2.3. Furthermore, the internal and external flows are related through continuity

\[
A_f U_i = A_{ch} U_o ,
\]

where \( A_{ch} = \pi(D_{ch}^2 - D_o^2)/4 \) is the cross-sectional area of the annular region.

The dynamics of the system depend both on the internal \( U_i \) and external \( U_o \) flows, as well as on the amount of confinement conveniently represented by the parameter \( \alpha_{ch} \equiv D_{ch}/D_o \), related to the familiar parameter \( \chi = (\alpha_{ch}^2 + 1)/(\alpha_{ch}^2 - 1) \). Letting \( \alpha = D_i/D_o \), equation (1.29) can be rewritten as

\[
U_o = \frac{(U_i \alpha^2)/(\alpha_{ch}^2 - 1)}{(U_i \alpha^2)/(\alpha_{ch}^2 - 1)}.
\]

Therefore for a given internal flow velocity \( U_i \), \( \alpha_{ch} \) affects not only the amount of confinement but also the magnitude of the external flow \( U_o \).

Paidoussis et al. (2008) obtained results for a system with parameters pertinent to a drill string, as well as for a bench-top sized system. In the two cases the dynamics
of the system was very similar. Low values of confinement \( \alpha_{ch} \equiv \frac{D_{ch}}{D_o} \geq 20 \), resulted in \( U_o \ll U_i \) and the system dynamics was dominated by the internal flow. Small values of \( U_i \) damped the motions of the cantilever, while for large enough internal flow velocity the system lost stability by flutter.

For a strongly confined system \( \alpha_{ch} \leq 1.2 \), the internal and external flow velocities were of the same magnitude \( U_i \approx U_o \). The reverse external flow dominated the dynamics, and strongly destabilized the system with flutter developing at very low flow velocities. The model of Paidoussis et al. (2008) formed the basis for some of the theoretical work done in this thesis, and is given some support by the experimental results contained herein.

Qian et al. (2008), investigated the dynamics of the tubular cantilever subjected to counter-current internal and external axial flow, but in this case the confined external flow was directed from the clamped to the free end, with a reverse (aspirating) internal flow. Their model was based on Paidoussis et al. (2008), and it is essentially equation (1.28) with the signs for the flow velocities reversed \( U_i \rightarrow -U_i \) and \( U_o \rightarrow -U_o \). The aspirating internal flow was taken into account in a simple manner, without incorporating any entrance effects such as those described in (1.2.2). Results were obtained for the same set of parameters as in Paidoussis et al. (2008). Two systems were investigated, a long drill-string like system, and a shorter bench-top sized system.

For the drill-string like system, results were obtained for \( 2 \leq \alpha_{ch} \leq 1.05 \). For sufficiently high internal flow velocities, a strongly confined system ( \( \alpha_{ch} \approx 1 \) ) was predicted to lose stability by divergence in the third or second-mode, depending on
confinement. However, it is this author’s opinion that in the investigation of Qian et al. (2008) the critical velocities for divergence have been underestimated, since the onset of divergence was deduced by the vanishing real part of the complex frequency of vibration \( \text{Re}(\Omega) \) (which corresponds to the regular “observable circular frequency”) thereby neglecting the imaginary part \( \text{Im}(\Omega) \), which if taken into account reveals that the corresponding modes are still heavily damped by virtue of \( \text{Im}(\Omega) \gg 0 \). Consequently if divergence does occur, which is likely, it will materialize at higher flow velocities \( U_i \) than those given in Qian et al. (2008). A vanishing real part of the frequency alone is not a sufficient criterion for the onset of static divergence. A relevant explanation of the types of instabilities and their associated Argand diagrams can be found in Paidoussis (2014), section 3.2.3.

For the bench-top sized system, for relatively low confinement (\( \alpha_{ch} = 2 \)) the system lost stability by flutter in the first and subsequent modes, the system dynamics being dominated by the internal flow. For a strongly confined system \( \alpha_{ch} = 1 \), the effect of the external flow became important, and the system lost stability by divergence, however the critical flow velocities of possible divergence were presumably underestimated due to the aforementioned reasons.
CHAPTER 2
Objectives

The purpose of the work described herein was to answer the following questions:

• How do the flow configurations under consideration compare to one another in terms of stability, and which are the most and least stable configurations? Since the flow configurations represent the different modes of operation of the salt caverns, the most stable configuration is also the most advantageous mode of operation.

• What is the qualitative behaviour of the unstable system?

• What is the effect of the length of annular confinement on the system?

• Can a linear theoretical model adequately predict the onset of instability, including the effect of confinement and annular length?

• What are the differences in system behaviour between short systems (such as those used in the experiments) and long systems (such as those used in real applications)?
CHAPTER 3
Experimental Investigation

The experimental investigation was undertaken in the Fluid-Structure Interactions Laboratory of McGill University. Experiments were performed in the Solution Mining Research Institute (SMRI) apparatus, a pressure vessel built in the previous years to simulate the brine-string and casing found in salt cavern hydrocarbon storage applications (Jamin, 2010).

In the apparatus, the brine-string was simulated using flexible rubber tubes; while the casing was simulated using rigid tubes of larger diameter placed concentrically around the flexible tube, forming an annulus. In the case of a narrow annulus, preliminary experiments had shown the tendency of the flexible tube to deflect and come in contact with the rigid outer tube, at relatively low flow velocities, effectively changing the system boundary conditions, prior to the development of any meaningful instability. As a result, the annuli were subsequently made wide enough to allow the flexible tube to become unstable before contacting outer tube.

The apparatus was operated with water as the working fluid, and allowed the motion-capture of the simulated brine string by various optical means for analysis, as well as the testing of several flow configurations.
3.1 Experimental Apparatus

3.1.1 Test-Section

The main component of the SMRI apparatus (figure 7–6) is a cylindrical stainless steel pressure vessel of approximately 0.11 m$^3$ capacity and of 0.48 m largest internal diameter. This allows for a 0.6 m tall, vertical test-section. On the sides of the main chamber four symmetrically placed, rectangular windows allow both viewing and access to the test chamber. The pressure vessel can sustain pressures of up to 205 kPa (40 psi) before significant leakage occurs. The maximum attainable flow velocities are dictated by the pressure capacity of the vessel. Two manual bleed ports in the test-section allow de-aeration of the system.

In addition to the test-section, the SMRI apparatus consists of a water holding tank which doubles as a settling tank for removing debris and air bubbles from the system.

3.1.2 Motive Power

Two 2.2 kW (3 HP) electric centrifugal pumps, provide up to 690 kPa (100 psi) of pressure each. When operating the apparatus, only one of the pumps is used, while the second one is retained as backup. The pumps draw water from the bottom of the water holding tank. The height of the water (approximately 1 m) in the holding tank serves to slightly pressurize the pump inlet, so as to ensure an uninterrupted water supply and avoid cavitation.

The pumps are controlled by dedicated digital controllers. The pump set-points are manually set during experimentation.
3.1.3 Flexible tubes: brine String Specimen

The brine string was simulated by flexible tubes of approximately 0.44 m in length. The dimensions and key properties of these tubes are listed in Table 7–1. Two different types of material where used for the construction of the tubes: (i) Silastic RTV, a castable two-part silicone mixture widely used in the research of Paidoussis et al. through the years; (ii) Santoprene™, a thermoplastic elastomer (TPE) that closely resembles rubber and is widely used in medical and food applications.

3.1.4 Annulus

A set of rigid plexiglas tubes of larger diameter than the flexible tube, with an internal diameter of $D_{ch} = 31.5$ mm (where “ch” stands for “channel”) were used to simulate the outer casing of the brine-string as seen in figure 7–7. Modifications to the test-section were made to allow for these plexiglas tubes to be secured concentrically to the flexible tube, simply by screwing them in place by hand. The inner flexible tube and outer plexiglas tube thus forming an annulus. Three annuli with nominal lengths of 10, 20, and 30 cm were used. The exact size, length, and fraction of tube-length in confinement for these annuli are given in Table 7–2. This set-up allowed for the length of the annulus to be varied between experiments, without removing the flexible inner tube. The rigid tubes were transparent, allowing observation of the portion of the tube inside the annulus.

3.2 Data Acquisition

3.2.1 Flow-rate measurement

The average flow velocity was measured with a magnetic flow-meter with an integrated display. The volumetric flow-rate was measured in litres per second (Lt/s),
with a resolution of 0.001 Lt/s, according to the flow-meter specifications. The flow-meter was connected in-line, with either the inlet or outlet of the apparatus, depending on the flow configuration. In all experimental configurations, there was only one inlet, one outlet, and no accumulation in the system. The flow velocity in both the annulus and the flexible tube could be determined from the single flow-rate measurement provided by the flow-meter. Selection of a given flow-rate was achieved by setting the pump controls at a certain level, observing the volumetric flow-rate displayed on the flow-meter, and making adjustments as needed.
3.2.2 Motion capture

Motion capture was the main form of data acquisition of the system. Measurements were made through the transparent plexiglas windows of the test-section. Two types of motion capture system were used for experiments: an Optron 5600, analog bi-axial follower, single-point vibrometer; and a pair of Unibrain, Fire-i, firewire (IEEE 1394) digital cameras.

**Optron 5600 Follower vibrometer.** The Optron 5600 (figure 7–8) is an analog system capable of capturing both small and large amplitude motion in a plane perpendicular to the axis of the device (very much like a camera). It outputs a voltage signal proportional to the displacement of a certain tracked point on the moving target. The Optron system was the main motion capture system used in our laboratory, and has proven very reliable over a number of years. Unfortunately, upon undertaking our experimental investigation, a calibration showed that the Optron was no longer capable of properly tracking, and repairs were in order. Due to the age and scarcity of the system, these costly repairs could only be performed in Milford CT, by a specialized company. An unfortunate shipping accident (the device was dropped in transit) further complicated the repair process.

**Dual camera system.** The temporary loss of the Optron system, necessitated the use of a cost-effective alternative. It was decided to make use of a pair of Unibrain, Fire-i, fire-wire (IEEE 1394), colour digital cameras. The cameras were set at a right angle to each other, so that the motions of the system could be captured in three dimensions 7–9. The cameras were capable of a maximum frame-rate of 30 frames per second (fps). Although the cameras did not have a built-in synchronization system,
they were externally synchronized by a pair of light emitting diodes (LED), visible by both cameras at the same time. This form of relatively crude synchronization allowed for a maximum asynchronism of 16.5 ms between the two video feeds. This was deemed acceptable for the purposes of this investigation. A Matlab script was written for extracting the displacements of the tube specimen from the video feeds. To facilitate this process, bright red bands were painted on the specimens with a mixture of general-purpose silicone and powder pigment. These would serve as targets for the image analysis.

The dual camera system had some advantages compared to the Optron system: it gave us the ability to track multiple points on the tube specimen; and it allowed the capture of the three-dimensional nature of the motion of the specimen. The tracking of multiple points enabled us to use some more advanced data-analysis techniques to gain insight into the system behaviour, while the three-dimensional motion capture gave a much better representation of system motion.

Unfortunately, the dual camera system also had several drawbacks compared to the Optron. Of these, the most significant were: the relatively low frame-rate of the camera, which was limited to a maximum of 30 fps; also a low resolution of 480 × 640 pixels, along with relatively weak optics. Luckily, for all configurations the system exhibited behaviour with frequencies well below the 15 Hz Nyquist limit imposed by our frame-rate. In configuration (ii) (tube aspirating and no flow in the annulus), the low resolution became a problem, as the small motion of the system could not be properly resolved. Additional magnification of the target was hindered by a combination of the size of the pressure vessel and distortion of the image. Because
of these issues, an additional Canon consumer camera with better optics was used to obtain more data for configuration (ii).

**Canon Powershot camera.** For configuration (ii) (aspirating string and no flow in the annulus) the vibratory motion was too small to be well-resolved by the dual camera system. Hence, the vibration was also captured by a Canon PowerShot SX260 consumer camera. This gave a better resolution of the motion, but without the benefit of the third dimension. The tube motion was extracted with Matlab, in the same manner as for the dual camera system.

### 3.2.3 Pressure Measurement

The mean pressure in the test-section was measured with a conventional Bourdon tube gauge, installed on the bleed line of the test-section, right before the manual bleed valve (figure 7–6(b)). This ensured that any air bubbles could be removed and a reliable pressure reading obtained. This rough pressure reading was used to monitor and ensure the integrity of the pressure vessel.

### 3.3 Experimental Protocol

In a very condensed manner, the experimental procedure was as follows.

1. The tube specimen and appropriate annulus were installed in the test-section and made vertical with the help of a spirit bubble-level and a plumb-line.

2. The outlet or inlet (depending on the configuration) of the system was connected to the Magnetic flow-meter and the flow-meter was set to display Lt/s.

3. The SMRI apparatus was filled with water, pressurized and run for a length of time while bleeding all air from the system.
4. The dual camera system was levelled, trained at the appropriate height, and the cameras made square to the windows of the test-section. The light levels were adjusted by trial and error. The camera settings were configured to 30 frames/second, with a 5 ms shutter to minimize motion blur.

5. The system was turned on, and approximately 210 seconds of the tube motion were captured at each flow velocity step. The footage also included an external trigger to aid in the video synchronization. The flow velocity was increased in steps from zero to some maximum value and subsequently reduced, retracing the steps down to zero. The rest (equilibrium) position of the specimen was recorded at the end of every experiment.

6. For an aspirating pipe with no external flow velocity (configuration (ii)) the specimen displacement was also recorded in a single direction with a Canon consumer camera, with a higher magnification and at 120 fps.

3.4 Data Analysis

3.4.1 Displacement Data Extraction

The use of cameras in place of a more traditional motion follower system introduced additional steps in the analysis, since it was necessary to first extract displacement data from the videos. The video feeds from the Fire-i cameras were recorded directly to a desktop computer through a fire wire (IEEE 1394) port in raw video format. Due to storage and processing limitations the videos were compressed in WMV3 format before any further processing. Although this practice is in general to be avoided, due the introduction of compression artefacts, the sheer size of the video files (10 GB each with approximately 1,330 recorded) made compression a necessity.
In the absence of an internal synchronization mechanism, a pair of light-emitting diodes (LED) wired on the same circuit were placed in such a way as to be recorded in the video feed of both cameras. Turning off these LEDs immediately after the start of every recording allowed for external synchronization of the two video feeds to within 16.5 ms.

The synchronized videos were loaded into Matlab for further processing. A two-dimensional, Gaussian, weighted averaging filter was employed to smooth out each video frame. The individual video frames were then scanned-through to identify and calculate the locations of the markers painted on the tubes, seen in figure 7–10(b). The centroid of the area of the markers was used as a representative point of their location. The pixel locations where related to displacements by using the known width of the tube as a reference. The displacement time series were then stored and processed with a separate Matlab script.

3.4.2 Displacement Data Analysis

For each internal flow velocity, we obtained a total of six time series, corresponding to the horizontal displacements of three points (the marker locations) on the tube surface, recorded from two perpendicular directions. The directions of observation were designated as “Front” and “Side”, and the corresponding tube displacements as “Front Displacement” and “Side Displacement”. The tracked points were designated “point 1”, “point 2”, and “point 3”, counting up from the marker closest to the tube free end. As shown in figure 7–10, point 1 was located 11 mm from the free end, with subsequent points spaced roughly 45 mm apart.
The mean displacements were subtracted from the displacement time series to separate the static and dynamic response of the tube. The rest configuration, i.e. without flow, after each experimental run was used as a datum for the static displacement. In order to reconstruct the real tube displacement and compensate for the limited resolution and frame-rate, the time series were smoothed using either a polynomial spline, or a low-pass finite impulse response (FIR) filter.

Since the displacements were largest at the downstream end, point 1 was used as representative of the system displacement. The root-mean-square (rms) displacement of point 1 gave information on the dynamic system behaviour. When plotted against flow velocity, a sudden increase of the rms amplitude indicated the onset of oscillatory instability. An example is given in figure 7–13(a).

With available displacement data from two perpendicular directions, the total horizontal movement of point 1 could be plotted with time. This was the tube trajectory—an example given in figure 7–19(d)— and provided insight of the qualitative aspects of the motion.

From the tube trajectory, it was seen that in many cases the vibration was symmetric about a vertical plane. The orientation of this plane, i.e. the azimuthal angle, would change with time. The plane of symmetry was denoted as the “principal plane”. The orientation of the principal plane was calculated from successive maxima of the tube motion, and provided insight as to whether the tube vibration was regular or random. With a known orientation, the system displacement could be projected onto the principal plane, giving the “principal transverse displacement”. An example is shown in figure 7–19(a). In certain cases, vibration that showed
amplitude modulation when observed from either the front or side directions, was more regular when projected onto the principal plane; indicating that at least some modulation was due to the changing orientation of the vibration.

The frequencies of vibration were obtained by computing the power spectral density (PSD) for each tracked point, from each direction of observation, and then averaging the results. The PSDs were calculated both by direct fast Fourier transform (FFT), as well as Welch’s method with eight windows. From the PSD, the frequency of vibration was identified, as the largest (or dominant) peak. An example is shown in figure 7–19(b).

The front and side displacements were plotted against the corresponding tube velocities to create phase plots (phase portraits). These provided insight on the qualitative aspects of the vibration, as an elliptic phase portrait was indicative of limit-cycle oscillation, i.e. regular oscillation with a constant amplitude. Examples of a phase portraits are given in figures 7–19(e) and 7–19(f).
3.5 Experimental Results

3.5.1 Configuration (i) : Pipe discharging in the tank; no flow in the annulus

**Effect of increasing the flow velocity.** For this configuration, starting from zero and with increasing flow velocity, we observed the emergence of second-mode flutter with a well defined frequency. A summary of the results is given in Table 7–3.

The onset of instability was indicated by a noticeable jump in the root-mean-square (rms) amplitude of vibration. This point also coincided well with the emergence of a single sharp peak in the power spectrum of the motion, shown in Figure 7–16(b)). The power spectral density was obtained by averaging the PSDs of the time histories from each point of measurement, as described in section 3.4.2.

To estimate the critical velocity we plotted the response rms amplitude versus the internal flow velocity. The system response was then separated into three regions: the pre-instability region ; the post-instability region; and (if impacting was observed) the impacting region. The critical flow velocity for second mode flutter $U_{f2}$ was estimated by fitting the post-instability region with a line and calculating the intercept on the velocity axis. For an illustration of this method see Figures 7–13 and 7–14.

Before the onset of instability the system behaviour was disorganised, with small amplitude three-dimensional motions (figure 7–18(d) for Tube 1, and 7–15(d) for Tube 2), of no particular orientation (figures 7–18(a) for tube 1, and 7–15(a) for tube 2), and without a dominant frequency but rather an overall increasing level of noise in the system (figure 7–18(b) for tube 1, and 7–15(b) for tube 2).
Both Tube 1 (Silastic) and Tube 2 (Santoprene) exhibited very similar behaviour, with a sharp onset of instability at a well-defined frequency (figure 7–19(b) for tube 1 and 7–16(b) for tube 2).

Once the system became unstable, the motion occurred around a plane of symmetry, the principal plane, with a slowly varying azimuthal angle (figure 7–19(a) for tube 1, and 7–16(a) for tube 2). The vibration of both tubes resembled a “tight” ellipse, one whose major axis was much larger than the minor axis. The principal plane was aligned with the major axis. As evidenced by the system trajectories (shown in figure 7–19(d) for tube 1, and figure 7–16(d) for tube 2) the motions of tube 1 are noticeably flatter than those of tube 2. We could argue that the vibration of tube 1 is essentially planar, whereas that of tube 2 is quasi-planar.

Phase portraits of the system undergoing second-mode flutter (figures 7–19(e) and 7–19(f) for tube 1; figures 7–16(e) and 7–16(f) for tube 2), show an elliptic envelope, implying that the underlying motion is regular and simple periodic. To further illustrate this, and for the cases where the rotation of the plane of symmetry is “nice enough”, the phase diagram of the motion projected onto the plane of symmetry shows a clearly delineated ellipse (figure 7–21). This indicates a limit-cycle oscillation that is regular and periodic.

**Effect of impacting.** The effects of impacting with the rigid outer tube were clearly noticeable in the frequency spectrum, with the appearance of multiple frequency peaks and a significant increase of the amount of noise in the system.

Once impacting began, the behaviour of tube 2 and tube 1 was markedly different. Tube 2 tended to “stick” and move tangentially along the inner surface of the
annulus. The trajectory of the tube (Figure 7–17(d)) reveals a motion tangent to a circular path. As a result, the system vibration has retained a plane of symmetry with a slowly varying orientation. It is not simple harmonic, but it is regular.

Tube 1, on the other hand, did not “stick” to the annulus, but continued to impact perpendicularly to the inner annular surface. This resulted in a more chaotic motion in a randomly varying plane of vibration. Figures 7–19(a) and 7–20(a) illustrate this progression from the well-oriented planar motion of second mode flutter, to the disorientation and randomness of the post impacting behaviour.

**Effect of the annular length on the evolution of instability.** For both tubes, the increase of the annular length had but a small effect on the measured critical flow velocities for flutter. The effect on the frequency of vibration, however, was more significant. There was a net decrease of the critical frequency for flutter with increasing annular length, by approximately 0.25 Hz for both tubes (Table 7–3), reflecting the effect of added mass. This amounts to a total change of 15% and 9% for tube 1 and tube 2, respectively.

**Effect of the annular length on the progress of instability.** In the absence of an annulus or in the case of the short 10 cm annulus, the amplitude of vibration increased exponentially until the motion was fully three-dimensional (orbital) and sufficiently large to strike the walls and windows of the test-section.

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1 These two types of motion are well known in the dynamics of heat exchanger tubes impacting on the baffle plates. See, e.g., Mureithi et al. (1994a,b).
When the 20 and 30 cm annuli were used, the amplitude of the motion increased until impacting with the internal surface of the annulus occurred. In this case the amplitude of the motion was bound by the presence of the annulus, and thus limited to between 15 and 10 mm, respectively (see figures 7–13(c) and 7–13(d)).

**Effect of annular confinement (annular gap).** Due to time constraints, the effect of varying degrees of confinement on the same tube were not investigated separately. However, due to the similarity of the tube specimens, some conclusions on the effect of confinement can be drawn. Nevertheless, the observed effects are invariably compounded with the approximately 21% (Table 7–1) difference in the flexural rigidity of the tubes.

The investigated tube specimens had the same inner diameter \(D_i = 6.35\) mm but different outer diameters \(D_o\). Tube 1 had an annular gap of 7.75 mm with a ratio of annular to inner flow areas \(A_{ch}/A_f\) of 18.26. Tube 2 was less confined, with an annular gap of 11 mm and an \(A_{ch}/A_f\) of 22.35. It is noted that the annular confinement was similar in these two cases. Nevertheless, we observed a correlation of confinement and the system response: the greater the annular length and the smaller the annular gap, the larger was the departure from the critical values for an unconfined tube. As was previously noted, the overall dependence in critical values

\[2\] It is recognized that these area ratios are far in excess of the desired value of \(~ 1\). However, experiments with smaller \(A_{ch}/A_f\) “failed”, in the sense that, after the first large-amplitude excursion, the tube would “stick” to the annulus for the duration of the experiment. In real applications the brine string is very much longer than the annulus, so that, after sticking, there is ample length of the tube free to undergo vibrations and other dynamics.
with annulus length was slight; however, the more confined tube 1 experienced a larger percent change in the critical frequency of vibration (15% versus 9% for tube 2).

A more significant effect of confinement was that the greater confinement of tube 1 promoted an earlier onset of impacting once second-mode flutter had began. This, of course, greatly affected the post-instability dynamics of the system by limiting the amplitude of vibration for tube 1 earlier than for tube 2, as well as leading to earlier nonlinear behaviour due to impacting.

3.5.2 Configuration (ii) : Pipe aspirating fluid; no flow in the annulus

Effect of increasing flow velocity. In this configuration, both tubes exhibited extremely small amplitude vibrations, on the order of 1 mm at the maximum achievable internal flow velocities of 7.58 m/s for tube 1 and 8.4 m/s for tube 2. This amplitude is comparable with what has been classified as pre-instability behaviour for configurations (i) and (iii), and reminiscent of the previously experimentally observed behaviour of an aspirating tube (see section 1.2.2 here, also Rinaldi (2009); Kuiper and Metrikine (2008)) of exhibiting a “feeble” form of flutter of very low amplitude. The internal flow velocities were limited by the pressurization limit of the test-section. Figure 7–23 shows some representative rms vibration amplitudes with increasing flow velocity.

The extremely small rms amplitude of vibration (88% to 98% smaller than for Configuration (i)) presented a problem for our dual camera data acquisition system, which had an average resolution of 0.6 mm for this experimental set-up. It was able to only just discern the vibration (figure 7–24(c) for example). An additional Canon
consumer camera with a 0.2 mm resolution was used, giving more detailed results; unfortunately, these results did not contain significant new information.

These results suggest the following two possibilities:

- the rms amplitude variation demonstrates an instability with a very gradual onset, namely one that would continue to grow in the absence of flow velocity limitations;
- the observed amplitude is the full extent of the vibration; the system does not become severely unstable for pipes of similar shape and properties as those tested.

To facilitate discussion of the above two possibilities, we present some representative time-dependent results for tube 1 in figure 7–23(a), and for tube 2 in figure 7–23(b).

If the response of the system is an instability that will continue to grow, then we have to conclude that the critical internal flow velocity for instability is greater than the velocities achieved. Tube 1 would become “more unstable” at $U_i > 7.58$ m/s, and tube 2 would become more unstable at $U_i > 8.40$ m/s. These values are greater than the critical velocities observed for all other configurations. As a result, in this scenario, configuration (ii) would have to be judged as the most stable configuration.

Alternatively, if the observations represent the full extent of the instability, the amplitude of vibration is much smaller than all other configurations. From figures 7–23(a) and 7–23(b), the critical velocities are approximately 4 m/s for both
tubes. Hence, the onset of vibration for configuration (ii) is earlier than for configurations (i) and (iii); however the extremely small amplitude of vibration makes this configuration weakly unstable.

A look at the calculated frequencies of vibration given in Table 7–4 reveals that on average, the frequency is 0.37 Hz for tube 1, and 0.41 Hz for tube 2. The system vibration is in the first mode.

**Static deflection.** Despite the small dynamic component of the motion, there was observed a large static deflection. For tube 1 and tube 2 this deflection was comparable, approximately 10 mm (figure 7–26). Increasing flow velocity resulted in greater deflection, with an amount of bowing persisting even after the internal flow was stopped. This curvature relaxed after a considerable amount of time.

The cause of this deflection could be either a manifestation of underlying permanent bow in the tubes, or, despite our best efforts, an asymmetric inlet flow field caused by imperfections at the free end of the tube.

**Effect of confinement and annulus length.** Table 7–4 reveals that there is no discernible effect of the annular length on the frequency of vibration.

The origin of the static deflection of the tube has not been determined; however, in the absence of external flow there is no reasonable correlation between a static instability and confinement. This is because, in the absence of annular flow, the confinement does not affect the stiffness of the system. As a result, any variation of the static deflection cannot be attributed to the varying annulus length.

Configuration (ii) is identical to configuration (i) in this respect. A look at the theoretical treatment of external fluid forces for configuration (i), sections 4.4
and 4.10.1, shows that the confinement parameter ($\chi$) and annular length ratio ($r_{ann}$) do not appear in the stiffness terms of the system (equation (4.64)).

3.5.3 Configuration (iii) : Pipe discharging in the tank; the annulus aspirating fluid

Effect of increasing flow velocity. With increasing flow velocity, we observed the emergence of second-mode flutter. The response of the system may be separated into three regions: the pre-instability region; the post-instability region; and (where applicable) the impacting region. The critical flow velocities for second-mode flutter were estimated in the same manner as for configuration (i). The process is illustrated in figures 7–27 and 7–28. The results are summarized in table 7–5, along with the corresponding critical frequencies. The same data is given graphically in figure 7–29.

Effect of the annular length on the onset of instability. The increase of the annulus length had a slight effect on the quantitative aspects and a large effect on the qualitative aspects of the instability. The annulus length corresponds to the length of the cantilever subjected to reverse (towards the clamped end) annular flow. Since in our experiments the tubes were not severely confined, the external flow velocity is only 5% of the internal flow velocity on average.

Successively longer annuli altered the critical flow velocities for flutter, but without a net stabilizing or destabilizing effect (figure 7–29(a)). The maximum variation in the critical flow velocity in comparison to the unconfined tube, was 9% for tube 1, and 3% for tube 2. The effect of the annulus length was much more pronounced on the critical frequency of vibration (figure 7–29(b)). Increasing the annular length resulted in a decrease in the critical frequency. The 30 cm annulus gave a net frequency
reduction of 31% and 26% for tube 1 and tube 2, respectively. For both tubes, the 10 and 20 cm annuli gave an average decrease of 5% and 10%, respectively.

The 10 cm annulus did not modify the qualitative characteristics of the motion significantly. For both tube 1 (figure 7–30), and tube 2 (figure 7–31), we observed the emergence of second-mode flutter, with a well-defined and sharp frequency. The motion was planar or quasi-planar with a slow precession in the plane of symmetry.

On the other hand, use of the 20 and 30 cm annuli heavily influenced the motion. The system lost stability in the second-mode. The frequencies of vibration were well-defined, but with broader peaks in the power spectrum. In addition, the motion became increasingly non-planar (depending on the annulus) and the orientation of the principal plane became increasingly random. To illustrate these observations, we present results with the 20 cm annulus for tube 1 (figure 7–32) and tube 2 (figure 7–33).

For tube 1, one can see the “broadening” of the frequency peaks between figures 7–30(b) and 7–32(b). The reduction in flatness and increasing three-dimensionality of the motion is illustrated in figures 7–30(d) and 7–32(d). The orientation of the plane of symmetry is slowly varying in figure 7–30(a), and randomly changing in figure 7–32(a). For tube 2, the same effects can be observed between figures 7–31 and 7–33.

Effect of confinement on the onset of instability. Similarly to configuration (i), see section 3.5.1, the effect of confinement is invariably compounded with the difference in flexural rigidity of the two tubes. The confinement of tubes 1 and
2 is identical to that of configuration (i); the associated parameters are given in table 7–2. For the larger confinement, a change in annular length has a greater effect on the system, a result of both the increased added mass, as well as the increased annular flow velocity. This is consistent with the more confined tube 1 exhibiting a 31% reduction in critical frequency when compared to the unconfined case. There is a 26% reduction in frequency for the less confined tube 2. As mentioned in the previous paragraph for the level of confinement in these experiments, there was no net change in the critical velocity for flutter. Therefore, no conclusions could be drawn regarding the effect of confinement on the critical flow velocity.

**Effect of confinement on the evolution of instability.** The effects of the annulus on the qualitative aspects of the vibration are described in section 3.5.3. In addition, and similar to configuration (i) (section 3.5.1), once the system became unstable the most prominent effect of confinement was to influence the onset of impacting between the tube and the inner annular surface. A more confined tube tends to impact earlier, because of the smaller annular gap.

**Effect of impacting.** If the flow velocity is increased enough, then impacting between the tube and inner annular surface begins. This results in the introduction of noise, and multiple peaks in the system frequency response. These are accompanied by the loss of a coherent plane of symmetry for the vibration, and by an increase in the three-dimensionality and randomness of the motion. Due to the similarity between configurations (i) and (iii), we refer the reader to figures 7–17 and 7–20 for a representative example of the effects of impacting. Further illustrations are omitted, as they provide no additional insight.
Comparison of configuration (iii) and configuration (i). The critical flow velocities and amplitude response of configuration (iii) (table 7–5) were found to be very similar to those of configuration (i) (table 7–3). For tube 1 the critical velocities for flutter differed by as much as 8% with no net stabilizing or destabilizing effect. For tube 2, the external annular flow resulted in a slight net destabilization, with critical velocities for configuration (iii) up to 2.3% below those of configuration (i).

On the other hand, the effect of the external flow on the critical frequencies for flutter was much more pronounced. Configuration (iii) exhibited lower frequencies than configuration (i), up to 19.5% for tube 1 and 19% for tube 2. This difference in frequency was dependent on the annulus length, with a 1% difference for the 10 cm annulus, and the maximum difference for the 30 cm annulus.

Despite strong similarities between the amplitude response of the system in configurations (iii) and (i), some qualitative aspects of the motion were significantly different. The differences were observed with the use of the long 20 and 30 cm annuli; the most important being the non-planar and randomly precessing vibration for configuration (iii), at the onset of second-mode flutter, shown in figures 7–32 and 7–33 for tube 1 and tube 2, respectively. This was in stark contrast with the planar (or quasi-planar) and slowly precessing vibration observed in configuration (i), regardless of the annulus length. An example for configuration (i) with the 30 cm annulus is shown in figures 7–19 and 7–16 for tube 1 and tube 2, respectively.
3.5.4 Configuration (iv) : Pipe aspirating fluid; the annulus discharging in the tank

**Effect of increasing flow velocity.** The annular flow caused the system to behave in a manner very different from that of the aspirating tube with no external flow. For both tube 1 and tube 2 the system lost stability by flutter in the first mode, at internal flow velocities $U_i$ less than half of the critical flow velocities for all other configurations.

The very early onset of instability made the estimation of critical flow velocities very difficult. The lowest experimental flow velocities used were selected by judging the system stability visually. Analysis of the results, however, showed that the system was already unstable at the lowest flow velocities. Therefore, in lieu of one single estimate, we provide a range in which the critical flow velocity lies in each case. The results are summarized in Table 7–6, along with the first-mode critical frequencies of the system. The amplitude versus internal flow velocity response is shown in figure 7–34 for tube 1 and figure 7–35 for tube 2.

The internal and external flow velocities are related by: $U_o = 0.055U_i$ for tube 1, and $U_o = 0.045U_i$ for tube 2. The resulting critical internal flow velocities and frequencies are summarized in table 7–6. The external flow velocity is much smaller than the internal flow velocity; yet it has a strongly destabilizing effect on the system. This can be seen in comparison to configuration (ii) (table 7–4, section 3.5.2), where the tube is aspirating but with no discharging flow in the annulus.

For tube 1, further increase of the flow velocity led to the emergence of first- and second-mode flutter together. This was evident in the power spectral density of the signal (figure 7–37(b)), with the largest peak corresponding to a first-mode
frequency of 0.51 Hz, and a much less prominent peak at 2.78 Hz. The existence of first-and-second-mode flutter was noticeable—as a small amplitude and higher frequency vibration superimposed on the large amplitude vibration of the tube—in the videos of the experiment recorded by the dual-camera system. From the footage, it was determined that the higher frequency component of the vibration had a node at approximately one-fifth of the tube length ($L_{\text{tube}}/5$) above the free end. This corresponded to the approximate location of the node for second-mode flutter observed in configurations (i) and (iii), as was obtained from the corresponding videos. This observation, along with the magnitude of the frequency, led to the conclusion that the less prominent peak ($f_2$) in figure 7–37(b) corresponded to the second-mode.

For Tube 2, only first-mode flutter was observed with an average frequency of 0.72 Hz. From the power spectral density (figure 7–38(b)), the emergence of an additional higher frequency component was evident; however the amplitude of this component was very small and not discernible in the recorded videos of the experiment. This additional component had an approximate frequency of 5.2 Hz. The corresponding mode-shape remains unknown, with the possibility of it being either the second or third mode.

In all cases, the system vibration was fully three-dimensional, with no well-defined plane of symmetry. This is evident in the orientation of the principal plane (figures 7–36(a) and 7–37(a) for tube 1, figure 7–38(a) for tube 2) and trajectories of the system (figures 7–36(d) and 7–37(d) for tube 1, and figure 7–38(d) for tube 2).
The phase portraits were accordingly very heavily distorted (7–36(e) and 7–36(f) for tube 1; 7–37(e) and 7–37(f) for tube 2), with no semblance to a limit-cycle oscillation.

**Effect of the annular length on the onset of instability.** Since we were unable to obtain exact values for the critical flow velocities, we are unable to quantify the effect of the annulus. Information of the variation of frequency with annular length is given in Table 7–6, and graphically in Figure 7–39. From these, we can see a slight increase of the critical frequency of vibration for tube 2, and no net effect on the frequency for tube 1.

**Effect of impacting.** For both Tube 1 and Tube 2, and similarly to configurations (i) and (iii), an even further increase in flow velocity caused impacting with the internal surface of the 20 cm and 30 cm annuli. Labelled as “Impacting” in Figures 7–34 and 7–35, we see the limiting of the amplitude of vibration due to the physical barrier of the annulus. The amplitude and frequencies of the system were disrupted in a very similar manner as for configurations (i) and (iii), with a large increase in system noise and multiple peaks in the frequency spectrum. Figures for these results are not shown, as they provide no additional insight.
3.6 Conclusions drawn from the experimental results

The following conclusions stem from our experimental analysis.

In three out the four configurations investigated, configurations (i), (ii), and (iv), the system lost stability by flutter past a certain flow-velocity. Configuration (ii) has shown vibration with very low amplitude; therefore, it remained undetermined whether the system will progress to a larger amplitude vibration or instability, or stay at that level of amplitude, if the flow velocity could be increased sufficiently.

In terms of structural stability, the relative stability of the flow configurations determines which configuration is most advantageous for system operation. For the given application, system operation in a non-optimal configuration can often be necessary due to other considerations, it is then useful to know which is one is the next optimal flow configuration.

Quantitatively, configuration (ii) appeared to be the most stable as it had the largest critical internal flow velocity $U_i$, or if unstable the smallest amplitude of vibration. This was followed by the less stable configurations (i) and (iii), with configuration (iii) being slightly more unstable than (i) only in the case of tube 2. Configuration (iv) was the least stable, surprisingly perhaps, with critical internal flow velocities less than half of those of configurations (i) and (iii).

The amplitude of the flow induced vibrations is relevant in assessing the potential damage that an instability in a particular configuration may cause. For example, small amplitude motions occurring at low flow velocities are not likely to cause a catastrophic failure. However, there would be cause for concern regarding fretting wear or fatigue failure at the supports.
In terms of the rms amplitudes of dynamic instability, configuration (ii) exhibited very small vibration. Configurations (i) and (iii) showed exponential growth past a certain critical flow velocity, with the potential for very violent, large amplitude oscillations. This effect was curbed by the longer 20 cm and 30 cm annuli, as the surrounding rigid pipe sufficiently restrained the tube. Configuration (iv) exhibited a more gradual increase of the amplitude of vibration. Hence, as the flow velocity increased, the vibration reached an amplitude which was smaller but still comparable to that of configurations (i) and (iii). Configuration (iv) did not have a tendency for violent oscillations.

In configurations (i) and (iii), the tube lost stability by second-mode flutter, while configuration (iv) exhibited first-mode flutter for both tubes, as well as concurrent first-and-second-mode flutter when tube 1 was used. The extent of instability for Configuration (ii) remains undetermined; however the system vibrated in the first mode.

Configurations (i) and (iii) showed planar and quasi-planar limit-cycle oscillations in the second mode, with well-defined and sharp frequencies of vibration. For configuration (iii), the use of the longer 20 and 30 cm annuli led to three-dimensional motions of random orientation. In configuration (iv) the flutter occurred in a random, fully three-dimensional manner, with less well-defined frequencies; the same occurred in configuration (ii), if that was indeed flutter.

Comparing configurations (i) and (iii), the annular flow affected the quantitative aspects of the system dynamics only slightly. There was a small net destabilization, observed only in the case of tube 2. A stronger effect of the annular flow was
observed on the critical frequencies for flutter, with configuration (iii) having lower critical frequencies.

Between configurations (ii) and (iv) the effects of the external annular flow were substantial, as the system was strongly destabilized in the latter case. The importance of the annular flow is further illustrated when one considers that the external flow velocities were only about 5% of the internal flow velocities.

Beyond the onset of instability, all the unstable configurations (i,iii, and iv) were subject to impacting between the tubes and the inner surface of the annulus. The effect of impacting was twofold: (a) it introduced disorder and chaos in the system dynamics, with the appearance of multiple frequencies and loss of any regularity in the motion; (b) for sufficiently long annuli, it resulted in a limitation of the motions of the system to a certain amplitude; thus preventing a further growth of the oscillation.
CHAPTER 4
Theoretical Investigation

Linear theoretical models for configurations (i) and (iii) were derived. No theoretical models were formulated for configurations (ii) and (iv); given the inconclusive experimental results for configuration (ii), and the inexact determination of the critical flow velocities for configuration (iv), which provided a poor basis of comparison for any such models.

An important consideration was that the length of the annulus was shorter than that of the tube. The forces related to the flow velocity and the inertia of the fluid surrounding the tube differed between the confined and unconfined parts. For the sake of simplicity, the effects of the annulus were considered local.

In configuration (iii), the acceleration of the fluid between the confined and unconfined sections was ignored. It was assumed that the surrounding fluid gains a non-zero flow velocity immediately upon entering the annulus.

The derivation of the theoretical models is given in what follows.
4.1 Derivation of Linear Theoretical Models

The derivation of the system dynamics closely follows the work done in Paidoussis et al. (2008). The equations of motion of the system were formulated for each configuration by a Newtonian approach, a balance of both structural and fluid-dynamical forces as shown in figure 7–11. Motions are assumed to be planar (no 3-D or whirling).

Summing the structural and hydrodynamic forces acting on a tubular element of length $\delta x$ (figure 7–11(b)), one obtains for the $x$ and $z$ directions,

$x$-direction:

$$\frac{\partial T + Q}{\partial x} - \frac{\partial}{\partial x} \left( Q \frac{\partial w}{\partial x} \right) + \frac{M_t g - (F_{in} + F_{en})}{\partial x} \frac{\partial w}{\partial x} + F_{it} - F_{et} = 0 ; \quad (4.1)$$

$z$-direction:

$$\frac{\partial T + Q}{\partial x} - \frac{\partial}{\partial x} \left( T \frac{\partial w}{\partial x} \right) + \frac{M_t \partial^2 w}{\partial t^2} + F_{in} + F_{en} + (F_{in} - F_{et}) \frac{\partial w}{\partial x} = 0. \quad (4.2)$$

In equations (4.1) and (4.2), $w$ is the $z$ displacement of the tube, $T$ the tension in the tube, $Q$ is the shear force, $g$ the gravitational acceleration, $M_t$ the mass per unit length of the tube, $F_{in}$ and $F_{it}$ respectively the normal and tangential hydrodynamic forces due to the internal flow, and $F_{en}$ and $F_{et}$ the normal and tangential forces due to the external flow. These equations are valid for both configurations (i) and (iii), i.e. discharging tube, up to the first order.

These forces are considered separately in the following subsections.
4.2 Structural Forces (Euler-Bernoulli beam theory)

To calculate the structural forces, we make use of Euler-Bernoulli (E-B) beam theory. The tube is quite slender and of uniform cross-section; hence, this is a reasonable approximation. For a tube immersed in water, the energy dissipation due to the surrounding fluid is far greater than the internal dissipation in the tube material; consequently, as suggested in (Paidoussis et al., 2008), neglect of the internal dissipation is a reasonable approximation.

The shear force in an E-B beam is given by

\[ Q = -\frac{\partial}{\partial x} \left( EI \frac{\partial^2 w}{\partial x^2} \right), \]  

where \( EI \) [N \cdot m^2] is the flexural rigidity of the beam.

Substitution of equation (4.3) in (4.1) and (4.2), and retention of first-order terms yields

\[ \text{x-direction:} \quad \frac{\partial T}{\partial x} + M_t g + (F_{it} - F_{in} \frac{\partial w}{\partial x}) - F_{en} \frac{\partial w}{\partial x} - F_{et} = 0; \]  

\[ \text{z-direction:} \quad EI \frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x} \left( T \frac{\partial w}{\partial x} \right) + M_t \frac{\partial^2 w}{\partial t^2} - (F_{in} + F_{it} \frac{\partial w}{\partial x}) - F_{en} + F_{et} \frac{\partial w}{\partial x} = 0. \]

4.3 Hydrodynamic Forces due to Internal Flow \((F_{in}, F_{it})\)

The forces due to the internal flow are unaffected by the external flow and hence are the same for configurations (i) and (iii). We begin by considering the rate
of change of internal fluid momentum given by Paidoussis (2004)

\[
\frac{d\mathcal{M}}{dt} = 0 \hat{i} + M_f \left( \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \right) \left( \frac{\partial w}{\partial t} + U_i \frac{\partial w}{\partial x} \right) \hat{k}, \tag{4.6}
\]

where \( \mathcal{M} \) is the fluid momentum due to internal flow, and \( \hat{i} \) and \( \hat{k} \) are the unit vectors in the \( x \) and \( z \) directions, respectively. \( M_f \) is the mass of the fluid per unit length associated with the internal flow. For our system, this is \( M_f = \rho_f A_f \), where \( \rho_f \) is the fluid density, and \( A_f \) the uniform cross-sectional area of the internal flow.

A force balance on an internal fluid element of length \( \delta x \) (figure 7–11(d)) yields in the \( x \)-direction:

\[
F_{it} - F_{in} \frac{\partial w}{\partial x} = M_f g - \frac{\partial}{\partial x} \left( A_f p_i \right) ; \tag{4.7}
\]

in the \( z \)-direction:

\[
- \left( F_{in} + F_{it} \frac{\partial w}{\partial x} \right) = M_f \left( \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \right)^2 w + \frac{\partial}{\partial x} \left( A_f p_i \frac{\partial w}{\partial x} \right) ; \tag{4.8}
\]

where \( p_i \) is the internal pressure in the tube, and \( \left( \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \right)^2 \) denotes repeated application of the operator in parentheses.

Substitution of equations (4.4) and (4.5), in equations (4.1) and (4.2), gives the following:

\( x \)-direction:

\[
\frac{\partial T}{\partial x} + M_t g + \left[ M_f g - \frac{\partial}{\partial x} \left( A_f p_i \right) \right] - F_{en} \frac{\partial w}{\partial x} - F_{et} = 0 ; \tag{4.9}
\]
z-direction:

\[ EI \frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x} \left( T \frac{\partial w}{\partial x} \right) + Mt \frac{\partial^2 w}{\partial t^2} \]

\[ - \left[ M_f \left( \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x} \right)^2 w + \frac{\partial}{\partial x} \left( A_f p_i \frac{\partial w}{\partial x} \right) \right] - F_{en} + F_{et} \frac{\partial w}{\partial x} = 0 . \]  

(4.10)

The effects of the internal flow are accounted for in equations (4.9) and (4.10). We now proceed to analyse the forces related to the external flow.

4.4 Hydrodynamic Forces due to External Flow \((F_{en}, F_{et})\)

The forces \(F_{en}\) and \(F_{et}\) are comprised of: \(F_A\) the lateral inviscid hydrodynamic force (lift); \(F_{px}\) and \(F_{pz}\), the resultant forces due to external pressure; and \(F_N\) and \(F_L\), the frictional viscous forces.

In accordance with E-B beam theory, the centreline displacements of the tube from the \(x\)-axis are presumed small. The flow-field outside the tube can then be simplified. More specifically, the external flow can be modelled as a potential flow, superimposing the effects of the mean fluid-flow, on the vibratory motion of the tube (Lighthill, 1960; Paidoussis, 2004). The effects of fluid viscosity are thus treated as distinct from this potential flow, and the viscosity-related forces \((F_N\) and \(F_L\)) are added to the system separately. This is a common approach in linear studies of problems of fluid-structure interactions. More details can be found in Paidoussis (2004); Hannoyer and Paidoussis (1978).

The external flow velocity \(U_o\), and tube confinement, differ for configurations (i) and (iii). Hence, the forces related to the external flow merit separate analysis. A
force balance on the external surface of a tube element (figure 7–11(c)) gives

\[ F_{et} = F_L + F_{px} - F_{pz} \frac{\partial w}{\partial x}; \]  \hspace{1cm} (4.11)

and

\[ F_{en} = - (F_A + F_N) + F_{px} \frac{\partial w}{\partial x} + F_{pz}. \]  \hspace{1cm} (4.12)

Equations (4.11) and (4.12) represent the sum of the external hydrodynamic forces in the directions tangent and normal to the tube centreline, respectively. They are valid for both configurations (i) and (iii). In the following subsections we consider these forces individually.

4.5 Lateral Inviscid Hydrodynamic Force (lift) \( (F_A) \)

4.5.1 Configuration (i): discharging pipe; no flow in the annulus

In configuration (i) (figure 7–5(a)) there is no mean flow over the external surface of the tube. The relative velocity of the surrounding fluid with respect to the tube is given by \( \mathbf{v} = -(\partial w/\partial t) \hat{k} \). The fluid momentum per unit length due to the tube motion is thus \( \mathbf{M} = M_a (\partial w/\partial t) \hat{k} \), where \( M_a \) is the virtual mass of external fluid per unit length associated with the tube motions, commonly referred to as the “added fluid mass” (Paidoussis, 2004).

The rate of change of fluid momentum, per unit length, for the surrounding fluid is given by

\[ F_A = M_a \frac{\partial^2 w}{\partial x^2}, \]  \hspace{1cm} (4.13)

as per Lighthill (1960) and later Paidoussis (2004).
The fluid added mass $M_a$ is given by $M_a = \chi \rho_f A_o$, where $A_o = \pi D_o^2 / 4$ is the outer cross-sectional area of the tube, and $\chi$ accounts for the confinement by

$$\chi = \frac{(D_{ch}/D_o)^2 + 1}{(D_{ch}/D_o)^2 - 1}.$$  

$D_{ch}$ is the internal diameter of the channel (the pipe surrounding the flexible tube, forming the annulus), and $D_o$ the outer (or external) tube diameter shown in figure 7–11(a). For an unconfined tube, $D_{ch} \to \infty$. Taking this limit

$$\lim_{D_{ch} \to \infty} \left[ \frac{(D_{ch}/D_o)^2 + 1}{(D_{ch}/D_o)^2 - 1} \right] = 1,$$

hence for an unconfined tube $M_a = \rho_f A_o$, equal to the fluid mass displaced by the tube per unit length.

In configuration (i) the annulus covers only a fraction of the length of the tube; hence the tube is partially confined. Letting $\chi$ be the confinement parameter pertinent to the confined part of the tube, we can express the lateral inviscid hydrodynamic force $F_A$ as

$$F_A = \left[ \chi + (1 - \chi) H(x - L') \right] \rho_f A_o \frac{\partial^2 w}{\partial t^2}.$$  \hspace{1cm} (4.14)

Here $H(x - L')$ is the Heaviside step function, and $L'$ is the length of the annulus (figure 7–11(a)). In this way we have accounted for the spatial variation of $M_a$ and consequently $F_A$ over the length of the tube.
4.5.2 Configuration (iii): discharging pipe; aspirating annulus

In this configuration (figure 7–5(c)) both the degree of confinement and the external flow velocity vary along the length of the tube. The external flow velocity is $U_o$ over the confined part of the tube, and zero over the unconfined portion. Similar to configuration (i), we let the mass of the fluid associated with the tube motions be $M_a = [\chi + (1 - \chi) H(x - L')] \rho_f A_o$. We can then express the external inviscid hydrodynamic force as

$$F_A = \left( \frac{\partial}{\partial t} - U_o \frac{\partial}{\partial x} + U_o H(x - L') \frac{\partial}{\partial x} \right) \times \left\{ [\chi - (1 - \chi)] \rho_f A_o \left( \frac{\partial w}{\partial t} - U_o \frac{\partial w}{\partial x} + U_o H(x - L') \frac{\partial w}{\partial x} \right) \right\},$$

where $U_o$ is the mean flow velocity over the confined part of the tube. In this way we have accounted for the spatial variation of both the added mass, and velocity of the surrounding fluid. After various simplifications, $F_A$ can be expressed as

$$F_A = -A_o U_o^2 \rho_f H(x - L') \frac{\partial^2 w}{\partial x^2} + A_o U_o^2 \rho_f  \frac{\partial^2 w}{\partial x^2} + 2A_o U_o \rho_f \chi H(x - L') \frac{\partial^2 w}{\partial x \partial t}$$
$$- 2A_o U_o \rho_f \chi \frac{\partial^2 w}{\partial x \partial t} + (1 - \chi) \rho_f A_o H(x - L') \frac{\partial^2 w}{\partial t^2} + A_o \rho_f \chi \frac{\partial^2 w}{\partial t^2}. \quad (4.15)$$

4.6 Forces due to friction with the surrounding fluid ($F_L$, $F_N$)

The frictional forces are: $F_L$ in the longitudinal direction, and $F_N$ in the direction normal to the tube. These forces are added to the system, considering the effects of viscosity as separate from those of the other hydrodynamic forces (Paidoussis, 2004; Paidoussis et al., 2008).
4.6.1 Configuration (i): discharging pipe; no flow in the annulus

For configuration (i) there is no mean flow over the tube \( U_o = 0 \) and, as a result, no frictional forces appear in the longitudinal direction, giving

\[
F_L = 0. \tag{4.16}
\]

The frictional force in the direction normal to the tube is approximated by

\[
F_N = k \frac{\partial w}{\partial t}. \tag{4.17}
\]

This is the simplified form of the expressions developed by Paidoussis (2004), Hanno yer and Paidoussis (1978), and Paidoussis et al. (2008), based on work by Taylor (1952) and later Sinyavskii et al. (1980). In the above expression, \( k \) is a (frequency-dependent) viscous drag coefficient given by

\[
k = \frac{2\sqrt{2}}{\sqrt{S}} \frac{1 + \bar{\gamma}^3}{(1 - \bar{\gamma}^2)^{7/2}} \rho_f A_o \Omega.
\]

Here \( S = \Omega D_o^2/4 \nu \) is the Stokes number (oscillatory Reynolds number), \( \Omega \) the circular frequency of the oscillation, \( \nu \) the kinematic viscosity of the fluid, and \( \bar{\gamma} = D_o/D_{ch} \) a measure of confinement.

For configuration (i), \( k \) varies along the tube. To account for this variation we let

\[
k = k_u = \frac{2\sqrt{2}}{\sqrt{S}} \rho_f A_o \Omega.
\]
where $k_u$ is the frequency-dependent friction coefficient applicable to the unconfined part of the tube. The frictional force thus becomes

$$F_N = k \frac{\partial w}{\partial t} = k_u \left[ \left( \frac{1 + \tilde{\gamma}^3}{(1 - \tilde{\gamma}^2)^2} \right) + H(x - L') \left( 1 - \frac{1 + \gamma^3}{(1 - \gamma^2)^2} \right) \right] \frac{\partial w}{\partial t}, \quad (4.18)$$

where $H(x - L')$ is the Heaviside step function, and $L'$ the annulus length.

### 4.6.2 Configuration (iii): discharging pipe; aspirating annulus

The longitudinal frictional force $F_L$ on the tube is given by

$$F_L = \frac{1}{2} C_f \rho_f D_o U_o^3 \left[ 1 - H(x - L') \right]. \quad (4.19)$$

Equation (4.19) is a modification of the expression given in Paidoussis et al. (2008), accounting for the spatial variation of the mean external flow velocity. The frictional coefficient was given a value of $C_f = 0.0125$, which was adopted from the aforementioned study as it was found to give a good estimate of the frictional force.

The frictional force in the direction normal to the tube $F_N$ is adapted from Paidoussis et al. (2008) to account for the velocity variation over the tube, and is given by

$$F_N = \frac{1}{2} C_f \rho_f D_o U_o \left[ 1 - H(x - L') \right] \left\{ \frac{\partial w}{\partial t} - \left[ 1 - H(x - L') \right] U_o \frac{\partial w}{\partial x} \right\} + k \frac{\partial w}{\partial x}, \quad (4.20)$$

where $k(\partial w/\partial t)$ is the same as in configuration (i) section 4.6.1. After appropriate simplifications and use of equation (4.18) we obtain

$$F_N = \frac{1}{2} C_f \rho_f D_o U_o \left\{ \frac{\partial w}{\partial t} \left[ 1 - H(x - L') \right] - U_o \left[ 1 - H(x - L') \right] \frac{\partial w}{\partial x} \right\} + \frac{k}{\partial x} \left[ \frac{1 + \tilde{\gamma}^3}{(1 - \tilde{\gamma}^2)^2} + H(x - L') \left( 1 - \frac{1 + \gamma^3}{(1 - \gamma^2)^2} \right) \right] \frac{\partial w}{\partial t}. \quad (4.21)$$
4.7 Forces due to Mean Tensioning and Pressurization \((F_{px}, F_{pz})\)

The forces \(F_{px}\) and \(F_{pz}\), are the force resultants of the mean external pressure acting on the external surface of the tube in the \(x\) and \(z\) directions, respectively. These were cleverly derived by Paidoussis (1973, 2004) and are given as follows:

\[ x\text{-direction:} \quad F_{px} = -\frac{\partial}{\partial x} (A_o p_o) + A_o \frac{\partial p_o}{\partial x}; \quad (4.22) \]
\[ z\text{-direction:} \quad F_{pz} = A_o \frac{\partial}{\partial x} \left( p_o \frac{\partial w}{\partial x} \right); \quad (4.23) \]

where \(p_o\) is the pressure outside the tube.

4.7.1 Configuration (i): discharging pipe; no flow in the annulus

The absence of external flow results in a purely hydrostatic external pressure distribution, unaffected by the length of the annulus. Hence we let

\[ \frac{\partial p_o}{\partial x} = \rho_f g. \quad (4.24) \]

After integration

\[ p_o(x) = \rho_f g x, \quad (4.25) \]

where the pressure at \(x = 0\) is the reference pressure, \(p_o|_{x=L}= 0\). As a consequence of equation (4.25), and the uniformity of the tube external cross-sectional area \(A_o\), \(F_{px}\) in equation (4.22) becomes zero. However, equation (4.22) is not discarded in the interests of consistency.
4.7.2 Configuration (iii): discharging pipe; aspirating annulus

For configuration (iii), there is external flow only over the confined part of the tube. For the purpose of this analysis, the entrance effects of the flow entering the annulus have been reduced to an infinitesimally thin region at \( x = L' \). Inside the annulus, there is a pressure loss due to both the friction of the flowing fluid, and gravity. Over the unconfined part of the tube, the pressure distribution is hydrostatic.

For \( 0 \leq x < L' \) we consider an annular fluid element shown in figure 7–12. The total frictional force acting on both the internal and external surfaces of such a fluid element is \( F_f \). A force balance per unit length yields

\[
-A_ch \frac{\partial p_o}{\partial x} + F_f + A_ch \rho_f g = 0 ,
\]

where \( A_ch = \pi(D_{ch}^2 - Do^2)/4 \) is the cross-sectional area of the annular flow.

Assuming that an equal shear stress acts on the external tube surface and the inside annulus surface, we have \( (F_f/S_{tot}) = (F_L/S_o) \); where \( S_{tot} = \pi(D_{ch} + Do) \) is the total wetted perimeter of the annular flow (the total wetted area per unit length), and \( S_o \) is the external wetted tube perimeter. Rearranging yields \( F_f = F_L(S_{tot}/S_o) \). Substitution of \( F_f \) into (4.26), multiplication by \( (A_o/A_ch) \), and rearranging yields

\[
A_o \frac{\partial p_o}{\partial x} = F_L \left( \frac{D_o}{D_h} \right) + A_o \rho_f g ,
\]

where \( D_h = 4A_o/S_{tot} = (D_{ch} - Do) \) is the hydraulic diameter of the annular channel. Integration of equation(4.27), where the external pressure at \( x = 0 \) is the reference
pressure \( (p_o|_{x=0} = 0) \), yields for \( 0 \leq x < L' \):

\[
P_o(x) = \left[ \frac{F_L}{A_o} \left( \frac{D_o}{D_h} \right) + \rho_f g \right] x .
\] (4.28)

For \( L' < x \leq L \), the outside pressure distribution is hydrostatic

\[
\frac{\partial p_o}{\partial x} = \rho_f g ,
\] (4.29)

which upon integration yields

\[
p_o(x) = \rho_f g x + C_1 ,
\] (4.30)

where \( C_1 \) is a constant of integration to be determined below.

In reference to figure 7–11(a), \( x_1 = L'^- \) is the location just inside the annulus while \( x_2 = L'^+ \) is the location just outside the annulus. Conservation of energy for the fluid from \( x_2 \) to \( x_1 \) yields

\[
p_o|_{x_2} = p_o|_{x_1} + \frac{1}{2} \rho_f U_o^2 + \rho_f g h_a .
\] (4.31)

The term \( h_a \) is the head loss associated with the stagnant fluid entering the “inwards-projecting, squared-cornered”\(^1\) annulus at \( x = L' \) and acquiring velocity \( U_o \). The value of \( h_a \) is determined by

\[
h_a = K_1 \frac{U_o^2}{2g} ,
\] (4.32)

\(^1\) The annulus is projecting into the body of water contained in the test-section (e.g. like Borda’s mouthpiece), with all sharp, ninety degree corners.
where for an inwards projecting, square-cornered tube $0.8 \leq K_1 \leq 0.9$, which is independent of flow velocity. Equation (4.32) is from Brater et al. (1996). Equation (4.31) yields

$$C_1 = \rho_f g x + \left[ \frac{F_L}{A_o} \left( \frac{D_o}{D_h} \right) L' \right] + \frac{1}{2} \rho_f U_o^2 + \rho_f gh_a,$$

where $|F_L| = \frac{1}{2} C_f \rho_f D_o U_o^2$.

For $0 \leq x \leq L$, the outside pressure gradient and outside pressure are:

$$\frac{\partial p_o}{\partial x} = \frac{|F_L|}{A_o} \left( \frac{D_o}{D_h} \right) [1 - H(x - L')] + \rho_f g + \left( \frac{1}{2} \rho_f U_o^2 + \rho_f gh_a \right) \delta_D(x - L'),$$

and

$$p_o = \frac{|F_L|}{A_o} \left( \frac{D_o}{D_h} \right) x - \frac{|F_L|}{A_o} \left( \frac{D_o}{D_h} \right) (x - L') H(x - L')
+ \rho_f g x + \left( \frac{1}{2} \rho_f U_o^2 + \rho_f gh_a \right) H(x - L'),$$

in which $\delta_D(x - L')$ is the Dirac delta function.

### 4.8 Resulting Equations of Motion

The sum of the external fluid forces in the $x$ and $z$ directions is given by:

**x-direction:**

$$-F_{en} \frac{\partial w}{\partial x} - F_{et} = -F_L - F_{px};$$

**z-direction:**

$$-F_{en} + F_{et} \frac{\partial w}{\partial x} = (F_A + F_N) - F_{pz} + F_L \frac{\partial w}{\partial x};$$

where we have retained up to first-order terms.

### 4.8.1 Configuration (i): discharging pipe; no flow in the annulus

Substituting equation (4.22) in (4.35) and equations (4.14),(4.18),(4.23) in equation (4.36), we obtain for the $x$ and $z$ directions:

**x-direction:**

$$-F_{en} \frac{\partial w}{\partial x} - F_{et} = \frac{\partial}{\partial x} (A_o p_o) - A_o \frac{\partial p_o}{\partial x};$$

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Equations (4.37) and (4.38) summarize the effects of the external flow on the system. Substituting these in (4.9) and (4.10) we obtain the following equations of motion:

\[ \frac{\partial T}{\partial x} + M_t g + A_f \rho_f g - \frac{\partial}{\partial x} (A_f p_i) + \frac{\partial}{\partial x} (A_o p_o) - A_o \frac{\partial p_o}{\partial x} = 0 \; ; \quad (4.39) \]

In the above equation, most of the system parameters are known, with the exception of tensioning and pressurization, i.e. the term \([T - A_f p_i + A_o p_o]\). To find this, we rearrange equation (4.39), giving

\[ \frac{\partial}{\partial x} (T - A_f p_i + A_o p_o) = -M_t g - A_f \rho_f g + A_o \frac{\partial p_o}{\partial x}. \]

Integrating the above equation from \(x\) to \(L\) yields

\[ (T - A_f p_i + A_o p_o) = (T - A_f p_i + A_o p_o)|_L + (M_t g + A_f \rho_f g - A_o \rho_f g)(x - L) \; , \quad (4.41) \]
where \((T - A_fp_i + A_o p_o)|_L\) denotes evaluation of the quantity in parentheses at \(x = L\).

Substitution of equation (4.41) in (4.40) yields the governing equation of motion:

\[
EI \frac{\partial^4 w}{\partial x^4} - \left[-M_t g - A_f \rho_f g + A_o \rho_f g\right] \frac{\partial w}{\partial x} \\
+ \left[-(T - A_fp_i + A_o p_o)|_L - (M_t g + A_f \rho_f g - A_o \rho_f g)(L - x)\right] \frac{\partial^2 w}{\partial x^2} \\
+ \{M_t + A_f \rho_f + A_o \rho_f \left[\chi + (1 - \chi)H(x - L')\right]\} \frac{\partial^2 w}{\partial t^2} + 2A_f \rho_f U_i \frac{\partial^2 w}{\partial x \partial t} \\
+ A_f \rho_f U_i^2 \frac{\partial^2 w}{\partial x^2} + k_u \left[\frac{1 + \gamma^3}{(1 - \gamma^2)^2} + H(x - L') \left(1 - \frac{1 + \gamma^3}{(1 - \gamma^2)^2}\right)\right] \frac{\partial w}{\partial t} = 0 ,
\]

with the associated boundary conditions

\[
w|_{x=0} = \left.\frac{\partial w}{\partial x}\right|_{x=0} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = 0 .
\]

In equation (4.42), explicit forms for \(T|_L\), \(P_o|_L\), and \(P_i|_L\) have yet to be determined. These are the applied tension, the external pressure, and internal pressure at the free end, respectively.

For this problem \(T|_L = 0\), since there is no externally applied tension at \(x = L\) (such as from base drag). This is a consequence of the downstream end of the tube being free and surrounded by stagnant fluid, as well as the lack of any end-piece.

The pressures \(P_o|_L\) and \(P_i|_L\), can be determined from the conventional conservation of energy equation for the fluid at the free end. Thus, \(P_o|_L\) and \(P_i|_L\) are related by

\[
p_i|_L = p_o|_L - \frac{\rho_f U_i^2}{2} + \rho_f g h_e ,
\]

in which \(h_e\) is the head loss at the free end due to sudden expansion of the flow exiting from the tube into the surrounding fluid.
The external pressure distribution has already been determined in equation (4.25); hence, \( p_o|_L = \rho_f g L \). The head loss \( h_e \) was calculated from
\[
h_e = \frac{K_2 U_i^2}{2g},
\]
with \( K_2 = 1 \). This expression, taken from Brater et al. (1996), is an empirical relationship valid for water flow and likely overestimates \( h_e \). Other equally applicable formulas for approximating the head loss are available in Brater et al. (1996) and other handbooks of hydraulics.

4.8.2 Configuration (iii): discharging pipe; aspirating annulus

Substitution of equations (4.19) and (4.22) in (4.35), equations (4.15), (4.21), (4.23), and (4.19) in equation (4.36), and subsequent substitution of the resulting equations in (4.9) and (4.10), yields for the \( x \) and \( z \) directions:

**\( x \)-direction:**
\[
\frac{\partial}{\partial x} (T - A_f p_i + A_o p_o) + M_i g + \rho_f A_f g - A_o \frac{\partial p_o}{\partial x} - \frac{1}{2} C_f \rho_f D_o U_o^2 [1 - H(x - L')] = 0,
\]
(4.46)
\[ z \text{-direction:} \]

\[
EI \frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x} \left[ (T - A_f p_i + A_o p_o) \frac{\partial w}{\partial x} \right] + M_t \frac{\partial^2 w}{\partial t^2} + A_f \rho_f \frac{\partial^2 w}{\partial t^2} \\
+ 2U_i A_f \rho_f \frac{\partial^2 w}{\partial x \partial t} + A_f \rho_f U_i^2 \frac{\partial^2 w}{\partial x^2} - A_o \rho_f \chi H(x - L') \frac{\partial^2 w}{\partial x^2} \\
+ A_o U_o^2 \rho_f \chi \frac{\partial^2 w}{\partial x \partial t} + 2A_o U_o \rho_f \chi H(x - L') \frac{\partial^2 w}{\partial x^2} - 2A_o U_o \rho_f \chi \frac{\partial^2 w}{\partial x \partial t} \\
+ (1 - \chi) \rho_f A_o H(x - L') \frac{\partial^2 w}{\partial t^2} + A_o \rho_f \chi \frac{\partial^2 w}{\partial t^2} \\
+ \frac{1}{2} C_f \rho_f D_o U_o \left[ 1 - H(x - L') \right] \frac{\partial w}{\partial t} \\
+ k_u \left[ \frac{1 + \tilde{\gamma}^3}{(1 - \tilde{\gamma}^2)^2} + H(x - L') \left( 1 - \frac{1 + \tilde{\gamma}^3}{(1 - \tilde{\gamma}^2)^2} \right) \right] \frac{\partial w}{\partial t} = 0 .
\]

In equation (4.47) the tensioning and pressurization term \((T - A_f p_i + A_o p_o)\) is as yet undetermined. To find it, similarly to configuration (i), integration of the \(x\)-direction equation (4.46) from \(x\) to \(L\) yields

\[
(T - A_f p_i + A_o p_o) = -(-M_t - \rho_f A_f + \rho_f A_o) g (L - x) \\
- \frac{1}{2} C_f \rho_f D_o U_o^2 \left( \frac{D_o}{D_h} + 1 \right) (L' - x) \left[ 1 - H(x - L') \right] \\
- A_o \left( \frac{1}{2} \rho_f U_o^2 + \rho_f gh_a \right) \left[ 1 - H(x - L') \right] + (T - A_f p_i + A_o p_o) |_{L} .
\]

(4.48)
Substitution of equation (4.48) in (4.47) gives the resulting equation of motion in the \( z \)-direction:

\[
EI \frac{\partial^4 w}{\partial x^4} + \left\{ (M_t + \rho_f A_f - \rho_f A_o) g - \frac{1}{2} C_f \rho_f D_o U_o^2 \left( \frac{D_o}{D_h} + 1 \right) [1 - H(x - L')] \right. \\
- A_o \left( \frac{1}{2} \rho_f U_o^2 + \rho_f g h_a \right) \delta_D(x - L') \right\} \frac{\partial w}{\partial x} + \left\{ (-M_t - \rho_f A_f + \rho_f A_o) g(L - x) \\
+ \frac{1}{2} C_f \rho_f D_o U_o^2 \left( \frac{D_o}{D_h} + 1 \right) (L' - x)[1 - H(x - L')] + A_o \left[ \frac{1}{2} \rho_f U_o^2 + \rho_f g h_a \right] \right. \\
\left. \times [1 - H(x - L')] - (T - A_f p_i + A_o p_o) \right|_L \left\{ \frac{\partial^2 w}{\partial x^2} + M_t \frac{\partial^2 w}{\partial t^2} + A_f \rho_f \frac{\partial^2 w}{\partial t^2} + 2U_i A_f \rho_f \frac{\partial^2 w}{\partial x \partial t} \\
+ A_f \rho_f U_i^2 \frac{\partial^2 w}{\partial x^2} + A_o \rho_f \chi U_o^2 [1 - H(x - L')] \frac{\partial^2 w}{\partial x^2} - 2A_o U_o \rho_f \chi [1 - H(x - L')] \frac{\partial^2 w}{\partial x \partial t} \\
+ (1 - \chi) \rho_f A_o H(x - L') \frac{\partial^2 w}{\partial t^2} + A_o \rho_f \chi \frac{\partial^2 w}{\partial t^2} + \frac{1}{2} C_f \rho_f D_o U_o [1 - H(x - L')] \frac{\partial w}{\partial t} \\
\right. \\
\left. + k_u \left\{ 1 + [1 - H(x - L')] \left( \frac{1 + \tilde{\gamma}^3}{(1 - \tilde{\gamma}^2)^2} - 1 \right) \right\} \right\} \frac{\partial w}{\partial t} = 0.
\]

(4.49)

The associated boundary conditions are identical to configuration (i), given in (4.43).

In equation (4.49) the relationship between \( p_o \mid_L \) and \( p_i \mid_L \) at the free end is identical to configuration (i), equation (4.44). Evaluation of equation (4.34) at \( x = L \) yields

\[
p_o \mid_L = \frac{1}{2A_o} C_f \rho_f D_o U_o^2 L' \left( \frac{D_o}{D_h} \right) + \rho_f g L + \frac{1}{2} \rho_f U_o^2 + \rho_f g h_a,
\]

(4.50)

where \( h_a \) is given by equation (4.32).
4.9 Equations of motion in non-dimensional form

With the introduction of the following parameters the equations of motion can be put in non-dimensional form:

\[ \xi = \frac{x}{L}, \quad \tau = \left[ \frac{EI}{Mt + \rho_f A_f + \rho_f A_o} \right]^{\frac{1}{2}} \frac{t}{L^2}, \quad \eta = \frac{w}{L}, \]

\[ u_i = \left( \frac{\rho_f A_f}{EI} \right)^{\frac{1}{2}} LU_i, \quad u_o = \left( \frac{\rho_f A_o}{EI} \right)^{\frac{1}{2}} LU_o, \quad \beta_o = \frac{\rho_f A_o}{Mt + \rho_f A_f + \rho_f A_o}, \]

\[ \beta_i = \frac{\rho_f A_i}{Mt + \rho_f A_f + \rho_f A_o}, \quad \gamma = \frac{(Mt + \rho_f A_f - \rho_f A_o) g L^3}{EI}, \quad \Gamma = \frac{T|L|L^2}{EI}, \]

\[ \Pi_{iL} = \frac{p_i|L|A_f L^2}{EI}, \quad \Pi_{oL} = \frac{p_o|L|A_o L^2}{EI}, \quad c_f = \frac{4C_f}{\pi}, \]

\[ \kappa_u = \frac{k_u L^2}{[\rho_f A_f + \rho_f A_o]^{\frac{1}{2}}}, \quad \varepsilon = \frac{L}{D_o}, \quad h = \frac{D_o}{D_h}, \]

\[ \alpha = \frac{D_i}{D_o}, \quad \alpha_{ch} = \frac{D_{ch}}{D_o}, \quad r_{ann} = \frac{L'}{L}. \]

(4.51)

Furthermore, the non-dimensional frequency of vibration is given by

\[ \omega = \left[ \frac{Mt + \rho_f A_f + \rho_f A_o}{EI} \right]^{\frac{1}{2}} L^2 \Omega. \]

(4.52)

With the exception of \( r_{ann} \), the resulting set of non-dimensional variables is well-established in the analysis of this type of system (refer to Paidoussis (2014) and Paidoussis (2004)).
4.9.1 Configuration (i): pipe discharging; no flow in the annulus

With the relations in (4.51), the non-dimensional form of equation (4.42) becomes

\[
\frac{\partial^4 \eta}{\partial \xi^4} + \gamma \frac{\partial \eta}{\partial \xi} + \left[ -\left( \Gamma - \Pi_{iL} + \Pi_{oL} \right) - \gamma(1 - \xi) \right] \frac{\partial^2 \eta}{\partial \xi^2} + \left\{ 1 - \beta_o(1 - \chi) \left[ 1 - H(\xi - r_{ann}) \right] \right\} \frac{\partial^2 \eta}{\partial \xi \partial \tau} + 2u_i \beta_i^2 \frac{\partial^2 \eta}{\partial \xi \partial \tau} + u_i^2 \frac{\partial^2 \eta}{\partial \xi^2} + \gamma \left( 1 - \xi \right) \frac{\partial^2 \eta}{\partial \xi^2} + \kappa \left\{ 1 + \left[ 1 - H(\xi - r_{ann}) \right] \left( \frac{1 + \alpha^{-3}}{(1 - \alpha^{-2})^2} - 1 \right) \right\} \frac{\partial \eta}{\partial \tau} = 0,
\]

with associated non-dimensional boundary conditions

\[
\eta \big|_{\xi=0} = \left. \frac{\partial \eta}{\partial \xi} \right|_{\xi=0} = \left. \frac{\partial^2 \eta}{\partial \xi^2} \right|_{\xi=1} = \left. \frac{\partial^3 \eta}{\partial \xi^3} \right|_{\xi=1} = 0.
\]

The relationship between the external and internal pressures at the free end, in non-dimensional form is

\[
\Pi_{iL} = \alpha^2 \Pi_{oL} - \frac{1}{2} u_i^2 + A_f \rho_f g h_e \left( \frac{L^2}{EI} \right),
\]

where

\[
\Pi_{oL} = A_O \rho_f g \frac{L^3}{EI},
\]

and \( h_e \) is given in equation (4.45).
4.9.2 Configuration (iii): pipe discharging; the annulus aspirating

Using (4.51) the non-dimensional form of equation (4.49) becomes

\[
\frac{\partial^4 \eta}{\partial \xi^4} + \left\{ \gamma - \frac{1}{2} c_f \varepsilon u_o^2 (1 + h) \left[ 1 - H(\xi - r_{ann}) \right] - \frac{1}{2} u_o^2 (1 + K_1) \delta_D (\xi - r_{ann}) \right\} \frac{\partial \eta}{\partial \xi} \\
- \left\{ (\Gamma - \Pi_{iL} + \Pi_{oL}) + \gamma (1 - \xi) - \frac{1}{2} c_f \varepsilon u_o^2 (1 + h)(r_{ann} - \xi) \left[ 1 - H(\xi - r_{ann}) \right] \\
- \frac{1}{2} u_o^2 (1 + K_1) \left[ 1 - H(\xi - r_{ann}) \right] \right\} \frac{\partial^2 \eta}{\partial \xi^2} + \{ 1 + \beta_o (\chi - 1) \times \\
\left[ 1 - H(\xi - r_{ann}) \right] \} \frac{\partial^2 \eta}{\partial \tau^2} + 2 \left\{ u_i \beta_i^2 - \chi u_o \beta_o^2 \left[ 1 - H(\xi - r_{ann}) \right] \right\} \frac{\partial^2 \eta}{\partial \xi \partial \tau} \\
+ \left\{ u_i^2 + \chi u_o^2 \left[ 1 - H(\xi - r_{ann}) \right] \right\} \frac{\partial^2 \eta}{\partial \xi^2} + \frac{1}{2} c_f \varepsilon u_o \beta_o^2 \left[ 1 - H(\xi - r_{ann}) \right] \frac{\partial \eta}{\partial \tau} \\
+ \kappa_u \left\{ 1 + [ 1 - H(\xi - r_{ann}) \left( \frac{1 + \alpha_{ch}}{(1 - \alpha_{ch})^2} \right) - 1 \right] \right\} \frac{\partial \eta}{\partial \tau} = 0 ,
\]

with the associated non-dimensional boundary conditions given in equation (4.54).

The relationship between the internal ($\Pi_{iL}$) and external ($\Pi_{oL}$) non-dimensional pressures at $\xi = 1$ is given by equation (4.55). The non-dimensional external pressure is given by

\[
\Pi_{oL} = \frac{1}{2} c_f h_e r_{ann} u_o^2 + \frac{1}{2} u_o^2 (1 + K_1) + \frac{A_o \beta_f g L^3}{EI} .
\]

4.10 Application of the Galerkin method to our derived models

Let an approximate solution to our equations of motion be

\[
\eta(\xi, \tau) \approx \tilde{\eta}(\xi, \tau) = \sum_{j=1}^{N} \Phi_j(\xi) q_j(\tau) ,
\]

where $N$ is finite and the above represents a truncated series. Here $N$ is the number of modes to be used in the approximation. The solution is presumed separable in terms of the non-dimensional spatial variable $\xi$ and the non-dimensional time $\tau$. 

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The $\Phi_j(\xi)$ are appropriate comparison functions satisfying the geometric and natural boundary conditions of the problem, in this case normalized beam eigenfunctions for the clamped-free Euler Bernoulli beam. The $q_j(\tau)$ are the generalised coordinates of the problem.

The Galerkin method is widely used for problems of this nature, so we do not provide any theoretical background here. For details on the use of the Galerkin method see Meirovitch (2001). To make our notation more compact, we define

$$
A_{ij}(a,b) \equiv \int_a^b \Phi_i \Phi_j \, d\xi,
$$

$$
B_{ij}(a,b) \equiv \int_a^b \Phi_i \left( \frac{d\Phi_j}{d\xi} \right) \, d\xi,
$$

$$
F_{ij}(a,b) \equiv \int_a^b \Phi_i \left( \frac{d^2\Phi_j}{d\xi^2} \right) \, d\xi,
$$

$$
J_{ij}(a,b) \equiv \int_a^b \xi \Phi_i \left( \frac{d^2\Phi_j}{d\xi^2} \right) \, d\xi.
$$

(4.60)

The application of the Galerkin method on a per configuration basis is shown in the following subsections.

4.10.1 Configuration (i): discharging pipe; no flow in the annulus

Substitution of equation (4.59) in equation (4.53), requirement that the integrated residuals be set to zero, and use of definitions (4.60) yields for configuration (i):

$$
\sum_{j=1}^{N} \left\{ \lambda_j^2 A_{ij(0,1)} q_j + \gamma B_{ij(0,1)} q_j - (\Gamma - \Pi_{iL} + \Pi_{oL}) F_{ij(0,1)} q_j - \gamma(F_{ij(0,1)} - J_{ij(0,1)}) q_j
+ A_{ij(0,1)} \ddot{q}_j - \beta_o(1-\xi) A_{ij(0,r_{ann})} \ddot{q}_j + 2u_i \beta_i^2 B_{ij(0,1)} \ddot{q}_j + u_i^2 F_{ij(0,1)} q_j
+ \kappa_u A_{ij(0,1)} \dot{q}_j + \kappa_u \left( \frac{1 + \alpha_{ch}^{-3}}{(1 - \alpha_{ch}^{-2})^2} - 1 \right) A_{ij(0,r_{ann})} \dot{q}_j \right\} = 0,
$$

(4.61)

for all $i = 1, 2, ..., N$.  

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Equation (4.61) may be put in standard matrix form: \([M] \ddot{q} + [C] \dot{q} + [K] q = 0\), where \([M]\), \([C]\), and \([K]\) are mass, dissipation, and stiffness matrices. Their elements are given by

\[
M_{ij} = A_{ij}(0,1) - \beta_o(1 - \chi) A_{ij}(0,r_{ann}), \quad (4.62)
\]

\[
C_{ij} = 2 u_i \beta_o^{\frac{3}{2}} B_{ij}(0,1) + \kappa_u A_{ij}(0,1) + \kappa_u \left( \frac{1 + \alpha^{-3}}{1 - \alpha^{-2}} - 1 \right) A_{ij}(0,r_{ann}), \quad (4.63)
\]

\[
K_{ij} = \lambda^2_{ij} A_{ij}(0,1) + \gamma B_{ij}(0,1) - (\Gamma - \Pi_{iL} + \Pi_{oL}) F_{ij}(0,1) - \gamma (F_{ij}(0,1) - J_{ij}(0,1))
+ u_i^2 F_{ij}(0,1). \quad (4.64)
\]

### 4.10.2 Configuration (iii): discharging pipe; aspirating annulus

For configuration (iii), substitution of equation (4.59) in (4.57) and proceeding in the same manner as for configuration (i) yields

\[
\sum_{j=1}^N \left\{ \lambda^4_{ij} A_{ij}(0,1) q_j + \gamma B_{ij}(0,1) q_j - \frac{1}{2} c_f \varepsilon u_o^2 (1 + h) B_{ij}(0,r_{ann}) q_j - \frac{1}{2} u_o^2 (1 + K_1) \times 
\left( \Phi_i(\xi=r_{ann}) \frac{\partial \phi_i}{\partial x} \right) q_j - (\Gamma - \Pi_{iL} + \Pi_{oL}) F_{ij}(0,1) q_j - \gamma (F_{ij}(0,1) - J_{ij}(0,1)) q_j
+ \frac{1}{2} c_f \varepsilon u_o^2 (1 + h) (r_{ann} F_{ij}(0,r_{ann}) - J_{ij}(0,r_{ann})) q_j + \frac{1}{2} u_o^2 (1 + K_1) F_{ij}(0,r_{ann}) q_j
+ A_{ij}(0,1) \ddot{q}_j + \beta_o(\chi - 1) A_{ij}(0,r_{ann}) \ddot{q}_j + 2 u_i \beta_o^{\frac{3}{2}} B_{ij}(0,1) \dot{q}_j
- 2 \chi u_o \beta_o^{\frac{3}{2}} B_{ij}(0,r_{ann}) \dot{q}_j + u_i^2 F_{ij}(0,1) q_j + \chi u_o^2 F_{ij}(0,r_{ann}) q_j
+ \frac{1}{2} c_f \varepsilon u_o \beta_o^{\frac{3}{2}} A_{ij}(0,r_{ann}) \dot{q}_j + \kappa_u A_{ij}(0,1) \dot{q}_j + \kappa_u \left( \frac{1 + \alpha^{-3}}{1 - \alpha^{-2}} - 1 \right) \right\} = 0,
\]

(4.65)

for all \(i = 1, 2, ..., N\).
The elements of the mass, dissipation, and stiffness matrices are given by

\[ M_{ij} = A_{ij}(0,1) - \beta_o(1 - \chi) A_{ij}(0,r_{ann}) , \]  
\[ C_{ij} = 2u_i\beta_o^2 B_{ij}(0,1) - 2\chi u_o\beta_o^2 B_{ij}(0,r_{ann}) + \frac{1}{2} c_f\varepsilon u_o\beta_o^4 A_{ij}(0,r_{ann}) + \kappa_u A_{ij}(0,1) \] 
\[ + \kappa_u \left( \frac{1 + \alpha_{ch}^{-3}}{(1 - \alpha_{ch}^2)^2} - 1 \right) A_{ij}(0,r_{ann}) , \]  
\[ K_{ij} = \lambda_j^4 A_{ij}(0,1) + \gamma B_{ij}(0,1) - \frac{1}{2} c_f\varepsilon u_o^2(1 + h) B_{ij}(0,r_{ann}) \] 
\[ - \frac{1}{2} u_o^2(1 + K_1) \left( \Phi_i|\xi=r_{ann} \frac{\partial\Phi}{\partial\xi}|\xi=r_{ann} \right) - (\Gamma - \Pi_i L + \Pi_o L) F_{ij}(0,1) \] 
\[ - \gamma(F_{ij}(0,1) - J_{ij}(0,1)) + \frac{1}{2} c_f\varepsilon u_o^2(1 + h) \left( r_{ann} F_{ij}(0,r_{ann}) - J_{ij}(0,r_{ann}) \right) \] 
\[ + \frac{1}{2} u_o^2(1 + K_1) F_{ij}(0,1) + u_i^2 F_{ij}(0,1) + \chi u_o^2 F_{ij}(0,r_{ann}) . \]
CHAPTER 5
Theoretical Results

To obtain solutions to equations (4.61) and (4.65), the sets of $N$ second-order coupled simultaneous equations are transformed to sets of $2N$ first-order coupled equations. Letting the generalized coordinates $q_j(\tau)$ be harmonic functions of the non-dimensional time, i.e. $q_j(\tau) = \exp(i\omega_j \tau)$, results in a standard algebraic eigenvalue problem. This yields the complex non-dimensional eigenfrequencies $\omega_j$, $j = 1, \ldots, 2N$, which are related to the system eigenvalues by $\omega_j = -i\lambda_j$. The real part $\text{Re}(\omega_j)$ is associated with the frequency of vibration of the $j^{\text{th}}$ mode, and the imaginary part $\text{Im}(\omega_j)$ is associated with damping in that mode. A negative imaginary part for any mode $\text{Im}(\omega_j) < 0$, results in negative damping and the system becomes unstable. The critical velocity is the non-dimensional internal flow velocity $u_i$ at which any of the modes become unstable.

As mentioned in section 4.10, the normalized E-B beam eigenfunctions for a clamped-free beam were used as comparison functions $\Phi_j(\xi)$, $j = 1, \ldots, N$. The integrals $A_{ij}(a,b)$, $B_{ij}(a,b)$, $F_{ij}(a,b)$, and $J_{ij}(a,b)$—defined in (4.60)—were calculated numerically for $a = 0$ and $b = r_{ann}$. In the special case of $a = 0$ and $b = 1$, closed form solutions are given in Paidoussis (2014).

The models derived in section 4.1 are used to generate numerical results for tube 1 and tube 2 in the laboratory set-up. These results are then compared with the experiments in non-dimensional form. In addition to our laboratory set-up, results
are also given for a pipe with the approximate dimensions of a brine string used in salt cavern hydrocarbon storage wells.

5.1 Theoretical results for the laboratory set-up

The relevant properties of the experimental set-up are given in tables 7–1 and 7–2. The corresponding non-dimensional parameters used (equation (4.51)) are listed in table 7–7. A value of $C_f = 0.0125$ was used for the frictional damping coefficient, as per Paidoussis et al. (2008), Rinaldi and Paidoussis (2012). Furthermore, $T = \Gamma = 0$ as there is no externally applied tension at the free end of the tubes.

Convergence of the solution was achieved by successively adding terms in the Galerkin approximation until a relative error of less than 5% was attained in the non-dimensional eigenfrequencies $\omega$ for the lowest three modes. This required up to $N = 8$ terms near the critical internal flow velocities $u_i$. The values for the frequency-dependent viscous damping $\kappa_u$ were calculated recursively for each mode and at every flow velocity.

Theoretical results for configuration (i), i.e. the critical non-dimensional internal flow velocity $u_i$ and associated circular frequency $\omega$, are shown in figures 7–40 and 7–41 in conjunction with the non-dimensional experimental results for tube 1 and tube 2, respectively. The results for configuration (iii) are given in figure 7–42 for tube 1 and figure 7–43 for tube 2.

The results are also summarized in Table 7–8 for configuration (i), and in Table 7–9 for configuration (iii).
5.1.1 Qualitative theoretical results

In agreement with the experiments, the theoretical models correctly predict second-mode flutter for both configurations (i) and (iii).

One can see from equations (4.53) and (4.57), that the presence of the annulus affects the system in three major ways:

I. The annulus increases the amount of fluid that is "dragged along" with the movement of the tube i.e. the "fluid added mass", effectively increasing the inertia of the tube; relevant for configurations (i) and (iii).

II. The annulus introduces Coriolis and gyroscopic forces due to the momentum of the external flow; relevant to configuration (iii).

III. The annulus changes the frictional forces acting on the external surface of the tube:
   a. By increasing the viscous forces associated with the lateral movements of the tube in still fluid; relevant to configurations (i) and (iii).
   b. By compressing the tube due to friction with the externally flowing fluid; relevant to configuration (iii).

The above effects are directly dependent on the degree of confinement (or narrowness) of the annular channel, i.e. the parameter $\chi$ in (4.53) and (4.57). Due to problems with the very early onset of impacting, the dimensions of the experimental set-up are such that the tube is only slightly confined. As a result, the aforementioned effects are greatly diminished.

For configuration (i), the model mimics the experimental results, giving no significant variation of the critical velocity $u_i$ with annulus length (figures 7–40(a) and
The model also predicts the decrease in frequency of vibration $\omega$ (figures 7–40(b) and 7–41(b)).

For configuration (iii), the theoretical model accurately predicts a small variation of $u_i$ with increasing annulus length (figures 7–42(a) and 7–43(a)), and a notable reduction of $\omega$ (figures 7–42(b) and 7–43(b)).

The qualitative agreement between the derived models and experimental results suggests that these simple models account for the presence of the annulus to some extent. However, there are discrepancies between the predicted and observed values, discussed in the following section.

5.1.2 Quantitative theoretical results

Despite qualitative agreement with experimental data, our models consistently overestimate the experimentally obtained critical flow velocities $u_i$ for both configurations (i) and (iii).

The linear theoretical models are valid for infinitesimally small displacements. Similarly, the experimentally determined values incorporate imperfections in the tube material and shape, and in the flow conditions. Hence, the critical flow velocity and frequency obtained by both the experiments and linear theory are estimates of the true value. Therefore, equations\(^1\) (5.1) and (5.2) are suitable for evaluating the agreement between the experimental and theoretical results. The error in the critical

\(^1\) The values obtained from equations (5.1) and (5.2) are to be used with the understanding that they provide an indicative rather than an exhaustive measure of the suitability of linear theory in the prediction of the critical flow velocity $u_i$ and frequency $\omega$ for the relevant configurations.
flow velocity is given by

\[
error(u_i) = \frac{|u_{i \text{theoretical}} - u_{i \text{experimental}}|}{\frac{1}{2} (u_{i \text{theoretical}} + u_{i \text{experimental}})} ,
\]

(5.1)

and the error in the associated frequency is given by

\[
error(\omega) = \frac{|\omega_{\text{theoretical}} - \omega_{\text{experimental}}|}{\frac{1}{2} (\omega_{\text{theoretical}} + \omega_{\text{experimental}})} ,
\]

(5.2)

following the approach of Langthjem (1995).

For configuration (i) the overestimation of the critical flow velocity ranges from 31% to 37% for tube 1 (figure 7–40(a)), and from 12% to 14% for tube 2 (figure 7–41(a)). For configuration (iii) the difference in the estimated critical flow velocity is from 31% to 36% for tube 1 (figure 7–42(a)) and from 10% to 14% for tube 2 (figure 7–43(a)).

To better understand the discrepancy between experiment and theory, an examination of the relative contributions of dissipation, gravity, and downstream end de-pressurization was undertaken. The downstream de-pressurization, given by the term \(- (A_f p_i + A_o p_o)|_L\) in equations (4.42) and (4.49) had the most significant effect on the predicted critical velocity \(u_i\).

To illustrate this effect, results were obtained for \((- A_f p_i + A_o p_o \equiv 0)\), thereby neglecting the downstream de-pressurization. This is valid in the limit of \(A_o \approx A_f\), and \(p_o|_L \approx p_i|_L\). These results are shown in figures 7–40, 7–41, 7–42, and 7–43 as “theory: no de-pressurization”. A large change in the predicted critical flow velocities \(u_i\) was obtained, whereas the change in the frequencies \(\omega\) was negligible.
The term \((-A_fp_i + A_o p_o)|_L\) is such that any error in the estimation of \((p_i|_L)\) and \((p_o|_L)\) is effectively “magnified” by \((A_o - A_f)\), the difference in internal and external cross-sectional areas. It can be seen that in the case of tube 2, for which \((A_o - A_f)\) is small (7-1), neglecting the downstream de-pressurization has a small effect on \(u_i\); whereas for tube 1, for which \((A_o - A_f)\) is four times larger than for tube 2, the downstream de-pressurization has a much greater effect. Since the value of \((-\Pi_{iL} + \Pi_{oL})\) depends on the internal \((p_i)\) and external \((p_o)\) pressures at the tube free end, the ability to directly measure or accurately estimate these pressures could yield better results.

The theoretical models provided a more accurate prediction of the critical frequencies \(\omega\). The error was 3% for tube 1 (figure 7–40(b)), and ranged from 2% to 8% for tube 2 (figure 7–41(b)) in configuration (i). For configuration (iii), the error ranged from 2% to 8% for tube 1 (figure 7–42(b)) and from 2% to 24% for tube 2 (figure 7–43(b)).

Figures 7–42(b) and 7–43(b) show that discrepancy between the experimental and theoretical \(\omega\) increases with increasing annular length. In addition, there is greater discrepancy for the cases with mean flow in the annulus(figures 7–42(b) and 7–43(b) ) and smaller discrepancy in the absence of mean flow in the annulus (figures 7–40(b) and 7–41(b)).

The predicted values of \(\omega\) correspond to the theoretical threshold for motions of very small amplitude. On the other hand, measurements of the frequency are obtained from oscillations that have already grown to some limiting amplitude. Furthermore, as the tube deflects, it approaches the wall of the annulus where local
wall-tube interaction occurs between the tube and the inner annular surface, and the surrounding annular flow becomes locally eccentric. An analysis of eccentric annular flow is given in El-Shaarawi et al. (1998). These local wall-tube interactions due to eccentric flow have the potential to modify the tube added mass (for example Mateescu et al. (1994)), and consequently affect the frequency of vibration, even before the onset of impacting.

Flow eccentricity is not accounted for by our theoretical model. These interactions could be more severe when there is mean flow in the annulus as compared to the annulus filled with stagnant fluid. This could be the reason for the differences seen between figure 7–41(b) where the discrepancy between the theoretical and measured values is small, and figure 7–43(b) where the discrepancy is larger. This effect is also noticeable between figures 7–40(b) and 7–42(b), albeit to a lesser extent.

Furthermore, when the tube is fluttering, the lateral displacement of the tube from the centreline increases along the $x$-direction. Therefore an increase in the annular length $L'$ leads to stronger wall-tube interaction along the $x$-direction. If true, this effect would be consistent with the increasing discrepancy between the theoretical and measured values with increasing annulus length, as can be seen figures 7–42(b) and 7–43(b).
5.2 Theoretical results for the brine string

Dimensions and properties of the brine-string under consideration were taken from Ratigan (2008)$^2$. A summary of the properties, along with the associated non-dimensional parameters, is given in table 7–10. In the brine string-layout (figure 7–44), $L_{total}$ is the total length of the brine-string and $L'$ is the length of the annulus, taken here as equal to the casing length. $L_{free}$ is the length of pipe hanging below the end of the annulus.

For very long and slender systems, such as the brine string under consideration, very interesting dynamics have been observed. For long horizontal flexible cylinders subjected to axial flow (de Langre et al., 2007), long hanging fluid-carrying pipes (Doaré and de Langre, 2002), and long ribbons hanging in axial flow (Lemaitre et al., 2005), it was found that as the length of system increases past a certain value, the instability (whether flutter or divergence) occurs only in a downstream segment of length $L_c$. The rest of the system is effectively made rigid by the tension induced through friction with the external fluid flow and/or through gravity.

5.2.1 Discharging unconfined brine-string immersed in fluid

The analysis of the discharging brine-string has been restricted to the unconfined portion of the brine-string of length $L = L_{free}$ hanging beyond the casing shoe. This is effectively configuration (i) with an annulus of zero length ($r_{ann} = 0$). The study

$^2$ These dimensions and properties are also relevant to the standard concentric leaching strings used in the construction of salt caverns for hydrocarbon storage. The flexible pipe corresponds to the “inner leaching string”, whereas the rigid outer tube corresponds to the “outer leaching string”
by Doaré and de Langre (2002) on long hanging fluid-conveying cantilevers forms the basis for this analysis.

Neglecting dissipation, the equation of small lateral motions for an unconfined, discharging, fluid-immersed cantilever becomes

\[ EI \frac{\partial^4 w}{\partial x^4} + \left[ -g(M_t + A_f \rho_f - A_o \rho_f)(L - x) + A_f \rho_f U_i^2 - (T - A_f p_i + A_o p_o) \right] \frac{\partial^2 w}{\partial x^2} \]
\[ + g(M_t + A_f \rho_f - A_o \rho_f) \frac{\partial w}{\partial x} + 2A_f \rho_f U_i \frac{\partial^2 w}{\partial x \partial t} + (M_t + A_f \rho_f + A_o \rho_f) \frac{\partial^2 w}{\partial t^2} = 0 \, . \]

(5.3)

This is the simplified form of equation (4.42). From equations (4.44) and (4.45), \( p_i|_L = p_o|_L \). Furthermore, for the brine-string under consideration, \( D_i \approx D_o \) (table 7–10) therefore \( A_f \approx A_o \). There is no applied tension at the tube free end, hence \( T = 0 \). As a result, \(- (T - A_f p_i + A_o p_o)|_L \approx 0 \) and can be neglected. The equation of motion now becomes

\[ EI \frac{\partial^4 w}{\partial x^4} + \left[ -g(M_t + A_f \rho_f - A_o \rho_f)(L - x) + A_f \rho_f U_i^2 \right] \frac{\partial^2 w}{\partial x^2} \]
\[ + g(M_t + A_f \rho_f - A_o \rho_f) \frac{\partial w}{\partial x} + 2A_f \rho_f U_i \frac{\partial^2 w}{\partial x \partial t} + (M_t + A_f \rho_f + A_o \rho_f) \frac{\partial^2 w}{\partial t^2} = 0 \, . \]

(5.4)

With the introduction of the following parameters, the equation of motion (equation (5.4)) can be put in a non-dimensional form. The parameters are as follows:

\[ \tilde{L} = \left[ \frac{EI}{(M_t + A_f \rho_f - A_o \rho_f) g} \right]^\frac{1}{2}, \quad v_i = U_i \tilde{L} \left( \frac{\rho_f A_f}{EI} \right)^\frac{1}{2}, \quad y = \frac{w}{\tilde{L}}, \]

\[ \tilde{\tau} = \left[ \frac{EI}{M_t + A_f \rho_f + A_o \rho_f} \right]^\frac{1}{2} \frac{t}{\tilde{L}^2}, \quad l = \frac{L}{\tilde{L}}, \quad \tilde{\xi} = \frac{x}{\tilde{L}}, \]

(5.5)
where $\tilde{L}$ is a reference length based on the ratio of flexural rigidity and gravity-buoyancy induced tension.

Furthermore, the non-dimensional circular frequency is given by

$$\tilde{\omega} = \left[ \frac{M_t + A_f \rho_f + A_o \rho_f}{EI} \right]^{\frac{1}{2}} \tilde{L}^2 \Omega. \tag{5.6}$$

The parameters in (5.5) are analogous to the parameters defined in (4.51) but with the reference length $\tilde{L}$ used instead of $L$ for non-dimensionalization. Using the above parameters, the non-dimensional form of the equation of motion becomes

$$\frac{\partial^4 y}{\partial \tilde{\xi}^4} + (v_i^2 - \mathcal{X}) \frac{\partial^2 y}{\partial \tilde{\xi}^2} - \frac{\partial \mathcal{X}}{\partial \tilde{\xi}} \frac{\partial y}{\partial \tilde{\xi}} + 2\beta_i^2 v_i \frac{\partial^2 y}{\hat{\tau}^2} = 0, \tag{5.7}$$

where $\beta_i$ is given in (4.51). The term $\mathcal{X} = (l - \tilde{\xi})$ is the non-dimensional tension induced by gravity-buoyancy, and $(\partial \mathcal{X}/\partial \tilde{\xi})$ strictly equals unity.

Equation (5.7) is identical to the non-dimensional equation considered by Doaré and de Langre (2002) and the conclusions reached therein are directly applicable to this problem. The results are summarized below.

It was found that the system remains stable in the region where $\mathcal{X}(\tilde{\xi}) > v_i^2 (1 - \beta_i)$ and is unstable outside that region, provided that $\beta_i \neq 0$ and $v_i \neq 0$. The induced tension $\mathcal{X}$ is $\tilde{\xi}$-dependent, and varies as $l \leq \mathcal{X} \leq 0$ for $0 \leq \tilde{\xi} \leq l$. Therefore, a downstream region of length $l_c = L_c/L = u_i^2 (1 - \beta)$ is identified in which the system can become unstable. As a result:

- a) if the length of the brine-string is such that $L \leq L_c$, when the system becomes unstable the entire length of the tube participates in the instability;
- b) if the length of the brine string is such that $L > L_c$, then only the downstream portion of length $L_c$, becomes unstable.
In light of the above, one would expect that for systems with length $L > L_c$, any further increase in length would not effectively change the critical velocity ($U_{cr}$) and frequency ($\Omega_{cr}$) of the system. This fact was verified both numerically and experimentally in Doaré and de Langre (2002).

In our investigation, and as suggested by de Langre et al. (2007), only the unstable portion of length $L_c$ of the brine-string was considered. The critical length, given by

$$L_c = \frac{A_f U_i^2 \rho_f (M_t + A_o \rho_f)}{g [M_t + (A_f - A_o) \rho_f] [M_t + (A_f + A_o) \rho_f]}$$

(5.8)

was used as a reference length ($L = L_c$) in (4.51). Using these parameters (with $L = L_c$), equation (4.53) was then solved with $r_{ann} = 0$, $\kappa_u = 0$, and $-(\Gamma - \Pi_{iL} + \Pi_{oL}) = 0$. This is a non-dimensional form of (5.4). The boundary conditions are given in (4.54). In this way, the brine-string was effectively truncated at a length $L = L_c$, and considered clamped at the upstream end, and free at the downstream end. Since $L_c$ (equation 5.8) is velocity-dependent, the value was recalculated for every flow velocity.

Results were obtained by the Galerkin method, as described in section 4.10.1, and convergence was achieved up to a relative error of less than 5% in the non-dimensional eigenfrequencies ($\omega$) of the lowest 8 modes. This required up to $N = 54$ E-B beam modes in the Galerkin approximation.

The results obtained with equation (5.8) are shown in figure 7–45 alongside calculations for brine-strings of increasing length from $10 \leq L \leq 400$ m. A summary of the results is also given in table 7–11 ($r_{ann} = 0$). Using the critical length approach, the brine string loses stability via flutter at $U_i = U_{cr} = 52.1$ m/s, with associated
circular frequency $\Omega_{cr} = 0.857$ rad/s, and associated critical length $L_c = 124.2$ m. It can be seen in figure 7–45 that the critical velocity (figure 7–45(a)) and frequency (figure 7–45(b)) obtained for a pipe of length $L = L_c$, describe the behaviour of an increasingly longer brine-string quite well.

The particular brine-string under consideration has a free hanging length of $L_{free} = 198$ m (table 7–10). Hence, $L_{free} > L_{cr}$ and as a result the predicted critical velocity $U_{cr}$ and frequency $\Omega_{cr}$, vary very little between $L = L_c = 124$ m , and $L = L_{free} = 198$ m.

It can be seen from figure 7–45, that a brine-string of length $L > L_{cr}$ can be effectively replaced with a clamped-free brine-string of length $L = L_{cr}$, without a significant change in the predicted critical values. A consequence of this is that one does not have to cope with the peculiarities of the boundary condition at the casing shoe.

In this configuration, the brine-string loses stability via flutter. For a short pipe $L = 10$ m flutter occurs in the second mode, for $L = L_c = 124.2$ m flutter occurs in the fourth mode, and for $L = 400$ m flutter occurs in the seventh mode. This is a result of the flutter instability taking place in a decreasing fraction of the pipe length, since $L_c$ remains the same while $L$ increases. As a result, progressively higher modes are required to properly localize the instability.

For this particular brine-string, the predicted critical velocity for flutter is $U_{cr} \approx 52$ m/s. This flow velocity is an order of magnitude greater than a typical flow velocity of 5 m/s encountered in the brine string of liquid product storage wells.
(Ratigan, 2008), making it unlikely that the free-hanging length of the brine-string will become unstable under normal operating conditions by simply discharging fluid.

### 5.2.2 Discharging brine-string, aspirating annulus

To analyse the behaviour of a discharging brine-string with an aspirating annulus (configuration (iii)), the brine-string was considered clamped at the well-head and free at the down-stream end. Therefore, the system dynamics for small lateral motions are described by equation (4.49) with associated boundary conditions (4.43), and equivalently in non-dimensional form by equation (4.57) with boundary conditions (4.54).

The brine string under consideration (table 7–10) has a total length \( L_{\text{total}} = 1283 \) m, and an annular length of \( L' = 1085 \) m. This is an annular ratio \( r_{\text{ann}} = L'/L = 0.85 \).

Attempts to solve the problem with the conventional Galerkin method, and clamped-free E-B eigenfunction comparison functions (section 4.10.2) proved unsuccessful, with accumulating numerical errors making convergence infeasible for a brine-string of length \( L > 450 \) m.

It was therefore decided to pursue an asymptotic analysis. Based on the results for the discharging unconfined brine-string, as well as the work of Doaré and de Langre (2002) and de Langre et al. (2007), it was possible that the system in this configuration would reach a point, past which the system length would no longer affect the critical velocity \( U_{\text{cr}} \) and frequency \( \Omega_{\text{cr}} \) for instability. Results were obtained for a given annular ratio \( r_{\text{ann}} \), by solving an incrementally longer system by the Galerkin method, as described in section 4.10.2, and convergence was achieved up
to a relative error of less than 5% in the non-dimensional eigenfrequencies ($\omega$) of the lowest 8 modes. This required between $N = 79$ E-B beam modes for $r_{ann} = 0.125$ and just $N = 20$ modes for $r_{ann} = 0.85$. The results are shown in figures 7–46 and 7–47, and a summary is given in table 7–11.

As the length of the system is increased, stability is lost first by flutter and finally by divergence, for all the annular ratios investigated. A look at figures 7–46(b), 7–46(d), 7–47(b), and 7–47(d) reveals that the circular frequency of vibration $\Omega_{cr}$ tends to zero for sufficiently long systems. This is the onset of divergence.

Average values of the critical internal flow velocities $U_i = U_{cr}$ after the onset of divergence (figures 7–46(a), 7–46(c), 7–47(a), and 7–47(c)), were used as representative critical velocities of the system (table 7–11) for a given $r_{ann}$ and can be expected to apply reasonably well to the full-length brine string.

As was mentioned above, the brine string under consideration has an annular ratio of $r_{ann} = 0.85$ which gives a predicted critical velocity of $U_{cr} = 27$ m/s for divergence. This velocity is approximately 50% of the predicted critical velocity $U_{cr} = 52$ m/s for flutter of the unconfined discharging brine-string. However, the predicted critical velocities are still much larger than the flow velocities encountered in the real application.

Furthermore, the effect of increasing the annular ratio $r_{ann}$ appears to be a slight stabilization for $0 < r_{ann} < 0.125$ and subsequently a destabilization of the system with increasing annular ratio for $0.125 < r_{ann} < 0.85$ (table 7–11).
CHAPTER 6
General conclusions and future work

6.1 Conclusions

The following overall conclusions stem from the experimental and theoretical work described in this thesis. These conclusions are based on a limited range of parameters, such as used in the theoretical and experimental work conducted, hence they may not apply generally.

- The potential for fluid-elastic instability depends strongly on the flow configuration. Experiments in a small-scale system revealed that in order of decreasing stability the flow configurations are as follows: (a) configuration (ii): aspirating pipe and no flow in the annulus, (b) configuration (i): discharging pipe and no flow in the annulus, (c) configuration (iii): discharging pipe with an aspirating annulus, (d) configuration (iv): discharging annulus and aspirating pipe.

  - The question of the exact nature of instability for (a), if any, remains unanswered due to limitations in our current apparatus. However, the vibration amplitudes for this flow configuration were between one and two orders of magnitude smaller than in the other flow configurations, for the same internal flow velocity $U_i$.

  - The presence of an external flow in (d) led to a profound decrease in stability, with critical internal flow velocities $U_i$ less than half those of the
all other configurations. This is despite the fact that the external flow velocity $U_o$ was only about 5% of the internal flow velocity.

The three-dimensional motion capture, revealed that the occurring vibrations are qualitatively very different in the various flow configurations, ranging from limit cycle motions with a single frequency, to random motions involving two or more frequencies.

- Impacting (and is some cases rubbing) of the tube with the surrounding simulated casing (rigid plexiglas pipe) was observed in all configurations except in the case of an aspirating pipe with no flow in the annulus. The effects of impacting are twofold.
  
  (i) When an instability occurs, the onset of impacting introduces randomness into the system behaviour giving rise to multiple frequency components and altering the orientation and amplitude of the vibrations.
  
  (ii) For a sufficiently long annulus, impacting can disrupt the growth of an instability, effectively limiting the amplitude of vibration.

- For the theoretically investigated flow configurations—namely a discharging pipe, with or without aspirating flow in the annulus—linear theory correctly predicted the type of instability that was observed. Qualitative agreement between the predicted and experimental critical flow velocities $u_i$, was within 14% for thin-walled tubes, and 37% for thick-walled tubes. It is thought that the poor agreement in the case of thick-walled tubes may be related to the action of the flow on the ring-shaped cross-sectional area of the tube $(A_o - A_f)$ at its free end, which is large enough to have a significant effect on stability.
(see for example Giacobbi et al. (2012)). This discrepancy is expected to be much smaller in the case of brine-strings, which are essentially, very thin-walled tubes, as $D_o \approx D_i$.

- The critical velocity $U_{cr}$ and associated circular frequency $\Omega_{cr}$ of the theoretically investigated flow configurations—namely a discharging pipe, with or without aspirating flow in the annulus—were shown to approach limiting values with increasing system length. This is consistent with the instability being confined to a downstream region of the pipe with length equal to a certain critical length $L_c$. For a discharging and unconfined pipe, an analytical expression for the critical length $L_c$ was derived, and the results were in agreement with the work of Doaré and de Langre (2002).

- It was shown that for systems longer than the critical length, only the downstream region of length $L_c$ need be considered, with clamped-free boundary conditions. In some cases this removes considerations about the exact nature of the boundary condition at the casing shoe or at the well-head. In addition, computational complexity is reduced, since convergence of the solutions for short systems is significantly easier.

- In our theoretical investigations of the simulated brine string, the presence of external annular flow was estimated to greatly reduce the critical flow velocities (by as much as 50% for the specific brine-string under consideration), and hence significantly destabilize the system. In addition, the type of instability (whether flutter or divergence) was greatly influenced by the amount of confinement, as
well as the presence, or not, of aspirating annular flow. This is consistent with
the predictions made by Paidoussis et al. (2008) on simulated drill-strings.

6.2 Future work

Looking back at the work presented in this thesis, there are several areas that
can be improved, and merit more work.

To begin with, it would be useful to expand the theoretical investigation to
include the configurations of an aspirating pipe with no flow in the annulus (for
which several models have been proposed), as well as the aspirating pipe with a
discharging annulus. It is recalled that the latter configuration proved to be the
most unstable in experiments.

Furthermore, it would be useful to pursue computational models and flow simu-
lations to aid in the estimation of the parameters used in the analysis. This approach
has already been implemented by Giacobbi et al. (2012), and has been shown to im-
prove the applicability of linear analysis without having to resort to more complex
techniques.

Practical improvements to the experimental apparatus, should be directed along
two main avenues:

Firstly, the internal end external counter-current flows need to be controlled
independently. In this way, the external flow velocity can be increased and play a
bigger role in the system dynamics, while avoiding the complexities of a very narrow
annulus, which tends to come in contact with the flexible tube thus modifying the
system support conditions.
Secondly, given the inconclusive nature of the experimental results of the aspirating pipe, appropriate modifications of the experimental apparatus could allow use of higher pressures. Consequently, higher flow velocities would also be possible, in order to further explore the nature of the instability.
CHAPTER 7
Tables and Figures

7.1 Tables

Table 7–1: Pipe properties

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Tube 1</th>
<th>Tube 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Material</td>
<td>silicone (Silastic RTV)</td>
<td>rubber (Santoprene)</td>
</tr>
<tr>
<td>$D_i$ [mm]</td>
<td>6.35</td>
<td>6.35</td>
</tr>
<tr>
<td>$D_o$ [mm]</td>
<td>16</td>
<td>9.525</td>
</tr>
<tr>
<td>Length [mm]</td>
<td>431</td>
<td>443</td>
</tr>
<tr>
<td>$EI$ [N·m²]</td>
<td>$7.37 \times 10^{-3}$</td>
<td>$9.33 \times 10^{-3}$</td>
</tr>
<tr>
<td>Density [kg·m⁻³]</td>
<td>$1.13 \times 10^3$</td>
<td>$1.03 \times 10^3$</td>
</tr>
<tr>
<td>First mode frequency (in air) [Hz]</td>
<td>1.13</td>
<td>1.66</td>
</tr>
</tbody>
</table>

Table 7–2: Annulus dimensions

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Annulus 10 cm</th>
<th>Annulus 20 cm</th>
<th>Annulus 30 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{ch}$ [mm]</td>
<td>31.5</td>
<td>31.5</td>
<td>31.5</td>
</tr>
<tr>
<td>Length ($L'$) [mm]</td>
<td>109&lt;sup&gt;a&lt;/sup&gt;</td>
<td>206.5&lt;sup&gt;a&lt;/sup&gt;</td>
<td>304.5&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>%Confined length (Tube 1)</td>
<td>25.3</td>
<td>47.8</td>
<td>70.5</td>
</tr>
<tr>
<td>%Confined length (Tube 2)</td>
<td>24.6</td>
<td>46.7</td>
<td>68.8</td>
</tr>
<tr>
<td>Annular gap [mm] (Tube 1)</td>
<td>7.75</td>
<td>7.75</td>
<td>7.75</td>
</tr>
<tr>
<td>Annular gap [mm] (Tube 2)</td>
<td>11</td>
<td>11</td>
<td>11</td>
</tr>
<tr>
<td>$A_{ch}/A_f$ (Tube 1)</td>
<td>18.26</td>
<td>18.26</td>
<td>18.26</td>
</tr>
<tr>
<td>$A_{ch}/A_f$ (Tube 2)</td>
<td>22.35</td>
<td>22.35</td>
<td>22.35</td>
</tr>
</tbody>
</table>

<sup>a</sup> Average values. Effective length of confinement varies slightly with tube installation.
Table 7–3: Critical flow velocities and frequencies for Configuration (i)

<table>
<thead>
<tr>
<th></th>
<th>Tube 1</th>
<th></th>
<th>Tube 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_{f2}$ [m/s]</td>
<td>$f_{f2}$ [Hz]</td>
<td>$U_{f2}$ [m/s]</td>
<td>$f_{f2}$ [Hz]</td>
</tr>
<tr>
<td>Unconfined</td>
<td>5.77</td>
<td>1.63</td>
<td>7.47</td>
<td>2.75</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>5.67</td>
<td>1.60</td>
<td>7.67</td>
<td>2.66</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>5.41</td>
<td>1.46</td>
<td>7.51</td>
<td>2.56</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>5.70</td>
<td>1.39</td>
<td>7.49</td>
<td>2.50</td>
</tr>
</tbody>
</table>

Table 7–4: Critical flow velocities and frequencies for Configuration (ii)

<table>
<thead>
<tr>
<th></th>
<th>Tube 1</th>
<th></th>
<th>Tube 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_f$ [m/s]</td>
<td>$f_f$ [Hz]</td>
<td>$U_f$ [m/s]</td>
<td>$f_f$ [Hz]</td>
</tr>
<tr>
<td>Unconfined</td>
<td>&gt; 7.58</td>
<td>0.37</td>
<td>&gt; 8.4</td>
<td>0.40</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>&gt; 7.58</td>
<td>0.37</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>&gt; 7.58</td>
<td>0.35</td>
<td>&gt; 8.4</td>
<td>0.41</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>&gt; 7.58</td>
<td>0.37</td>
<td>&gt; 8.4</td>
<td>0.41</td>
</tr>
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</table>

Table 7–5: Critical (internal) flow velocities and frequencies for Configuration (iii)

<table>
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<tr>
<th></th>
<th>Tube 1</th>
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<tr>
<td></td>
<td>$U_{f2}$ [m/s]</td>
<td>$f_{f2}$ [Hz]</td>
<td>$U_{f2}$ [m/s]</td>
<td>$f_{f2}$ [Hz]</td>
</tr>
<tr>
<td>Unconfined</td>
<td>5.77</td>
<td>1.63</td>
<td>7.47</td>
<td>2.75</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>5.68</td>
<td>1.56</td>
<td>7.65</td>
<td>2.63</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>5.78</td>
<td>1.44</td>
<td>7.27</td>
<td>2.48</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>5.24</td>
<td>1.12</td>
<td>7.31</td>
<td>2.03</td>
</tr>
</tbody>
</table>

Table 7–6: Critical (internal) flow velocities and frequencies for configuration (iv)

<table>
<thead>
<tr>
<th></th>
<th>Tube 1</th>
<th></th>
<th>Tube 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_{f1}$ [m/s]</td>
<td>$f_{f1}$ [Hz]</td>
<td>$U_{f1}$ [m/s]</td>
<td>$f_{f1}$ [Hz]</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>1.25 − 2.21</td>
<td>0.45</td>
<td>1.35 − 3.16</td>
<td>0.64</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>&lt; 2.21</td>
<td>0.47</td>
<td>0.54 − 3.16</td>
<td>0.73</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>1.83 − 2.21</td>
<td>0.44</td>
<td>&lt; 3.16</td>
<td>0.78</td>
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</tbody>
</table>
Table 7–7: Non-dimensional pipe properties for laboratory set-up

<table>
<thead>
<tr>
<th>Identifier</th>
<th>Tube 1</th>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>0.397</td>
<td>0.666</td>
</tr>
<tr>
<td>$\alpha_{ch}$</td>
<td>1.97</td>
<td>3.31</td>
</tr>
<tr>
<td>$\beta_i$</td>
<td>$7.41 \times 10^{-2}$</td>
<td>0.22</td>
</tr>
<tr>
<td>$\beta_o$</td>
<td>0.470</td>
<td>0.496</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>2.69</td>
<td>0.104</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>26.9</td>
<td>46.5</td>
</tr>
<tr>
<td>$\chi$</td>
<td>1.70</td>
<td>1.20</td>
</tr>
<tr>
<td>$h$</td>
<td>1.03</td>
<td>0.434</td>
</tr>
<tr>
<td>$EI$ [N·m²]</td>
<td>$7.37 \times 10^{-3}$</td>
<td>$9.33 \times 10^{-3}$</td>
</tr>
<tr>
<td>$M_t$ [kg·m⁻¹]</td>
<td>0.194</td>
<td>4.07×10⁻²</td>
</tr>
<tr>
<td>$r_{ann}$, Annulus 10 cm</td>
<td>0.253</td>
<td>0.246</td>
</tr>
<tr>
<td>$r_{ann}$, Annulus 20 cm</td>
<td>0.478</td>
<td>0.467</td>
</tr>
<tr>
<td>$r_{ann}$, Annulus 30 cm</td>
<td>0.705</td>
<td>0.688</td>
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Table 7–8: Theoretical results and comparison with experiment for configuration (i)

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Tube 1</th>
<th>Tube 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_i$ $\omega$</td>
<td>$u_i$ $\omega$</td>
</tr>
<tr>
<td>Unconfined</td>
<td>5.16 14.47</td>
<td>6.09 13.32</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>5.06 14.21</td>
<td>6.26 12.86</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>4.83 13.03</td>
<td>6.12 12.40</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>5.09 12.38</td>
<td>6.10 12.11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theory</th>
<th>Tube 1</th>
<th>Tube 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_i$ $\omega$</td>
<td>$u_i$ $\omega$</td>
</tr>
<tr>
<td>Unconfined</td>
<td>6.70 14.08</td>
<td>6.47 13.47</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>6.70 14.00</td>
<td>6.47 13.46</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>6.72 13.32</td>
<td>6.47 13.26</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>6.80 12.43</td>
<td>6.46 13.00</td>
</tr>
</tbody>
</table>
Table 7–9: Theoretical results and comparison with experiment for configuration (iii)

<table>
<thead>
<tr>
<th></th>
<th>Tube 1</th>
<th></th>
<th>Tube 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_i$</td>
<td>$\omega$</td>
<td>$u_i$</td>
<td>$\omega$</td>
</tr>
<tr>
<td>Unconfined</td>
<td>5.16</td>
<td>14.47</td>
<td>6.09</td>
<td>13.32</td>
</tr>
<tr>
<td>Annulus 10 cm</td>
<td>5.08</td>
<td>13.88</td>
<td>6.24</td>
<td>12.72</td>
</tr>
<tr>
<td>Annulus 20 cm</td>
<td>5.17</td>
<td>12.84</td>
<td>5.93</td>
<td>11.97</td>
</tr>
<tr>
<td>Annulus 30 cm</td>
<td>4.69</td>
<td>9.97</td>
<td>5.96</td>
<td>9.81</td>
</tr>
</tbody>
</table>

|                 | Tube 1 |            | Tube 2 |            |
|                | $u_i$  | $\omega$  | $u_i$  | $\omega$  |
| Unconfined     | 6.70   | 14.08      | 6.47   | 13.47      |
| Annulus 10 cm  | 6.69   | 14.18      | 6.44   | 13.45      |
| Annulus 20 cm  | 6.47   | 13.40      | 6.27   | 13.06      |
| Annulus 30 cm  | 6.44   | 10.80      | 6.15   | 12.48      |

Table 7–10: Brine-string properties and associated non-dimensional parameters

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_i$ [m]</td>
<td>0.159</td>
<td>$\alpha$</td>
<td>0.897</td>
</tr>
<tr>
<td>$D_o$ [m]</td>
<td>0.1778</td>
<td>$\alpha_{ch}$</td>
<td>1.676</td>
</tr>
<tr>
<td>$D_{ch}$ [m]</td>
<td>0.298</td>
<td>$\chi$</td>
<td>2.106</td>
</tr>
<tr>
<td>$L_{total}$ [m]</td>
<td>1283</td>
<td>$\beta_i$</td>
<td>0.239</td>
</tr>
<tr>
<td>$L'$ [m]</td>
<td>1085</td>
<td>$\beta_o$</td>
<td>0.297</td>
</tr>
<tr>
<td>$L_{free}$ [m]</td>
<td>198</td>
<td>$h$</td>
<td>1.479</td>
</tr>
<tr>
<td>$EI$ [N·m²]</td>
<td>$3.47 \times 10^6$</td>
<td>$\gamma^a$</td>
<td>$2.019 \times 10^5$</td>
</tr>
<tr>
<td>$M_t$ [kg·m⁻¹]</td>
<td>38.7</td>
<td>$r_{ann}$</td>
<td>0.85</td>
</tr>
</tbody>
</table>

$^a$ Based on $L_{total}$.  

104
Table 7–11: Theoretical results for the brine-string

<table>
<thead>
<tr>
<th>Instability</th>
<th>$r_{ann} = 0$</th>
<th>$r_{ann} = 0.125$</th>
<th>$r_{ann} = 0.25$</th>
<th>$r_{ann} = 0.5$</th>
<th>$r_{ann} = 0.85^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$U_{cr}$ [m/s]</td>
<td>$\Omega_{cr}$ [rad/s]</td>
<td>$L_c$ [m]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Flutter</td>
<td>52</td>
<td>0.86</td>
<td>124.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Divergence</td>
<td>59</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>45</td>
<td>0</td>
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<td></td>
<td>33</td>
<td>0</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>27</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$a$ Average value for increasingly longer brine-string, up to $L = 450$ m.

$b$ No theoretical expression for calculation of $L_c$ for this configuration.

$c$ Actual annular ratio of the brine-string under consideration.
Figure 7–1: Schematic of a salt cavern used for hydrocarbon storage. (a) Storage of liquid hydrocarbon. (b) Storage of gas (either hydrocarbon, compressed air, or carbon dioxide).
Figure 7–2: (a) Schematic of the articulated cantilevered tube considered by Benjamin (1961a,b). (b), Schematic of the continuously flexible cantilevered tube investigated by Gregory and Paidoussis (1966a,b).

Figure 7–3: Photos of the Dracone barge reproduced from Hawthorne (1961). (a) A Dracone barge being towed in the vicinity of the New York harbor. (b) Snapshots of a scale model of the Dracone barge over a period of oscillation.
Figure 7–4: Schematic of the simplified ocean mining system as considered by Paidoussis and Luu (1985). The system consists of a hanging, aspirating, tubular cantilever with an end-mass.
Figure 7–5: The investigated flow configurations: (a), tube discharging in the tank and no flow in the annulus dimensions; (b), tube aspirating and no flow in the annulus; (c), tube discharging in the tank and the annulus aspirating fluid, (d), tube aspirating fluid and the annulus discharging in the tank.
Figure 7–6: Layout of the experimental apparatus. (a) Simple schematic of the apparatus test-section. The flow configuration shown in detail corresponds to a discharging pipe and an aspirating annulus. (b) The real-life counterpart. The conventional Bourdon pressure gauge and bleed lines visible to the right of the apparatus.
Figure 7–7: Close-up of the experimental test-section, showing respectively: (a) the 10 cm, (b) 20 cm, and (c) 30 cm annuli installed over tube 1. These are nominal values, and the actual annular dimensions are available in table 7–2.
Figure 7–8: The Optron system in operation. (a) The system is tracking motions inside the test-section. The tracking head is visible in the left side of the image, while the amplifier is visible in the bottom centre. (b) Optron real-time output, captured with an analog-to-digital converter.
Figure 7–9: Dual camera system arrangement. (a) Simple schematic. The directions of observation are identified as “Front” and “Side”. (b) The cameras are trained onto tube 1 in the test-section.
Figure 7–10: (a) Diagram of the marker locations and definition of the tracked point 1, 2, and 3; (b), the markers as they appear at the free end of tube 1. The bright white semicircle visible near the top of the image is the reflection of the illumination source in the rear window of the test-section.
Figure 7–11: Problem set-up and forces in the system: (a), layout of the tube and annulus including main dimensions; (b), forces acting on an element of length $\delta x$ of the tube; (c), forces due to the outside fluid acting on the external surface of the tube element; (d), forces acting on an element of internally flowing fluid of length $\delta x$. 

\[ \text{(a)} \]

\[ \text{(b)} \]

\[ \text{(c)} \]

\[ \text{(d)} \]
Figure 7–12: Pressure, frictional, and gravity forces acting on an annular fluid element.
Figure 7–13: Configuration (i), Tube 1. Determination of the critical flow velocity for second-mode flutter $U_{f2}$. (a) unconfined tube; (b),(c) and (d), tube with annuli of increasing length. In (c) and (d), impacting limits the amplitude of vibration.
Figure 7–14: Configuration (i), Tube 2. Determination of the critical flow velocity for second-mode flutter $U_{f2}$. (a), unconfined tube; (b),(c) and (d) tube with annuli of increasing length. In (d), the onset of impacting visibly curtails the growth of the amplitude of vibration.
Figure 7–15: Configuration (i), Tube 2 with the 30 cm annulus installed. System behaviour just before the onset of second-mode flutter. Internal flow velocity of $U_i = 6.95$ m/s. The absence of a well defined plane of vibration is evident in (a). The power spectral density (b) shows that a certain frequency of vibration is emerging, but it is not yet definite. The time series of motion (c) and (d), as well as the phase portraits from each direction of observation, (e) and (f), further reinforce the notion that the system is not executing a well-defined periodic motion.
Figure 7–16: Configuration (i), Tube 2 with the 30 cm annulus installed. System in second-mode flutter. Internal flow velocity of $U_i = 7.89$ m/s. The power spectral density (b) shows a sharp peak at the second-mode frequency of 2.37 Hz. The three-dimensional trajectory of the motion (d) shows a quasi-planar vibration, with a slowly preceding plane of symmetry (a). The phase portraits from each direction of observation, (e) and (f), are approximately elliptic, indicating a possible limit-cycle oscillation. This is in agreement with (b).
Figure 7–17: Configuration (i), Tube 2 with the 30 cm annulus installed. Internal flow velocity of $U_i = 8.68$ m/s. The tube alternatively impacts or moves tangent to the inner surface of the annulus (a). The power spectral density (b) shows the emergence of multiple peaks in addition to a dominant frequency of vibration. The overall amount of noise in the signal is noticeably high. The phase portraits (f) and (e) are heavily distorted. This shows that the motion is not simple periodic, but potentially contains multiple frequencies.
Figure 7–18: Configuration (i), Tube 1 with the 30 cm annulus installed. Behaviour of the system before the onset of second-mode flutter. Internal flow velocity of $U_i = 5.37$ m/s. The system trajectory (d) indicates three-dimensional motions of random orientation (a). The power spectral density (b) shows a lack of a dominant frequency, further supported by (c). The phase portraits from each direction of observation, (e) and (f), further attest to the aperiodic response of the system.
Figure 7–19: Configuration (i), Tube 1 with the 30 cm annulus installed. System in second-mode flutter. Internal flow velocity of $U_i = 6.16$ m/s. Vibration is planar (d), in a slowly preceding plane (a). The power spectral density (b) shows a sharp peak at 1.39 Hz. Phase portraits from each direction of observation, (e) and (f), have elliptic envelopes, indicating an underlying stable limit cycle motion. The observed spiralling of the phase plots is predominantly due to the precession of the plane of vibration. Some non regular amplitude modulation is observed in the plane of symmetry (a).
Figure 7–20: Configuration (i), Tube 1 with the 30 cm annulus installed. Internal flow velocity of $U_i = 8.21$ m/s. The specimen impacts the inner surface of the annulus. The power spectral density (b) shows a very high level of noise in the signal, and multiple broad peaks. The motion is fully three-dimensional with no well defined plane of symmetry (a). The phase portraits (f) and (e) are heavily distorted. The motion is not simple periodic.
Figure 7–21: Configuration (i), Tube 1 with the 30 cm annulus installed. System in second-mode flutter. Internal flow velocity of $U_i = 6.16$ m/s. The motion is projected onto the principal plane of vibration. The phase portraits from either direction of observation are combined to reveal an ellipse. This reveals the underlying stable limit-cycle oscillation at a frequency of 1.39 Hz.
Figure 7–22: Configuration (i). Effect of increasing the annulus length on: (a) the critical velocity for second-mode flutter $U_{f2}$; and (b) the critical frequency of vibration $f_{f2}$.

Figure 7–23: Configuration (ii). Rms vibration amplitude versus internal flow velocity $U_i$ for both tube 1 (a), and tube 2 (b).
Figure 7–24: Configuration (ii), Tube 1 with the 30 cm annulus installed. Internal flow velocity of $U_i = 7.58$ m/s. Very low amplitude vibration. The power spectral density (b) reveals a vibration frequency of 0.35 Hz. The motion is three-dimensional (d), with no well-defined plane of symmetry (a). The phase portraits, (f) and (e), show that the motion is far from simple periodic.
Figure 7–25: Configuration (ii), Tube 2 with the 30 cm annulus installed. Internal flow velocity of $U_i = 8.21 \text{ m/s}$. The motion is of very low amplitude (c). The power spectral density (b) shows a vibration frequency of 0.41 Hz. The motion has no well defined plane of symmetry (a). The phase portraits (f) and (e), are extremely distorted, indicating that the motion is far from simple periodic.
Figure 7–26: Configuration (ii), static deflection versus internal flow velocity $U_i$: (a), for Tube 1; (b), for Tube 2.
Figure 7–27: Configuration (iii), Tube 1. Determination of the critical flow velocity for second-mode flutter $U_{f2}$. (a), (b) and (c), tube with annuli of increasing length. The onset of impacting limits the vibration amplitude in (c) and (d).
Figure 7–28: Configuration (iii), Tube 2. Determination of the critical flow velocity for second-mode flutter $U_{f2}$. (a), (b) and (c), tube with annuli of increasing length. In (c) the onset of impacting limits the vibration amplitude.
Figure 7–29: Configuration (iii). Effect of increasing the annulus length on: (a) the critical internal flow velocity for second-mode flutter $U_{f2}$; and (b) the critical frequency of vibration $f_{f2}$.
Figure 7–30: Configuration (iii), Tube 1 with the 10 cm annulus. Internal flow velocity of $U_i = 6.0$ m/s, external flow $U_o = 0.33$ m/s. System in second-mode flutter. The power spectral density (b) shows a sharp peak at 1.56 Hz. The motion is planar (a) in a slowly rotating plane. The phase portraits, (f) and (e), have an elliptic envelope, suggesting a simple periodic motion.
Figure 7–31: Configuration (iii), Tube 2 with the 10 cm annulus. System in second-mode flutter. Internal flow velocity of $U_i = 7.89$ m/s and external flow of $U_o = 0.35$ m/s. The power spectral density (b) shows a sharp peak at 2.56 Hz. The trajectory of the motion (d) reveals an almost planar vibration, with a preceding plane of symmetry (a). The phase portraits from each direction of observation, (e) and (f), have an elliptic envelope, indicating an underlying regular periodic motion.
Figure 7–32: Configuration (iii), Tube 1 with the 20 cm annulus. Internal flow velocity of $U_i = 6.47 \text{ m/s}$, external flow $U_o = 0.35 \text{ m/s}$. System in second-mode flutter. The power spectral density (b) shows a (relatively wide) peak at 1.25 Hz. (d) indicates a departure from planar motion, with a plane of symmetry rotating in an uncertain manner (a). The phase portraits (f) and (e), reveal some distortion, a result of the wandering the plane of symmetry.
Figure 7–33: Configuration (iii), Tube 2 with the 20 cm annulus. System in second-mode flutter. Internal flow velocity of $U_i = 7.58$ m/s and external flow of $U_o = 0.34$ m/s. The power spectral density (b) shows a wide peak at 2.56 Hz. The trajectory of the motion (d) reveals strongly three-dimensional motions, with an unpredictably varying plane of symmetry (a). The phase portraits from each direction of observation (e) and (f) are strongly distorted, revealing a non regular motion.
Figure 7–34: Configuration (iv), Tube 1. Determination of the critical flow velocities for first mode flutter $U_{f1}$. Amplitude data are fitted with a line or second order polynomial. (b) illustrates problems of estimating the critical velocity for flutter by such a fit. Effects of impacting with the annulus are visible in (b) and (c).
Figure 7–35: Configuration (iv), Tube 2. Determination of the critical flow velocities for first mode flutter $U_{f1}$. Amplitude data are fitted with a line or second order polynomial. (c) illustrates the shortcomings of estimating the critical velocity in this manner, as it results in either negative, or nonsensical values. Effects of impacting with the annulus are visible in (b) and (c).
Figure 7–36: Configuration (iv), Tube 1 with the 10 cm annulus installed. Internal flow velocity $U_i = 2.21$ m/s and external flow velocity $U_o = 0.12$ m/s. The tube is fluttering in the first mode. The power spectral density (b) shows a single dominant peak $f_1$ at 0.44 Hz. The motion is fully three-dimensional (d), with no well-defined plane of symmetry (a). The phase portraits (f) and (e) are far from elliptic, a consequence of the random orientation of vibration.
Figure 7–37: Configuration (iv), Tube 1 with the 10 cm annulus installed. Internal flow velocity $U_i = 6.32$ m/s and external flow velocity $U_o = 0.35$ m/s. The system is fluttering in both the first and second modes. The power spectral density (b) shows a dominant peak at the first-mode frequency ($f_1$) of 0.51 Hz and an additional peak at the second-mode frequency ($f_2$) of 2.78 Hz. The motion is fully three-dimensional (d), with a randomly preceding plane of symmetry (a). The phase portraits, (f) and (e), illustrate the effects of the randomness of the vibration.
Figure 7–38: Configuration (iv), Tube 2 with the 10 cm annulus installed. Internal flow velocity $U_i = 8.4$ m/s and external flow velocity $U_o = 0.375$ m/s. The motion is fully three-dimensional (d), with a random plane of symmetry (a). The power spectral density (b) shows a clear peak at the first-mode frequency of 0.6 Hz. The phase portraits (f), and (e), are very distorted, a consequence the randomness of the vibration.
Figure 7–39: Configuration (iv), Effect of increasing the annulus length on the frequency of vibration $f_{f1}$ for first-mode flutter.

Figure 7–40: Configuration (i), Tube 1 with the three available annuli. Critical internal non-dimensional flow velocity $u_i$ (a), and associated non-dimensional circular frequency $\omega$ (b), for second-mode flutter. Comparison of experimental results, and theoretical results with and without down-stream end de-pressurization.
Figure 7–41: Configuration (i), Tube 2 with the three available annuli. Critical internal non-dimensional flow velocity $u_i$ (a), and associated non-dimensional circular frequency $\omega$ (b), for second-mode flutter. Comparison of experimental results, and theoretical results with and without down-stream end de-pressurization.

Figure 7–42: Configuration (iii), Tube 1 with the three available annuli. Critical internal non-dimensional flow velocity $u_i$ (a), and associated non-dimensional circular frequency $\omega$ (b), for second-mode flutter. Comparison of experimental results, and theoretical results with and without down-stream end de-pressurization.
Figure 7–43: Configuration (iii), Tube 2 with the three available annuli. (a), Critical internal non-dimensional flow velocity $u_i$, and (b) associated non-dimensional circular frequency $\omega$ for second-mode flutter. Comparison of experimental results, and theoretical results with and without down-stream end de-pressurization.
Figure 7–44: Schematic and definitions for the brine-string under investigation.
Figure 7–45: Theoretical results for an unconfined discharging brine-string and no flow in the annulus (configuration (i)), as a function of the tube length $L$. Stability is lost via flutter. (a), Dimensional critical flow velocity $U_{cr}$; (b), dimensional critical circular frequency $\Omega_{cr}$. (●), Numerical results; (- - -), $U_{cr}$ and $\Omega_{cr}$ based on the critical length $L_c = 124.2$ m calculated as proposed in Doaré and de Langre (2002).
Figure 7–46: Theoretical results, brine string discharging and annulus aspirating fluid, as a function of the tube length $L$. (a),(c), Dimensional critical flow velocity $U_{cr}$; (b),(d), dimensional critical circular frequency $\Omega_{cr}$. (●), Numerical results; (---), average values of $U_{cr}$ and $\Omega_{cr}$, for increasing $L$. In (b) and (d), instability is flutter when $\Omega_{cr} \neq 0$ and static divergence when $\Omega_{cr} = 0$. 

(a) $r_{ann} = 0.125$

(b) $r_{ann} = 0.125$

(c) $r_{ann} = 0.25$

(d) $r_{ann} = 0.25$
Figure 7–47: Theoretical results, brine string discharging and annulus aspirating fluid, as a function of the tube length $L$. (a),(c), Dimensional critical flow velocity $U_{cr}$; (b),(d) dimensional critical circular frequency $\Omega_{cr}$. (●), Numerical results; (---), average values of $U_{cr}$ and $\Omega_{cr}$ for $L$ large. In (b) and (d), instability is flutter when $\Omega_{cr} \neq 0$ and static divergence when $\Omega_{cr} = 0$. 

(a) $r_{ann} = 0.5$

(b) $r_{ann} = 0.5$

(c) $r_{ann} = 0.85$

(d) $r_{ann} = 0.85$
REFERENCES


Dyson Ltd. Dyson cinetic dc54; dyson’s most advanced vacuum cleaner technology, November 2014. URL https://www.youtube.com/watch?v=G1GwoT2QXbo.


S. Rinaldi. Experiments on the dynamics of cantilevered pipes subjected to internal and/or external axial flow. Master’s thesis, Department of Mechanical Engineering, McGill University, Montreal, Quebec, Canada, 2009.


D.S. Weaver. On the non-conservative nature of “gyroscopic conservative” systems.