ESTIMATION OF THE PICKANDS
DEPENDENCE FUNCTION FOR BIVARIATE
ARCHIMAX COPULAS

SIMON CHATELAIN

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Abstract

This M.Sc. thesis contributes to the use of Archimax copulas to model bivariate extremes. After a review of Archimedean and extreme-value copulas, Archimax copulas are defined and two corresponding estimators of the Pickands dependence function are proposed. These estimators require no knowledge of the margins, but the Archimedean generator is assumed to be known and to have certain properties of regular variation. Within these imposed conditions, the asymptotic behavior of the estimators is established and related to empirical copula processes and limiting Brownian processes. A simulation study is then conducted, using copulas that fulfill the requirements for consistency as well as copulas that do not. Many questions arise from Chapters 4 and 5, clearly showing where more work on the estimation of Archimax copulas is needed.
Résumé

Ce mémoire contribue à l’utilisation des copules Archimax pour modéliser les données extrêmes bi-variées. Après une revue des copules Archimédiennes et max-stables, les copules Archimax sont définies. Deux estimateurs pour la fonction de dépendance de Pickands sont proposés. Ces estimateurs ne requièrent aucun savoir sur les lois marginales, mais on doit supposer la connaissance du générateur Archimédien et imposer des conditions de variation régulière. Le comportement asymptotique de ces estimateurs est établi, en utilisant des outils tels que le processus de copule empirique et les processus Browniens. Une étude de simulation est menée, en utilisant des copules qui remplissent ou non les conditions nécessaires à la convergence des estimateurs. De cette étude il est évident où plus de travail de recherche est requis pour l’ajustement des copules Archimax.
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Copulas are becoming increasingly popular in multivariate analysis. They provide a complete description of the dependence structure that ties random variables together, and as such are employed in many applications of statistics. For example, hydrology, meteorology, climatology, finance and economics are sciences that often require the joint modeling of several quantities. There is an extensive and growing literature on copula modeling techniques; book treatments include Joe (2014) and Nelsen (2006). Inference techniques are summarized, e.g., in articles such as Genest & Favre (2007) and Genest & Nešlehová (2012).

The analysis of extremes is also an area of statistics and probability which is widely used in applications and currently growing. Natural catastrophes such as floods, hurricanes and heat waves can be seen as the occurrence of a random variable’s extreme value. The same can be said for large losses in a stock’s return. The main difficulty in the study of extremes is the fact that in practice there is very little data to analyze, since these events are by definition rare. This has led to developments in asymptotic theory that captures the tails of distributions as accurately as possible. See Gnedenko (1943) for important results regarding the Generalized Extreme Value (GEV) family. For the “peaks over threshold” (POT) method, see Balkema & de Haan (1974) and Pickands (1975). Comprehensive books on extreme-value theory include, for example, Coles (2001), Embrechts et al. (1997) and de Haan & Ferreira (2006).
Chapter 1. Introduction

Combining the previously mentioned areas, multivariate extremes is the subject of this thesis. In the applications already provided, extreme events are more often than not characterized by several variables jointly exceeding a high threshold. A financial crisis is due to several components crashing, and a hurricane occurs for specific temperature, pressure and wind speed conditions.

Max-stable copulas describe the dependence structure of extreme values, and are usually fitted to samples of component-wise maxima. It is at this step that statisticians make the so-called “leap of faith”, where asymptotic theory is used for inference in finite samples. In practice, there are difficulties with fitting extreme-value copulas to extreme data because we find ourselves in a pre-asymptotic setting. Archimax copulas, introduced by Capéraà et al. (2000), can help fix this problem. As a large class that generalizes both extreme-value copulas and Archimedean copulas, they can provide the extra flexibility needed in applications.

This thesis contributes to modeling techniques for multivariate extremes, specifically to Archimax copulas. Very little work has been done on inference for these models. Thanks to their construction, inspiration can be drawn from what is known about inference for Archimedean and extreme-value copulas. In this thesis, two new estimators of the so-called Pickands dependence function are proposed in the bivariate setting. These estimators are rank-based and therefore do not require knowledge of the marginal distributions. In a sense, they could be qualified as semi-parametric since they assume the knowledge of the Archimedean generator of the Archimax copula.

In Chapter 2, important definitions and concepts are reviewed. First, fundamental results such as the Decomposition Theorem of Sklar (1959) are presented. The motivation and definition of extreme-value copulas are presented, where the crucial Pickands dependence function is defined. Two non-parametric estimators from Genest & Segers (2009) for this function are presented as well as their limiting distribution. The Archimedean copula class is widely used in dependence modeling due to its flexibility. Important results related to its stochastic representation are given, as well as a random number generation algorithm and examples. Finally, we go over important notions of regular variation which will be used in Chapter 4.
By Chapter 3 we have the necessary elements from which we can define Archimax copulas. Indeed, this class of copulas is defined by an Archimedean generator and a stable tail dependence function (or Pickands dependence function). Many important results due to Charpentier et al. (2014) and Capéraà et al. (2000) are presented, particularly regarding the stochastic representation. This leads us to an algorithm that allows us to draw random observations from bivariate Archimax copulas. Finally, we conclude this chapter with examples and plots of data drawn from Archimax copulas. These examples of Archimax copulas will be used again in the simulation study in Chapter 5.

Two Pickands dependence function estimators for bivariate Archimax copulas are proposed and studied in depth in Chapter 4. Assuming the knowledge of the Archimedean generator means that we can impose certain conditions of regular variation at zero and infinity. This allows for two estimators whose uses are complementary since the conditions required for their asymptotic behavior are different. The case of generators with a finite support is also studied. This chapter establishes consistency and weak convergence of the estimators to a Gaussian process.

Chapter 5 consists of an extensive Monte-Carlo type simulation study in which the Mean Integrated Square Error of the two different estimators studied in Chapter 4 is compared. The study includes data simulated from copulas that are known and whose properties are easily tuned by altering the values of their parameters. This allows us to explore cases in which the estimators are consistent by the results obtained in Chapter 4 as well as cases where the conditions from Chapter 4 do not apply. This leads to interesting comparisons between families and shapes of the Pickands dependence function.

Chapter 6 concludes the thesis with a summary of the results and findings in this thesis. Many different paths of research are outlined, resulting from both the assumptions made in Chapter 4 and the simulation study in Chapter 5.
2 Background Material

2.1 Copulas

The joint distribution of several random variables contains the marginal effects as well as the dependence structure which ties them together. While the univariate marginal distributions will describe each individual variable, they do not give any information on how the variables behave together. In general, this behaviour cannot be entirely explained by a single measure such as linear correlation. Instead, copulas provide all the information about the dependence structure with marginal effects eliminated. In this thesis we will be limiting ourselves to the bivariate case. Note that all the results in this section, including Sklar's theorem, can be extended to an arbitrary number of dimensions.

Definition 2.1.1. A bivariate copula is the distribution function of a random vector \((U_1, U_2)\), where \(U_1\) and \(U_2\) are uniformly distributed on the interval \((0, 1)\).

The immediate properties of a bivariate copula \(C\) are:

1. \(C\) is increasing in each component;

2. For all \(u_1, v_1 \in [0, 1]\), \(C(u_1, 0) = C(0, u_2) = 0\);
3. for all \( u_1, u_2, u_1^*, u_2^* \in [0, 1] \) such that \( u_1 \leq u_1^* \) and \( u_2 \leq u_2^* \),

\[
C(u_1^*, u_2^*) - C(u_1^*, u_2) - C(u_1, u_2^*) + C(u_1, u_2) \geq 0;
\]

4. For all \( u_1, u_2 \in [0, 1] \), \( C(u_1, 1) = u_1, C(1, u_2) = u_2 \).

Properties 1-3 are due to the fact that \( C \) is a bivariate distribution function on the unit square. Property 4 is due to the fact that the marginal distributions are uniform. In fact, these conditions are necessary and sufficient for \( C \) to be a bivariate copula. Next, we present a theorem which allows to not only retrieve the copula from a joint distribution but also to build joint distributions using copulas.

### 2.1.1 Sklar’s Theorem

Let \((X, Y)\) be a pair of random variables with joint distribution function \(H\) and marginal distributions \(F\) and \(G\). That is, for all \(x, y \in \mathbb{R}\),

\[
H(x, y) = P(X \leq x, Y \leq y), \quad F(x) = P(X \leq x), \quad G(y) = P(Y \leq y).
\]

We can then define the inverses of \(F\) and \(G\). For all \(u_1, u_1' \in (0, 1]\),

\[
F^{-1}(u_1) = \inf\{x \in \mathbb{R} : F(x) \geq u_1\}, \quad G^{-1}(u_2) = \inf\{y \in \mathbb{R} : G(y) \geq u_2\} . \quad (2.1)
\]

These are also known as the quantile functions. Note that they are left-continuous on \((0, 1]\).

The power of copulas for multivariate statistics is mainly due to the following result of Sklar (1959).

**Theorem 2.1.1.** Given a bivariate distribution function \(H\) with marginal distributions \(F\) and \(G\), there exists a copula \(C\) such that for all \(x, y \in \mathbb{R}\),

\[
H(x, y) = C(F(x), G(y)) . \quad (2.2)
\]
Moreover if the marginal distributions $F$ and $G$ are continuous, $C$ is unique and is the cumulative distribution function of the pair $(U, V) = (F(X), G(Y))$. It is given, for all $u, v \in (0, 1]$, by

$$C(u, v) = H\left(F^{-1}(u), G^{-1}(v)\right). \quad (2.3)$$

Conversely, given any copula $C$ and univariate distribution functions $F$ and $G$, the function $H$ defined in (2.2) is a joint distribution function with marginal distributions $F$ and $G$.

In sections pertaining to the stochastic representation of Archimedean and Archimax copulas, the notion of survival copula is used. Here we present the survival copula version of Sklar’s theorem. Recall that if $X, Y$ are random variables, their bivariate survival function is given by $\bar{H}(x, y) = P(X > x, Y > y)$.

**Theorem 2.1.2.** The following statements are true.

1. If $C$ is a bivariate copula and $\bar{F}$ and $\bar{G}$ are univariate survival functions, then

$$\bar{H}(x, y) = C \left(\bar{F}(x), \bar{G}(y)\right), \quad x, y \in \mathbb{R}, \quad (2.4)$$

is a bivariate survival function with marginal survival functions $\bar{F}$ and $\bar{G}$.

2. Conversely, if $\bar{H}$ is a bivariate survival function with margins $\bar{F}$ and $\bar{G}$, there exists a copula $C$ satisfying (2.4). If $F$ and $G$ are continuous, then $C$ is unique and has the form

$$C(u, v) = \bar{H}\left(\bar{F}^{-1}(u), \bar{G}^{-1}(v)\right) \quad \text{for all } u, v \in [0, 1],$$

where $\bar{F}^{-1}(u) = \inf\{x : \bar{F}(x) \leq u\}$ and $\bar{G}^{-1}(v) = \inf\{y : \bar{G}(y) \leq v\}$.

### 2.1.2 Empirical Copulas

Let $(X_i, Y_i)_{i=1}^n$ be a random sample from a bivariate distribution function $H$ with continuous margins $F$ and $G$. First suppose that the margins $F$ and $G$ are known. We then have access to the copula sample, which is given by

$$(U_i, V_i)_{i=1}^n = (F(X_i), G(Y_i))_{i=1}^n.$$
The unknown copula can then be estimated by the empirical distribution function of the sample $(U_i, V_i)_{i=1}^n$ from $C$:

$$C_n(u, v) = \frac{1}{n} \sum_{i=1}^n 1 \{(U_i \leq u, V_i \leq v)\}.$$ 

The empirical process $\alpha_n = (C_n - C)$ has a well known limit. The following theorem can be found in Stute (1984), Gaenssler & Stute (1987), Chapter 5, van der Vaart & Wellner (1996), page 389, Tsukahara (2005), Fermanian et al. (2004), Ghoudi & Rémillard (2004) and van der Vaart & Wellner (2007).

**Remark:** Weak convergence is denoted by $\rightarrow_w$ throughout this thesis.

**Theorem 2.1.3.** The empirical process $\alpha_n = n^{1/2}(C_n - C)$ converges weakly to the bivariate pinned $C$-Brownian sheet $\alpha$, i.e. a centered Gaussian random field on the unit square whose covariance is given by

$$\text{cov}\{\alpha(u, v), \alpha(u', v')\} = C(u \wedge u', v \wedge v') - C(u, v)C(u', v').$$

for all $u, v, u', v' \in [0, 1]$.

In practice however, the margins are rarely known. When this is the case, a sample from $C$ is no longer observable. To circumvent this problem, we can estimate $F$ and $G$ by the marginal empirical distribution functions. This leads to a pseudo-sample $(\hat{U}_i, \hat{V}_i)_{i=1}^n$ from $C$, where $\hat{U}_i$ and $\hat{V}_i$ are the scaled component-wise ranks of the observations from $H$:

$$\hat{U}_i = \frac{1}{n+1} \sum_{j=1}^n 1 \{X_j \leq X_i\}, \quad \hat{V}_i = \frac{1}{n+1} \sum_{j=1}^n 1 \{Y_j \leq Y_i\}.$$

The unknown copula $C$ can then be estimated from this pseudo-sample. In this thesis, we will be using the same definition of empirical copulas as in Genest & Segers (2009).

**Definition 2.1.2.** For all $u, v \in [0, 1]$, let the empirical copula be defined as the empirical
distribution function of the scaled component-wise ranks:

\[ \hat{C}_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} 1(\hat{U}_i \leq u, \hat{V}_i \leq v). \]

As was the case in the situation when \( F \) and \( G \) are known, it is of interest to study the empirical process \( n^{1/2}(\hat{C}_n - C) \). In order to exploit the result from Theorem 2.1.3, we define the remainder term \( R_n(u, v) \) implicitly by the following equation.

\[ n^{1/2}\left\{ \hat{C}_n(u, v) - C(u, v) \right\} = \alpha_n(u, v) - \hat{C}_1(u, v)\alpha_n(u, 1) - \hat{C}_2(u, v)\alpha_n(1, v) + R_n(u, v), \]  

(2.5)

where \( \hat{C}_1 \) and \( \hat{C}_2 \) are the two first order partial derivatives of \( C \), that is

\[ \hat{C}_1(u, v) = \frac{\partial C(u, v)}{\partial u}, \quad \hat{C}_2(u, v) = \frac{\partial C(u, v)}{\partial v}. \]

According to Stute (1984) and Tsukahara (2005), provided that the second-order partial derivatives of \( C \) exist and are continuous on \( [0, 1]^2 \),

\[ \sup_{(u, v) \in [0,1]^2} |R_n(u, v)| = O\left\{ n^{-1/4}(\log n)^{1/2}(\log \log n)^{1/4} \right\} \]  

(2.6)

almost surely as \( n \to \infty \).

Note that other definitions of empirical copulas exist, for example the following definition is based on the fact that \( C(u, v) = H(F^{-1}(u), G^{-1}(v)) \).

**Definition 2.1.3.** Deheuvels (1979) defines the empirical copula as follows. For \( u, v \in [0, 1] \),

\[ \hat{C}_n^D(u, v) = H_n\left\{ F_n^{-1}(u), G_n^{-1}(v) \right\}. \]

\( F_n \) and \( G_n \) are the empirical distributions of \( (X_1, \ldots, X_n) \) and \( (Y_1, \ldots, Y_n) \) respectively, while \( H_n \) is the empirical distribution of the pairs \( (X_i, Y_i)_{i=1}^n \).

Equation (2.6) holds true when \( \hat{C}_n \) is replaced by \( \hat{C}_n^D \) in Equation (2.5).

In fact, when the marginal distributions are continuous, the difference between these two
definitions of the empirical copula is asymptotically negligible.

From Equations (2.5) and (2.6), one can obtain the following result. More details can be found in the same reference material given for Theorem 2.1.3, for example in Tsukahara (2005).

**Theorem 2.1.4.** The empirical copula process \( C_n = n^{1/2} \left( \hat{C}_n - C \right) \) defined in Equation (2.5) converges weakly to

\[
C(u, v) = \alpha(u, v) - \hat{C}_1(u, v)\alpha(u, 1) - \hat{C}_2(u, v)\alpha(1, v),
\]

for all \( u, v \in [0, 1] \), where \( \alpha \) is as defined in Theorem 2.1.3.

### 2.2 Extreme Value Copulas

A particular class of copulas are extreme-value (EV) copulas, used to model the dependence of multivariate extremes. Again in this section we limit ourselves to the bivariate case. For the case of more dimensions \( d > 2 \), see McNeil et al. (2005).

Consider a random sample \((X_i, Y_i)_{i=1}^n\) from a bivariate distribution \( H \) with continuous margins \( F \) and \( G \). We can define the component-wise maxima as

\[
M_n = \max \{X_1, \ldots, X_n\}, \quad N_n = \max \{Y_1, \ldots, Y_n\}.
\]

Their distributions functions are simply given by

\[
P(M_n \leq x) = F^n(x), \quad P(N_n \leq y) = G^n(y), \quad (2.7)
\]

while their joint distribution function is

\[
P(M_n \leq x, N_n \leq y) = H^n(x, y).
\]

First, we can study the univariate case, looking at \( M_n \) and \( N_n \) separately. Equations (2.7) are not useful in practice since \( F \) and \( G \) are unknown. Non-parametric estimates of
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F and G are also discarded since raising them to the power of \( n \) tends to significantly increase error. The Fisher-Tippett Theorem below allows us to establish the limiting (univariate) distributions of \( M_n \) and \( N_n \) without assuming knowledge of \( F \) and \( G \). This result can be seen as an analogue of the central limit theorem where sample means can be approximately described by a normal distribution. Here, the generalized extreme value (GEV) distribution is used to describe sample maxima.

**Theorem 2.2.1.** If there exist sequences of constants \( \{a_n > 0\} \) and \( \{b_n > 0\} \) such that

\[
P \left( \frac{M_n - b_n}{a_n} \leq z \right) \to G(z)
\]

as \( n \to \infty \), for a non-degenerate distribution function \( G \), then \( G \) is a member of the GEV family

\[
G(z) = \exp \left\{ -\left[ 1 + \frac{\xi}{\sigma} \left( \frac{z - \mu}{\sigma} \right) \right]^{-1/\xi} \right\},
\]

defined on \( \{ z : 1 + \xi(z - \mu)/\sigma > 0 \} \), where \( -\infty < \mu < \infty \), \( \sigma > 0 \) and \( -\infty < \xi < \infty \).

This result is also called the extreme-value theorem. See Embrechts et al. (1997), Theorem 3.2.3, for a sketch of the proof. Note that depending on the shape parameter \( \xi \), the GEV distribution is one of the following three distributions: Fréchet when \( \xi > 0 \), Weibull when \( \xi < 0 \) and Gumbel when \( \xi = 0 \). The GEV law is related to the important property of max-stability, which we define below for univariate distributions.

**Definition 2.2.1.** A distribution \( G \) is said to be max-stable if, for every \( n = 2, 3, \ldots \), there are constants \( \alpha_n > 0 \) and \( \beta_n \) such that

\[
G^n(\alpha_n z + \beta_n) = G(z).
\]

The following theorem is essential to the proof of Theorem 2.2.1, since it states that the GEV distribution and univariate max-stable are in fact the same class of distributions. See, for example, Coles (2001), Theorem 3.2.

**Theorem 2.2.2.** A (univariate) distribution is max-stable if and only if it is a generalized extreme-value (GEV) distribution.
2.2. Extreme Value Copulas

We now come back to the bivariate setting. By Sklar’s theorem, \( H \) (the joint distribution function of \( X \) and \( Y \)) has a copula \( C_1 \) such that

\[
H^n(x, y) = C_1^n \left( \left\{ F^n(x) \right\}^{\frac{1}{n}}, \left\{ G^n(y) \right\}^{\frac{1}{n}} \right),
\]

and it follows that the unique copula of \( H^n \) is given by

\[
C_n(u, v) = C_1^n \left( u^{\frac{1}{n}}, v^{\frac{1}{n}} \right).
\] (2.8)

It can be seen that if \( C_n(u, v) \) has a limit for all \( u, v \in [0, 1] \) as \( n \to \infty \), this limit, say \( C \), must be an extreme-value copula, as defined below.

**Definition 2.2.2.** A bivariate copula is called extreme-value if and only if it has the max-stable property (for copulas), which states that for any \( t > 0 \) and any \( u, v \in [0, 1] \),

\[
C(u, v) = C^t \left( u^{\frac{1}{t}}, v^{\frac{1}{t}} \right).
\]

Indeed, suppose that the copula \( C_n \) given in Equation (2.8) has a limit \( C(u, v) \) as \( n \to \infty \). Let \( t \in \mathbb{N} \), then

\[
\lim_{n \to \infty} C_{nt}(u, v) = \lim_{n \to \infty} C_1^{nt} \left( u^{\frac{1}{nt}}, v^{\frac{1}{nt}} \right) = C(u, v).
\]

Therefore,

\[
C(u, v) = \lim_{n \to \infty} C_1^{nt} \left( u^{\frac{1}{nt}}, v^{\frac{1}{nt}} \right) = \lim_{n \to \infty} \left( C_1^{n} \left( \left( u^{\frac{1}{n}} \right)^{\frac{1}{nt}}, \left( v^{\frac{1}{n}} \right)^{\frac{1}{nt}} \right) \right)^t = C^t \left( u^{\frac{1}{t}}, v^{\frac{1}{t}} \right).
\]

With appropriate shifting and scaling of \( M_n \) and \( N_n \), these two component-wise maxima jointly converge to an extreme-value distribution, say \( L \). That is, the limiting marginal distributions \( L_1 \) and \( L_2 \) are GEV by Theorem 2.2.1 and the underlying limiting copula \( C \) is an extreme-value copula as in Definition 2.2.2. \( L \) is also max-stable, in the sense that for
every $n$ there exists constants $a_n$, $b_n$, $c_n$ and $d_n$ such that for any $x, y \in \mathbb{R}$,

$$L^n(a_n x + b_n, c_n y + d_n) = L(x, y). \quad (2.9)$$

Indeed, fixing $n$ and picking the constants $a_n$, $b_n$, $c_n$ and $d_n$ such that the max-stability property from Definition 2.2.1 holds for the margins $L_1$ and $L_2$ of $L$, we obtain

$$L^n(a_n x + b_n, c_n y + d_n) = C^n \left( L_1(a_n x + b_n), L_2(c_n y + d_n) \right) \quad (2.10)$$

$$= C^n \left( [L_1^n(a_n x + b_n)]^{1/n}, [L_2^n(c_n y + d_n)]^{1/n} \right)$$

$$= C^n \left( [L_1(x)]^{1/n}, [L_2(y)]^{1/n} \right) \quad (2.11)$$

$$= C \left( L_1(x), L_2(y) \right) = L(x, y). \quad (2.12)$$

Equation 2.10 is due to Sklar’s Theorem. Equation 2.11 follows from the max-stable property of the marginal distributions $L_1$ and $L_2$ which are GEV. Finally, Equations 2.12 are due to the fact that $C$ is a max-stable copula as well as to Sklar’s Theorem.

### 2.2.1 Pickands Representation

**Definition 2.2.3.** A Pickands dependence function $A$ defined on the unit interval $[0, 1]$ is a real valued function with the following two properties.

1. $A$ is convex.
2. $\max\{t, 1 - t\} \leq A(t) \leq 1$ for all $t \in [0, 1]$.

As shown by Pickands (1981), bivariate extreme-value copulas can be expressed conveniently using a Pickands dependence function $A$.

**Theorem 2.2.3.** A bivariate copula $C$ is an extreme-value copula if and only if there exists a unique Pickands dependence function $A$ such that for $u, v \in (0, 1)$,

$$C(u, v) = \exp \left[ \log(uv) A \left( \frac{\log(u)}{\log(uv)} \right) \right].$$
The upper bound \( A(t) = 1 \) for all \( t \in [0, 1] \) corresponds to the independence copula \( C(u, v) = uv \) for \( u, v \in [0, 1] \). The lower bound \( A(t) = \max\{t, 1-t\} \) corresponds to the comonotone copula \( C(u, v) = \min\{u, v\} \).

### 2.2.2 Estimation

In this subsection we present two common estimators for the Pickands dependence function. Suppose \((X, Y)\) is a pair of continuous random variables with a continuous joint distribution function \( H \) and marginal distributions \( F \) and \( G \). Suppose also the underlying copula \( C \) is extreme-value with Pickands dependence function \( A \). Since the variables \( U = F(X) \) and \( V = G(Y) \) are then uniformly distributed with copula \( C \), it then follows that \( S = -\log U \) and \( T = -\log V \) are exponentially distributed with mean 1. For \( t \in (0, 1) \), let

\[
\xi(t) = \frac{S}{t} \land \frac{T}{1-t}
\]

and set \( \xi(0) = T \) and \( \xi(1) = S \). The survival function of \( \xi(t) \) can be computed, for all \( t \in (0, 1) \), as follows:

\[
P(\xi(t) > x) = P(S > tx, T > (1-t)x) = P(U < \exp\{-tx\}, V < \exp\{-(1-t)x\}) = \exp\{-xA(t)\},
\]

where the last step follows from Theorem 2.2.3. Therefore \( \xi(t) \) is an exponential random variable with mean \( \frac{1}{A(t)} \). Furthermore,

\[
E\{\log \xi(t)\} = \int_0^\infty \log(x) A(t) e^{-A(t)x} \, dx = \int_0^\infty \log\left(y/A(t)\right) e^{-y} \, dy = \int_0^\infty \log(y) e^{-y} \, dy - \log(A(t)) = -\gamma - \log(A(t)),
\]

where \( \gamma \) denotes the Euler-Mascheroni constant. These two observations can now be used to construct non-parametric estimators of the Pickands dependence function \( A \).

Suppose that \((X_i, Y_i)_{i=1}^n\) is a random sample from a continuous bivariate distribution \( H \) with an unknown extreme-value copula \( C \). In order to estimate the unknown Pickands de-
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dependence function $A$, recall first that if $F$ and $G$ were known, $(U_i, V_i)_{i=1}^n = (F(X_i), G((Y_i)))_{i=1}^n$ would constitute a sample from $C$. Consequently, for any $t \in [0, 1]$, $(\xi_i(t))_{i=1}^n$ where

$$\xi_i(1) = \log U_i, \quad \xi_i(0) = -\log V_i, \quad \xi_i(t) = \frac{-\log U_i}{t} \wedge \frac{-\log V_i}{1-t}, \quad t \in (0, 1)$$

would constitute a random sample from $\xi(t)$. The unknown Pickands dependence function $A$ of $C$ could then be estimated by either one of the following statistics:

$$\frac{1}{A_n^P(t)} = \frac{1}{n} \sum_{i=1}^n \xi_i(t),$$

$$\log A_n^{CFG}(t) = -\gamma - \frac{1}{n} \sum_{i=1}^n \log \xi_i(t),$$

for $t \in [0, 1]$. The first estimator is due to Pickands (1981) while the second one, where CFG stands for Capéraà-Fougères-Genest, is due to Capéraà et al. (1997).

The above estimators of $A$ rely on the knowledge of the margins. When $F$ and $G$ are unknown, i.e. when we cannot observe the pairs $(U_i, V_i) = (F(X_i), G((Y_i)))$, we can work with the scaled component-wise ranks instead, viz.

$$\hat{U}_i = \frac{1}{n+1} \sum_{j=1}^n \mathbf{1}(X_j \leq X_i), \quad \hat{V}_i = \frac{1}{n+1} \sum_{j=1}^n \mathbf{1}(Y_j \leq Y_i),$$

as explained in Section 2.1.2. Now for every $i = 1, \ldots, n$, let

$$\hat{S}_i = -\log \hat{U}_i = \hat{\xi}_i(1), \quad \hat{T}_i = -\log \hat{V}_i = \hat{\xi}_i(0), \quad \hat{\xi}_i(t) = \frac{\hat{S}_i}{t} \wedge \frac{\hat{T}_i}{1-t}, \quad t \in (0, 1).$$

The rank-based Pickands and Capéraà-Fougères-Genest estimators as found in Genest & Segers (2009) are given by

$$\frac{1}{A_{n,r}^P(t)} = \frac{1}{n} \sum_{i=1}^n \hat{\xi}_i(t),$$

and

$$\log A_{n,r}^{CFG}(t) = -\gamma - \frac{1}{n} \sum_{i=1}^n \log \hat{\xi}_i(t).$$
Among other results, Genest & Segers (2009) establish the asymptotic behavior of these two estimators. Define the corresponding empirical processes as follows,

\[ A_{n,t}^P(t) = n^{1/2} \left( A_{n,t}^P(t) - A(t) \right), \quad A_{n,t}^{CFG}(t) = n^{1/2} \left( A_{n,t}^{CFG}(t) - A(t) \right), \]

for \( t \in [0, 1] \).

**Theorem 2.2.4.** If \( A \) is twice continuously differentiable, then \( A_{n,t}^P \to_w A^P \) and \( A_{n,t}^{CFG} \to_w A^{CFG} \) as \( n \to \infty \) in the space \( \mathcal{C}([0, 1]) \) of continuous functions on \([0, 1]\) equipped with the topology of uniform convergence, where

\[
A^P_r(t) = -A^2(t) \int_0^1 C(u', u^{1-t}) \frac{du}{u}, \\
A^{CFG}_r(t) = A(t) \int_0^1 C(u', u^{1-t}) \frac{du}{u \log u},
\]

where \( C \) denotes the weak limit of \( n^{1/2}(\hat{C}_n - C) \) as in Section 2.1.2.

In Chapter 4, we will extend these results of Genest & Segers (2009) to a broader class of estimators.

### 2.3 Archimedean Copulas

Another widely used copula class is class of Archimedean copulas. A \( d \)-variate Archimedean copula has the following form:

\[
C_\psi(u_1, \ldots, u_d) = \psi \left( \psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d) \right), \quad (u_1, \ldots, u_d) \in [0, 1]^d.
\]  

(2.13)

The right-hand side in Equation (2.13) does not need to be a copula for an arbitrary function \( \psi \). Multivariate Archimedean copulas of dimension \( d \) are defined in terms of \( d \)-varying Archimedean generators; their existence has been established in McNeil & Nešlehová (2009). We now give the definition of Archimedean generators.

**Definition 2.3.1.** Let \( d \geq 2 \) be a given integer. The function \( \phi = \psi^{-1} : [0, \infty) \to [0, 1] \) is called a \( d \)-variate Archimedean generator, if \( \psi(0) = 1 \), \( \psi(x) \to 0 \) as \( x \to \infty \) and if \( \psi \) is
Table 2.1: Common Bivariate Archimedean Copulas

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Name} & \text{ } & \text{ } & \text{ } & \Theta \\
\hline
\text{Independence} & C(u, v) : u, v \in (0, 1) & \phi, t \in [0, 1] & \psi, t \in (0, \infty) & \text{N/A} \\
\hline
\text{Clayton} & \left[ u^{-\theta} + v^{-\theta} \right]^{-1/\theta} & \frac{1}{\theta} \left( t^{-\theta} - 1 \right) & (1 + \theta t)^{-1/\theta} & \theta \in [-1, \infty) \setminus \{0\} \\
\text{Gumbel} & \exp \left[ - \left( -\log u^{\theta} + (-\log v^{\theta}) \right)^{1/\theta} \right] & (-\log t)^{\theta} & \exp \left( -t^{1/\theta} \right) & \theta > 1 \\
\text{Ali-Mikhail-Haq} & \frac{1}{1-\theta}(1-u)(1-v) & \frac{1}{1-\theta} \exp(1-t) & \log \left( 1 - (1 - t)^{1/\theta} \right) & \theta \in (-1, 1) \\
\text{Joe} & \frac{1}{\theta} \log \left( 1 + \frac{(1-u)^{\theta}-1}{e^{\theta t} - 1} \right) & -\log \left( e^{\frac{\theta t}{\theta - 1}} - 1 \right) & \frac{1}{\theta} \log \left( 1 + e^{-t} \left( e^{-\theta} - 1 \right) \right) & \theta \in \mathbb{R} \setminus \{0\} \\
\text{Frank} & \frac{1}{\theta} \log \left( 1 + \frac{e^{-\theta t}}{e^{-\theta} - 1} \right) & -\log \left( e^{-\frac{\theta t}{\theta - 1}} - 1 \right) & \frac{1}{\theta} \log \left( 1 + e^{-t} \left( e^{-\theta} - 1 \right) \right) & \theta \in \mathbb{R} \setminus \{0\} \\
\hline
\end{array}
\]

d-monotone on \((0, \infty)\), i.e. if \(\psi\) has \(d-2\) derivatives satisfying \((-1)^k \psi^{(k)} \geq 0\) on \((0, \infty)\) for \(1 \leq k \leq d-2\) and that \((-1)^{d-2} \psi^{(d-2)}\) is non-increasing and convex on \((0, \infty)\). By convention, \(\psi^{-1}(0) = \phi(0) = \inf \{ x \in [0, \infty) : \psi(x) = 0 \} \).

McNeil & Nešlehová (2009) prove the following result.

**Theorem 2.3.1.** If \(\phi = \psi^{-1}\) is a \(d\)-variate Archimedean generator, then

\[ C(u_1, \ldots, u_d) = \psi \left( \psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d) \right), \quad (u_1, \ldots, u_d) \in [0, 1]^d, \]

is a copula.

The Archimedean class encompasses many well-known copula families. Letting the generator be \(\psi(t) = e^{-t}\) we recover the independence copula \(C(u_1, \ldots, u_d) = u_1 \cdots u_d\) for \(u_1, \ldots, u_d \in [0, 1]\). Other well-known Archimedean families are the Clayton, Gumbel, Frank and Joe copulas (see Table 2.1).

### 2.3.1 Stochastic Representation

In order to obtain a stochastic representation of the Archimedean family, we use the following transformation due to Williamson (1956). This representation brings a different perspective on Archimedean copulas and allows us to generate samples using a radial random variable rather than an Archimedean generator. Additionally this stochastic representation is used to construct examples of generators which are \(d\)-monotone.
2.3. Archimedean Copulas

**Definition 2.3.2.** If $R$ is a non-negative random variable with distribution function $F_R$ satisfying $F_R(0) = 0$ and $d \geq 2$ is an integer, then the Williamson $d$-transform of $F_R$ is a real function on $[0, \infty)$ given by

$$W_d F_R(x) = \int_x^\infty \left(1 - \frac{t}{x}\right)^{d-1} dF_R(t) = E\left(1 - \frac{R}{x}\right)^{d-1}, \quad x \in [0, \infty),$$

where $E(Y)_+$ denotes the expectation of the non-negative part of the random variable $Y$.

As shown in McNeil & Nešlehová (2009), a non-negative random variable is uniquely characterized by its Williamson $d$-transform for any $d \geq 2$. In particular, if $\psi = W_2 F_R$, then for $x \in (0, \infty)$, $F_R(x) = W_2^{-1} \psi(x)$, where

$$W_2^{-1} \psi(x) = 1 - \psi(x) + x \psi'(x). \quad (2.14)$$

In dimensions $d > 2$, Equation (2.14) generalizes to

$$W_d^{-1} \psi(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k \psi^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} \psi^{(d-1)}(x)}{(d-1)!}.$$

McNeil & Nešlehová (2009) link simplex distributions to Archimedean copulas via the Williamson $d$-transform. In the bivariate case, a random vector $X = (X_1, X_2)$ is said to have a simplex distribution if $X \equiv_d R S_2$ where $S_2$ is a random vector uniformly distributed on the unit simplex $S_2 = \{x \in \mathbb{R}_+^2 : \|x\|_1 = 1\}$, and $R$ is an independent, strictly positive scalar random variable. The radial distribution refers to $F_R$, the distribution function of the independent strictly positive scalar random variable $R$. $\|\cdot\|_1$ is the $\ell_1$ norm, i.e. for $x = (x_1, \ldots, x_d)$, $\|x\|_1 = \sum_{i=1}^d |x_i|$.

**Theorem 2.3.2.** The following statements hold:

- If $X = (X_1, X_2)$ has a simplex distribution with radial distribution $F_R$ satisfying $F_R(0) = 0$, then $X$ has a bivariate Archimedean survival copula with generator $\psi = W_2 F_R$.  

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- If \((U, V)\) is distributed as a bivariate Archimedean copula \(C\) with generator \(\psi\), then \((\psi^{-1}(U), \psi^{-1}(V))\) has a simplex distribution with radial distribution \(F_R = 2^{\psi^{-1}}\).

2.3.2 Examples

McNeil & Nešlehová (2010) use the characterization in Theorem 2.3.2 to create new copula families, including the following.

The Pareto-Simplex Family

Suppose that \(R\) is Pareto with distribution function \(F_R(r) = 1 - r^{-\alpha}\) for \(r \geq 1\) and \(F_R(r) = 0\) for \(r < 1\) and parameter \(\alpha > 0\). As shown in McNeil & Nešlehová (2010), the Archimedean copula of the \(d\)-dimensional simplex distribution has the following generator:

\[
\psi_{\alpha, d}(x) = \alpha x^{-\alpha} B(\min(x, 1), \alpha, d)
\]

where \(B\) denotes the incomplete beta function.

In two dimensions the generator simply reduces to

\[
\psi_{\alpha, 2}(x) = \begin{cases} 
1 - \frac{ax}{a+1} & \text{for } x \leq 1 \\
\frac{x^{-\alpha}}{a+1} & \text{for } x > 1
\end{cases}
\]

The Inverse Pareto-Simplex Family

If \(1/R\) is Pareto with parameter \(\gamma > 0\), then the density of \(R\) is given by \(f_r(r) = \gamma r^{\gamma-1}\) for \(r \in (0, 1]\) and \(f_r(r) = 0\) otherwise. In two dimensions, the resulting generator can easily be found. For \(x \in [0, 1]\),

\[
\psi_{\gamma, 2}(x) = \int_x^1 \left(1 - \frac{x}{r}\right) (\gamma r^{\gamma-1}) dr = \gamma \left[\int_x^1 r^{\gamma-1} dr - x \int_x^1 r^{\gamma-2} dr\right] = 1 - x \frac{\gamma}{\gamma-1} + x^\gamma \left(\frac{\gamma}{\gamma-1} - 1\right)
\]
2.3. Archimedean Copulas

\[ 1 - x \frac{\gamma}{\gamma - 1} + \frac{x^{\gamma}}{\gamma - 1}, \quad (2.15) \]

and \( \psi_{\gamma,2}(x) = 0 \) for \( x > 1 \).

**The Mixture Family**

Let \( R \) have distribution function

\[ F_R(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^\gamma}{2} & \text{for } 0 \leq x \leq 1 \\ 1 - \frac{x^{-\alpha}}{2} & \text{for } x > 1 \end{cases}, \]

meaning \( R \) is drawn with probability 1/2 from a Pareto distribution with parameter \( \alpha \) and with probability 1/2 from an inverse Pareto distribution with parameter \( \gamma \). In two dimensions, the generator is given by the next expression.

\[ \psi_{\alpha,\gamma,2}(x) = \begin{cases} 1 - \frac{x}{2} \frac{\gamma}{\gamma - 1} + \frac{\alpha}{2} \frac{x^{\gamma}}{\gamma - 1} - \frac{x\alpha}{2(\alpha+1)} & \text{for } x \leq 1 \\ \frac{x^{-\alpha}}{2} & \text{for } x > 1 \end{cases}. \quad (2.16) \]

The derivations are shown below. First, for \( x > 1 \),

\[ \psi_{\alpha,\gamma,2}(x) = \mathcal{M}_2 F_R(x) = \int_x^\infty \left(1 - \frac{x}{r}\right) \frac{ar^{-a-1}}{2} dr = \frac{\alpha}{2} \left[ \int_x^\infty r^{-a-1} dr - x \int_x^\infty r^{-a-2} dr \right] = \frac{\alpha}{2} \left[ \frac{x^{-\alpha}}{a} - x \left( \frac{x^{\alpha-1}}{a+1} \right) \right] = \frac{x^{-\alpha}}{2(a+1)}. \]

Now if \( x \leq 1 \) we obtain

\[ \psi_{\alpha,\gamma,2}(x) = \mathcal{M}_2 F_R(x) = \int_x^1 \left(1 - \frac{x}{r}\right) \left( \frac{\gamma}{2} r^{a-1} \right) dr + \int_1^\infty \left(1 - \frac{x}{r}\right) \frac{ar^{-a-1}}{2} dr = \frac{\gamma}{2} \left[ \int_x^1 r^{\gamma-1} dr - x \int_x^1 r^{\gamma-2} dr \right] + \frac{\alpha}{2} \left[ \frac{1-x}{a} \right]. \]
\[ = \left\{ \frac{1}{2} - \frac{x}{2} \right\} \gamma - 1 + \frac{1}{2} x^{\gamma} \left( \frac{\gamma}{\gamma - 1} - 1 \right) \right\} \right\} + \left\{ \frac{1}{2} - \frac{\alpha x}{2(\alpha + 1)} \right\} \right\} 
\]

\[ = 1 - \frac{x}{2} \gamma - 1 + \frac{1}{2} x^{\gamma} - \frac{x}{2} - \frac{\alpha x}{2(\alpha + 1)}. \]

### 2.3.3 Random Number Generation

The stochastic representation of Archimedean copulas from Theorem 2.3.2 makes it easy to generate random samples. To generate a sample from a bivariate Archimedean copula \( C_{\psi} \) with generator \( \psi \), proceed as following.

1. Generate \( R \), usually by using inverse probability sampling. If we start from the generator \( \psi \), we can use Equation (2.14).

2. Independently, generate a random vector \( S_2 \) uniformly distributed on the unit simplex \( \mathcal{S}_2 \) by using

\[ S_2 = (S_1, S_2) = d \left( \frac{Y_1}{Y_1 + Y_2}, \frac{Y_2}{Y_1 + Y_1} \right) \]

where \( Y_1, Y_2 \) are independent exponential random variables with unit mean.

3. The observation from \( C_{\psi} \) is given by

\[ \left( \psi(RS_1), \psi(RS_2) \right). \]

This algorithm was first introduced by Whelan (2004), but only justified for completely monotone generators. McNeil & Nešlehová (2009) show that this procedure can be used for any \( d \)-monotone generator.

### 2.4 Regular Variation

Regular variation is a concept crucial to the estimators used in this thesis. Regular variation of the Archimedean generator was found to be one of the sufficient conditions to
ensure the consistency of the proposed estimators.

**Definition 2.4.1.** A measurable function \( f : (0, \infty) \to (0, \infty) \) is called slowly varying at infinity if for all \( t > 0 \),

\[
\lim_{x \to \infty} \frac{f(tx)}{f(x)} = 1.
\]

Slowly varying functions include any power of the logarithmic function, as well as any function with a positive limit at infinity.

**Definition 2.4.2.** A measurable function \( f : (0, \infty) \to (0, \infty) \) is called regularly varying function at infinity with index \( \alpha \in \mathbb{R} \) (written \( f \in RV_\alpha \)), if for any \( x > 0 \),

\[
\lim_{t \to \infty} \frac{f(tx)}{f(x)} = x^\alpha.
\]

\( \mathbb{R} \) denotes the extended real number line. The following result is often used in Chapter 4. For a proof, see Theorem 0.8(v) from Resnick (1987).

**Theorem 2.4.1.** Suppose that \( f \) is non decreasing, \( f(\infty) = \infty \), and \( f \in RV_\rho \), \( 0 \leq \rho \leq \infty \). Then

\[
f^{-1} \in RV_{\rho^{-1}}.
\]

where \( f^{-1}(y) = \inf \{x : f(x) \geq y\} \), as in Equation (2.1).

Another important result is Karamata’s theorem, see Theorem 0.6 in Resnick (1987).

**Theorem 2.4.2.** The following statements hold.

1. If \( \rho \geq 1 \) then \( f \in RV_\rho \) implies \( \int_0^x f(t)dt \in RV_{\rho+1} \) and

\[
\lim_{x \to \infty} \frac{xf(x)}{\int_0^x f(t)dt} = \rho + 1.
\]

If \( \rho < -1 \) (or if \( \rho = -1 \) and \( \int_{-\infty}^0 f(s)ds < \infty \)) then \( f \in RV_\rho \) implies \( \int_{-\infty}^x f(t)dt \) is finite, \( \int_{-\infty}^x f(x) \in RV_{\rho+1} \) and

\[
\lim_{x \to -\infty} \frac{xf(x)}{\int_{-\infty}^x f(t)dt} = -\rho - 1.
\]
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2. If $f$ satisfies
\[ \lim_{x \to \infty} \frac{x f(x)}{\int_0^x f(t) \, dt} = \lambda \in (0, \infty), \]
then $f \in RV_{\lambda - 1}$. If $\int_0^\infty f(t) \, dt < \infty$ and
\[ \lim_{x \to \infty} \frac{x f(x)}{\int_x^\infty f(t) \, dt} = \lambda \in (0, \infty), \]
then $f \in RV_{-\lambda - 1}$.

We finish this section with another important result called Karamata's characterization theorem.

**Theorem 2.4.3.** If $f : (0, \infty) \to (0, \infty)$ is regularly varying at infinity with index $\rho$, then there exists a slowly varying function $L$ such that
\[ f(x) = x^\rho L(x). \]

**Proof.** Let $L(x) = f(x)/x^\rho$. This function has index of regular variation equal to zero, meaning that it is slowly varying. \( \square \)

Combining the previous two theorems, we have that if $f \in RV_\rho$ with $\rho < -1$, then for a suitable slowly varying function $L$,
\[ \int_0^\infty f(t) \, dt = x^{\rho + 1} L(x), \quad (2.17) \]
which will prove to be quite useful in Chapter 4.
In the previous chapter, definitions of the Pickands dependence function and the Archimedean generator were given. These two functions are used jointly in the definition of a bivariate Archimax copula.

**Definition 3.0.3.** A bivariate Archimax copula is given by the following expression, valid for all \( u_1, u_2 \in [0, 1] \):

\[
C_{\psi, A}(u_1, u_2) = \psi \left[ \left( \psi^{-1}(u_1) + \psi^{-1}(u_2) \right) A \left( \frac{\psi^{-1}(u_1)}{\psi^{-1}(u_1) + \psi^{-1}(u_2)} \right) \right],
\]

where \( A : [0, 1] \to [1/2, 1] \) is a Pickands dependence function and \( \phi = \psi^{-1} \) a 2-variate Archimedean generator.

The proof that \( C_{\psi, A} \) is indeed a copula can be found in Appendix A of Capéraà et al. (2000).

This family of copulas is a generalization of both Archimedean copulas and extreme value copulas. Indeed, using the generator

\[
\phi_{EV}(t) = \psi_{EV}^{-1}(t) = -\log(t), \quad t \in [0, \infty),
\]
we obtain the extreme-value copula with Pickands dependence function $A$:

$$C_{\psi, E V, A}(u_1, u_2) = \exp \left\{ \left[ \log(u_1) + \log(u_2) \right] A \left( \frac{\log(u_1)}{\log(u_1) + \log(u_2)} \right) \right\} = (u_1 u_2)^{A \left( \log(u_1) / \log(u_1) + \log(u_2) \right)} .$$

When using the identity function $A_I = 1$, we recover the Archimedean copula with generator $\psi$:

$$C_{\psi, A_I}(u_1, u_2) = \psi \left[ (\psi^{-1}(u_1) + \psi^{-1}(u_2)) \right] .$$

Capéraà et al. (2000) establish the maximum domain of attraction (MDA) of certain bivariate Archimax copulas in the next theorem. That is, they determine the limit $C^*$ of

$$C_n(u, v) = C_{\psi, A}^n(u^{1/n}, v^{1/n}), \quad u, v \in [0, 1]$$

as $n \to \infty$. As seen in Section 2.2, if the limit exists, then $C^*$ is an extreme-value copula and $C$ is said to be in the maximum domain of attraction of $C^*$.

**Theorem 3.0.4.** If $C_{\psi, A}$ is a bivariate Archimax copula with Pickands dependence function $A$ and Archimedean generator $\phi = \psi^{-1}$ such that $\phi(1 - 1/t) \in RV_{-m}$ for some $m \geq 1$, then $C_{\psi, A}$ belongs to the maximum domain of attraction of an extreme value copula with Pickands dependence function $A^*$ given by

$$A^*(t) = \left\{ t^m + (1 - t)^m \right\} A^{1/m} \left\{ \frac{t^m}{t^m + (1 - t)^m} \right\}, \quad t \in [0, 1] .$$

Analogously, Capéraà et al. (2000) also establish the minimum domain of attraction of certain bivariate Archimax copulas. This is done by computing the limit of

$$\tilde{C}_n(u, v) = \tilde{C}_{\psi, A}^n \left( 1 - u^{1/n}, 1 - v^{1/n} \right) ,$$

as $n \to \infty$, where $\tilde{C}$ denotes a survival copula.

**Theorem 3.0.5.** If $C_{\psi, A}$ is a bivariate Archimax copula with Pickands dependence function $A$ and Archimedean generator $\phi = \psi^{-1}$ such that $\lim_{t \to 0} \phi(t) = \infty$ and $\phi \in RV_{1/m}$ for some $m \geq 0$, then $C_{\psi, A}$ belongs to the minimum domain of attraction of the copula $C_*(u, v) = u + v - 1 + (1 - u)(1 - v)/C_{\psi, A}(1 - u, 1 - v)$, where $\psi^{-1}(t) = \log^{-1/m}(1/t)$.
3.1 Stochastic representation

Note that $C_{\psi^*, A}(1-u, 1-v)$ in Theorem 3.0.5 is not a copula because $\psi^{-1}(t) = \log^{-1/m}((1/ t))$ is not a 2-variate Archimedean generator.

Theorems 3.0.4 and 3.0.5 are generalized to $d > 2$ dimensions in Propositions 5.1 and 5.4 respectively of Charpentier et al. (2014). They are important in that they motivate the construction of Archimax copulas. Indeed, they allows us to construct parametric families of bivariate copulas with given limiting extreme-value attractors. As such, Archimax copulas provide extra flexibility when compared to extreme-value copulas. They can be used in pre-asymptotic settings, i.e., when the data at hand do not yet exhibit the limiting extreme-value dependence.

3.1 Stochastic representation

In analogy to Archimedean copulas in Subsection 2.3.1, we present a stochastic representation of Archimax copulas. For the sake of generality, we will present results for any number of dimensions first and focus on the special case of two dimensions subsequently. To do so, a new type of function is called upon to characterize Archimax copulas. This following function is defined in Huang (1992).

**Definition 3.1.1.** A function is $l : [0, \infty) \rightarrow [0, 1]$ is called a $d$-variate stable tail dependence function if there exists a $d$-variate extreme-value copula $D$ such that for all $x_1, \ldots, x_d \in [0, \infty)$,

$$
l(x_1, \ldots, x_d) = -\log \left\{ D(e^{-x_1}, \ldots, e^{-x_d}) \right\}.
$$

In the bivariate case, the stable tail dependence function is linked to the Pickands dependence function by the following equation, valid for all $x, y \in [0, \infty)$,

$$
l(x, y) = (x + y) A \left( \frac{x}{x+y} \right).
$$

Charpentier et al. (2014) prove that for any choice of $d$-variate stable tail dependence function $l$ and $d$-variate Archimedean generator $\psi$,

$$
C_{\psi, l}(u_1, \ldots, u_d) = \psi \circ l \left\{ \phi(u_1), \ldots, \phi(u_d) \right\}
$$
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is an Archimax copula.

Charpentier et al. (2014) then give two stochastic representations for Archimax copulas. The first relies on a random vector \((T_1, \ldots, T_d)\) with survival function given for all \(t_1, \ldots, t_d \in [0, \infty)\) by

\[ P(T_1 > t_1, \ldots, T_d > t_d) = \exp \{-\ell(t_1, \ldots, t_d)\} . \]  

(3.1)

Each \(T_i\) for \(i \in \{1, \ldots, d\}\) is exponentially distributed with unit mean, while the survival copula of the random vector \((T_1, \ldots, T_d)\) is an extreme-value copula with stable tail dependence function \(\ell\). In the bivariate case, the joint survival function of \((T_1, T_2)\) is then

\[ P(T_1 > t_1, T_2 > t_2) = \exp \left( -(t_1 + t_2)A\left(\frac{t_1}{t_1 + t_2}\right) \right) , \quad t_1, t_2 \geq 0 . \]

Theorem 3.1.1. The copula \(C_{\psi, \ell}\) is Archimax with \(d\)-variate stable tail dependence function \(\ell\) and completely monotone Archimedean generator \(\psi\) if and only if it is the survival copula of the random vector

\[(X_1, \ldots, X_d) = (T_1/\Theta, \ldots, T_d/\Theta),\]

where \(\Theta\) has Laplace transform \(\psi\) and is stochastically independent of the random vector \((T_1, \ldots, T_d)\) defined in Equation (3.1).

This representation only holds for completely monotone generators \(\psi\). As we saw in Theorem 2.3.1, we require only \(d\)-monotonicity. For this more general case, consider a random vector \((S_1, \ldots, S_d)\) of strictly positive random variables such that for all \(s_1, \ldots, s_d \in [0, \infty),\)

\[ P(S_1 > s_1, \ldots, S_d > s_d) = \tilde{G}_l(s_1, \ldots, s_d) = \left[ \max \{0, 1 - \ell(s_1, \ldots, s_d)\} \right]^{d-1} . \]  

(3.2)

Note that in the bivariate case,

\[ P(S_1 > s_1, S_2 > s_2) = \tilde{G}_l(s_1, s_2) = \left[ \max \left\{ 0, 1 - (s_1 + s_2)A\left(\frac{s_1}{s_1 + s_2}\right) \right\} \right] , \quad s_1, s_2 \geq 0 . \]
3.2. Random Number Generation

The support of this joint survival function is

\[ \Omega_d(l) = \left\{ (s_1, \ldots, s_d) \in [0,1]^d : l(s_1, \ldots, s_d) \leq 1 \right\}, \]

and \( S_1, \ldots, S_d \) are dependent Beta random variables with parameters \((1, d-1)\). Proof that Equation (3.2) is indeed a joint survival function can be found in Charpentier et al. (2014).

We can now present Theorem 3.2 from Charpentier et al. (2014).

**Theorem 3.1.2.** The following statements hold.

1. Suppose that

\[(X_1, \ldots, X_d) = \varphi(RS_1, \ldots, RS_d),\]

where \( R \) is a strictly positive random variable and \((S_1, \ldots, S_d)\) has a joint survival function as in Equation (3.2). Then its survival copula is the Archimax copula \( C_{\psi, \ell} \), where \( \psi \) is the Williamson \( d \)-transform of \( R \) (as in Definition 2.3.2).

2. Let \( \ell \) be a \( d \)-variate stable dependence function and \( \psi \) be a \( d \)-monotone Archimedean generator. Then \( C_{\psi, \ell} \) is the survival copula of a random vector \((X_1, \ldots, X_d)\) of the form (3.3), where the distribution function \( F_R \) of \( R \) is the inverse Williamson \( d \)-transform of \( \psi \), as given in Equation (2.3.1).

The next remark follows from the proof of Theorem 3.3 in Charpentier et al. (2014).

**Remark.** If \((U_1, \ldots, U_d)\) is distributed according to an Archimax copula \( C_{\psi, \ell} \) with Archimedean generator \( \psi \) and stable tail dependence function \( \ell \), then

\[(X_1, \ldots, X_d) = \left( \psi^{-1}(U_1), \ldots, \psi^{-1}(U_d) \right)\]

is distributed as \((R(S_1), \ldots, R(S_d))\) from Equation (3.3).

### 3.2 Random Number Generation

In this section we present an algorithm to generate a bivariate sample from any Archimax copula \( C_{\psi, A} \) which is due to Capéraà et al. (2000); see also Ghoudi et al. (1998) and Genest
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& Nešlehová (2013). This algorithm assumes that \( A \) is twice continuously differentiable. Two more random number generation algorithms based on the two stochastic representations seen in Section 3.1 can be found in Charpentier et al. (2014). To draw a pair \((U, V)\) from an Archimax copula \( C_{\psi, A} \) with Archimedean generator \( \psi \) and Pickands dependence function \( A \), proceed as follows.

1. Draw \((U_1, V_1)\) from an extreme-value copula with Pickands dependence function \( A \) and set \( Z = \log(U_1) / \log(U_1 V_1) \).

2. Given the first and second order derivatives \( A' \) and \( A'' \) of \( A \), compute
   \[
   p(Z) = \frac{Z(1-Z)A''(Z)}{A(Z) + (1-2Z)A'(Z) + Z(1-Z)[A''(Z) - A'(Z)^2 / A(Z)]}
   \]

3. Generate a pair \((U_2, V_2)\) from the Archimedean copula \( C_\phi \) with generator \( \phi \) and a random variable \( U_3 \) uniformly distributed over \((0, 1)\).

4. Set \( W = U_2 \) if \( U_3 \leq p(Z) \), \( W = C_\phi(U_2, V_2) \) otherwise. The pair \((U, V)\) from the Archimax copula \( C_{\psi, A} \) is given by
   \[
   U = \psi\left(\frac{Z\phi(W)}{A(Z)}\right), \quad V = \psi\left(\frac{(1-Z)\phi(W)}{A(Z)}\right).
   \]

To show that the above algorithm indeed generates a random pair \((U, V)\) from an Archimax copula \( C_{\psi, A} \), define \( W = C_{\psi, A}(U, V) \) and \( Z = \psi^{-1}(U) / \{\psi^{-1}(U) + \psi^{-1}(V)\} \). It is easy to see that the relationship between \((U, V)\) and \((W, Z)\) is one-to-one, since

\[
U = \psi\left(\frac{Z\phi(W)}{A(Z)}\right), \quad V = \psi\left(\frac{(1-Z)\phi(W)}{A(Z)}\right).
\]

Therefore, once the pair \((W, Z)\) has been generated, \((U, V)\) can be recovered from it as in Step 4.

To generate the pair \((W, Z)\), observe first that because of the stochastic representation of Archimax copulas described in Theorem 3.1.2 and the subsequent remark, \( Z \) has the same distribution as \( \log(U_1) / \log(U_1 V_1) \), where \((U_1, V_1)\) is distributed as an extreme-value
3.3. Examples

copula with the same Pickands dependence function \( A \) as \( C_{\psi,A} \). This explains Step 1. Steps 2 and 3 generate \( W \) using Proposition 5.1 of Capéraà et al. (1997). As the latter Proposition states, the conditional distribution of \( W \) given \( Z \) is a mixture distribution of a uniform distribution on \((0, 1)\) and \( C_{\psi}(U_2, V_2) \) where \((U_2, V_2)\) is distributed according to the Archimedean copula \( C_{\psi} \) with generator \( \psi \).

This algorithm will allow us to generate samples in the next section, and will also be used for the simulation study in Chapter 5.

3.3 Examples

In this section we present plots of Archimax data simulated with the algorithm presented above. The families we choose to illustrate here will be the ones used in the simulation study later in the thesis. We begin by choosing the Pickands dependence function.

3.3.1 Asymmetric Gumbel-Hougaard Pickands Dependence Function

The Gumbel-Hougaard copula family has the following symmetric Pickands dependence function:

\[
A = \beta(t) = \left( t^\beta + (1 - t)^\beta \right)^{1/\beta},
\]

for \( t \in [0, 1] \) and parameter \( \beta \geq 1 \). The higher \( \beta \) is, the closer \( A(t) \) is to its lower bound, thus increasing the strength of dependence. Indeed, recall the definition of Kendall’s tau.

**Definition 3.3.1.** Let \((X, Y)\) and \((X^*, Y^*)\) be independent and identically distributed bivariate random vectors, each with joint distribution \( H \). Kendall’s tau is defined to be the probability of concordance minus the probability of discordance, viz.

\[
\tau(H) = P_c - P_d = P\{ (X - X^*)(Y - Y^*) > 0 \} - P\{ (X - X^*)(Y - Y^*) < 0 \}.
\]

For a bivariate extreme-value copula \( C_\beta \) with Gumbel Pickands dependence function \( A_\beta \), Kendall’s tau is given by \( \tau(C_\beta) = 1 - \beta^{-1} \).
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$A$ is called symmetric because $A(t) = A(1 - t)$. The extreme-value copula $C_A$ with this particular $A$ is exchangeable due to this symmetry, i.e. $C_A(u, v) = C_A(v, u)$ for $u, v \in [0, 1]$.

To remove this restriction we can extend the above Pickands dependence function $A$ by two additional parameters $\kappa, \lambda \in [0, 1]$ in order to make it asymmetric. See Khoudraji’s device in Genest et al. (1998) for more details.

$$A_{\beta, \kappa, \lambda}(t) = (1 - \kappa) t + (1 - \lambda)(1 - t) + (\kappa t + \lambda(1 - t)) \left( \left( \frac{\kappa t}{\kappa t + \lambda(1 - t)} \right)^\beta + \left( 1 - \frac{\kappa t}{\kappa t + \lambda(1 - t)} \right)^\beta \right)^{1/\beta}.$$ (3.4)

Setting $\beta = 3$ and $(\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}$, we obtain the following plots of the Pickands dependence function. We can see from the plots that $\kappa$ and $\lambda$ dictate at which side the imbalance occurs, the magnitude of $\kappa$ and $\lambda$ also affects the strength of the dependence together with $\beta$. In fact, Genest et al. (2011) use the Fréchet-Hoeffding inequality to find the following upper bound on Kendall’s tau for an extreme-value copula with Pickands dependence function given by Equation (3.4).

$$\tau(C_{\beta, \kappa, \lambda}) \leq \frac{\kappa \lambda}{\kappa + \lambda - \kappa \lambda}.$$
3.3.2 Clayton Family

In this subsection, we will look at bivariate samples from Archimax copulas with Clayton generator given by $\psi_\theta(t) = (1 + \theta t)^{-1/\theta}$ and asymmetric Gumbel-Hougaard Pickands dependence function given by Equation (3.4).

In Figure 3.2, we can see the effect of the asymmetry parameters $(\kappa, \lambda)$. The more imbalanced they are, the less the copula appears to be exchangeable. The parameter $\theta$ from the Clayton generator has a noticeable effect of increasing the positive association. In the case of the Archimedean Clayton copula with generator $\psi_\theta$, we have the following Kendall’s tau: $\tau(C_{\psi_\theta}) = \theta/(\theta + 2)$. Capéraà et al. (2000) establish that Kendall’s tau an Archimax copula $C_{\psi, A}$ depends on both the values of Kendall’s tau associated with the extreme-value copula $C_A$ and the Archimedean copula $C_\psi$, viz.

$$\tau(C_{\psi, A}) = \tau(C_A) + (1 - \tau(C_A)) \tau(C_{\psi}).$$

3.3.3 Mixture Family

In this subsection, we will look at bivariate samples from Archimax copulas with an Archimedean generator given by Equation (2.16) and Pickands dependence function given by Equation (3.4). Recall that this Archimedean generator is a result of taking the Williamson $d$-transform of a mixture of a Pareto and inverse Pareto distribution $F_R$.

In Figure 3.3, we can observe the effect of $\alpha$ on the lower tail of the copula. Indeed, the index of regular variation of $\psi$ is $-\alpha$;

$$\lim_{x \to \infty} \frac{\psi(tx)}{\psi(x)} = \lim_{x \to \infty} \frac{(tx)^{-\alpha}/(2\alpha + 2)}{(x)^{-\alpha}/(2\alpha + 2)} = t^{-\alpha}.$$

As $\alpha$ becomes larger, $\psi$ converges more quickly to zero. Looking back at the Definition 3.0.3 of Archimax copulas, one can see that the regular variation of $\psi = \phi^{-1}$ becomes relevant when $u$ and $v$ are close to zero. Therefore it is not surprising to see less lower tail dependence when $\alpha$ grows.
Figure 3.2: Scatter plots ($n = 1000$) of random draws from a bivariate Archimax copula with Clayton Archimedean generator $\psi_\theta$ with parameters $\theta = [0.2, 0.9, 2]$ (top to bottom), with Gumbel-Hougaard asymmetric Pickands dependence function with parameters $(\kappa, \lambda) = [(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)]$ (left to right) and $\beta = 3$.

In Figure 3.4, $\gamma$ is the variable of interest that varies from row to row. From the expression of $\psi$, it is clear that the regular variation index of $1 - \psi(1/x)$ is equal to $-\min\{\gamma, 1\}$. As $u$ and $v$ both tend to 1, this regular variation comes into play. The smaller the $\gamma$, the faster $\psi$. 

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Figure 3.3: Scatter plots ($n = 1000$) of random draws from a bivariate Archimax copula with Pareto & Inverse Pareto mixture generator $\psi_{\alpha, \gamma}$ with parameters $\alpha = \{1, 3, 6\}$ (top to bottom) and $\gamma = 0.3$, with Gumbel-Hougaard asymmetric Pickands dependence function with parameters $(\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}$ (left to right) and $\beta = 3$.

converges to 1, that is why we see more upper tail dependence in the plots for smaller $\gamma$. 

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Figure 3.4: Scatter plots \((n = 1000)\) of random draws from a bivariate Archimax copula with Pareto & Inverse Pareto mixture generator \(\psi_{\alpha, \gamma}\) with parameters \(\gamma = \{0.3, 0.6, 0.9\}\) (top to bottom) and \(\alpha = 3\), with Gumbel-Hougaard asymmetric Pickands tail dependence function with parameters \((\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}\) (left to right) and \(\beta = 3\).

### 3.3.4 Inverse Pareto-Simplex Family

In this Subsection, we will look at draws from bivariate Archimax copulas with generator given by Equation (2.15) and Pickands dependence function given by Equation (3.4). As
it is the case for the mixture family of Subsection 3.3.3, the index of regular variation of \(1 - \psi(1/x)\) is equal to \(-\min\{\gamma, 1\}\). Figure 3.5 illustrates the effect of \(\gamma\) on the shape of the copula. The larger \(\gamma\) is, the less strong dependence is in the upper tail.

Figure 3.5: Scatter plots \((n = 1000)\) of random draws from a bivariate Archimax copula with Inverse Pareto mixture generator \(\psi\), with parameters \(\gamma = \{0.3, 0.9, 5\}\) (top to bottom) and with Gumbel-Hougaard asymmetric Pickands tail dependence function with parameters \((\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}\) (left to right) and \(\beta = 3\).
3.3.5 Gumbel Generator

Finally, we are interested in bivariate Archimax copula random draws with Gumbel generator $\psi_\theta$ given in Table 2.1 and asymmetric Gumbel-Hougaard Pickands dependence function given by Equation (3.4). In Figure 3.6, we do not clearly see the effect of the asymmetry parameters $\kappa$ and $\lambda$. We notice that $\theta$ is positively associated with the strength of dependence, as it is the case in the regular Archimedean Gumbel copula.
3.3. Examples

Figure 3.6: Scatter plots \((n = 1000)\) of random draws from a bivariate Archimax copula with Gumbel Archimedean generator \(\psi_\theta\) with parameters \(\theta = \{2, 5, 10\}\) (top to bottom), with asymmetric Gumbel Pickands dependence function with parameters \((\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}\) (left to right) and \(\beta = 3\).
4 Estimating the Pickands dependence function in Archimax Copulas

In this Chapter, two new estimators for the Pickands dependence function for bivariate Archimax Copulas are motivated and described. Section 4.1 studies the Pickands type estimator while Section 4.2 investigates the Capéraà-Fougères-Genest type (CFG) estimator. The asymptotic behavior of both estimators is established by Theorems 4.1.1 and 4.2.1 for regularly varying Archimedean generators. Theorems 4.1.2 and 4.2.2 treat the case of generators which have a finite support. Because these new estimators are generalizations of the rank-based Pickands and Capéraà-Fougères-Genest estimators, the proofs of the aforementioned Theorems have a similar structure to the proofs from Genest & Segers (2009). However, the derivations in this Chapter required different tools and are original results.

Suppose that \((X, Y)\) has a continuous bivariate distribution with continuous margins \(F\) and \(G\) and an underlying Archimax copula \(C_{\psi, A}\), viz.

\[
C_{\psi, A}(u, v) = \psi \left[ \left( \psi^{-1}(u) + \psi^{-1}(v) \right) A \left( \frac{\psi^{-1}(u)}{\psi^{-1}(u) + \psi^{-1}(v)} \right) \right]
\] (4.1)

where \(A : [0, 1] \to \left[ \frac{1}{2}, 1 \right] \) is a bivariate Pickands dependence function and \(\phi = \psi^{-1}\) a 2-variate Archimedean generator. The purpose of this chapter is to construct estimators of the Pickands dependence function \(A\) given knowledge of \(\psi\), and to determine their asymptotic behavior.
Suppose first that the margins $F$ and $G$ are known. Recall that this allows us to access

$$(U, V) = (F(X), G(Y)) ,$$  \hspace{1cm} (4.2)

which is distributed according to $C_{\psi, A}$. Now define the random variables

$$\xi(t) = \frac{\psi^{-1}(U) \wedge \psi^{-1}(V)}{t} ,$$

for $t \in (0, 1)$, while setting $\xi(0) = \psi^{-1}(V)$ and $\xi(1) = \psi^{-1}(U)$.

$\xi(t)$ is a random variable with survival function given by

$$P(\xi(t) > x) = P\left(\frac{\psi^{-1}(U) \wedge \psi^{-1}(V)}{t} > x\right)$$
$$= P\left(\frac{\psi^{-1}(U)}{t} > x, \frac{\psi^{-1}(V)}{1-t} > x\right)$$
$$= P\left(U < \psi(tx), V < \psi((1-t)x)\right)$$
$$= C_{\psi, A}(\psi(tx), \psi((1-t)x))$$
$$= \psi(xA(t)) ,$$

for $x \geq 0$. Let $Z = A(t)\xi(t)$. Note that the distribution of $Z$ depends only on the Archimedean generator $\psi$, which is assumed to be known. Indeed,

$$P(Z > x) = P\left(\frac{\psi(tx)}{A(t)} > x\right) = \psi(x) .$$

If the expectation of $Z$ exists,

$$E(Z) = E(A(t)\xi(t)) = A(t)E(\xi(t)) .$$

Therefore,

$$A(t) = \frac{E(Z)}{E(\xi(t))} ,$$  \hspace{1cm} (4.3)
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Also, provided \( E(\log Z) \) exists,

\[
E(\log Z) = E(\log \xi(t)) + E(\log A(t)) = E(\log \xi(t)) + \log(A(t)) ,
\]

so that

\[
A(t) = \exp \{ E(\log Z) - E(\log \xi(t)) \} .
\]  \hspace{1cm} (4.4)

Equation (4.3) will motivate the Pickands type (P) estimator in Section 4.1, while Equation (4.4) will motivate the Capéraà-Fougères-Genest type (CFG) estimator in Section 4.2.

Now suppose \((X_i, Y_i)_{i=1}^n\) is a random sample from a joint distribution with continuous margins \(F\) and \(G\) and underlying bivariate Archimax copula \(C_{\psi, A}\) as in Equation 4.1. Suppose that the margins are unknown, as it is typically the case in practice. Recall the definition of the scaled ranks from Chapter 2, viz.

\[
\hat{U}_i = \frac{1}{n+1} \sum_{j=1}^{n} 1(X_j \leq X_i), \quad \hat{V}_i = \frac{1}{n+1} \sum_{j=1}^{n} 1(Y_j \leq Y_i) ,
\]

which constitute a pseudo-sample from \(C_{\psi, A}\), as well as the empirical copula

\[
\hat{C}_n(u, v) = \frac{1}{n} \sum_{i=1}^{n} 1(\hat{U}_i \leq u, \hat{V}_i \leq v) .
\]

For \(i = 1, \ldots, n\), define

\[
\hat{\xi}_i(t) = \frac{\psi^{-1}(\hat{U}_i)}{t} \wedge \frac{\psi^{-1}(\hat{V}_i)}{1-t} ,
\]

for \(t \in (0, 1)\), and set \(\hat{\xi}_i(0) = \psi^{-1}(\hat{V}_i)\) and \(\hat{\xi}_i(1) = \psi^{-1}(\hat{U}_i)\). For any \(t \in [0, 1]\), \(\hat{\xi}_i(t)\), \(i = 1, \ldots, n\) is then an approximation to a random sample from \(\xi(t)\) and can be used to estimate \(A\). Two ways in which this can be done are explored next.

4.1 Pickands type estimator for Archimax Copulas

Equation (4.3) leads us to define the following estimator, which is an extension of the Pickands estimator for extreme-value copulas.
4.1. Pickands type estimator for Archimax Copulas

If $\psi$ is integrable on $\mathbb{R}^+$ we can define the Pickands type estimator for Archimax copulas to be:

$$A_n^p(t) = \frac{E(Z)}{\frac{1}{n} \sum_{i=1}^{n} \hat{\xi}_i(t)}$$

In the remainder of this section, we investigate the limiting distribution of the empirical process $A_n^p = n^{\frac{1}{2}} (A_n^p - A)$, which will determine the asymptotic properties of the Pickands type estimator $A_n^p$ as a function of $t$.

In the following lemma, we first find an integral representation for our statistic.

**Lemma 4.1.1.** Recall that $Z = A(t)\xi(t)$ and suppose that $Z$ has a finite expectation. Then for $t \in [0, 1]$,

$$\frac{1}{A_n^p(t)} = \frac{1}{E(Z)} \int_0^\infty \hat{C}_n (\psi(t x), \psi((1-t)x)) \, dx .$$  \hspace{1cm} (4.5)

**Proof.**

$$\frac{1}{A_n^p(t)} = \frac{1}{E(Z)} \times \frac{1}{n} \sum_{i=1}^{n} \hat{\xi}_i(t)$$

$$= \frac{1}{E(Z)} \times \frac{1}{n} \int_0^\infty \hat{C}_n (\psi(t x), \psi((1-t)x)) \, dx .$$

If we replace $\hat{C}_n$ by $C$ in the right hand side of Equation (4.5) we obtain $1/A(t)$. Indeed,

$$\frac{1}{E(Z)} \int_0^\infty C(\psi(t x), \psi((1-t)x)) \, dx = \frac{1}{E(Z)} \int_0^\infty \psi(x A(t)) \, dx$$

$$= \frac{1}{E(Z)} \int_0^\infty P(\xi(t) > x) \, dx$$

$$= \frac{1}{E(Z)} E(\xi(t))$$

$$= \frac{1}{E(Z)} E\left(\frac{Z}{A(t)}\right) = \frac{1}{A(t)} .$$

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Because the empirical copula \( \hat{C}_n \) is an unbiased estimator of \( C_{\psi,A} \), we expect the Pickands type estimator \( A_n^P \) to be asymptotically unbiased. To establish this and other results, define \( B_n^P = n^{-\frac{1}{2}} \left( \frac{1}{A_n^P} - \frac{1}{A} \right) \). Then

\[
A_n^P = -A^2 B_n^P 1 + n^{-\frac{1}{2}} B_n^P A_n^P. \tag{4.6}
\]

Although \( A_n^P \) is the process of interest, the process \( B_n^P \) is easier to work with. The limit of \( B_n^P \) will provide us with the limit of \( A_n^P \) due to Equation (4.6) in view of Slutsky’s Lemma.

4.1.1 Regularly Varying Generators

**Theorem 4.1.1.** Suppose that \( \psi(x) \in RV_\rho \) with \( \rho < -4 \). Let \( \hat{C}_1 \) and \( \hat{C}_2 \) be the first order partial derivatives of \( C_{\psi,A} \). Then

\[
A_n^P \xrightarrow{w} \frac{-A^2 B_n^P}{1 + n^{-\frac{1}{2}} B_n^P A_n^P} \int_0^\infty C(\psi(tx),\psi((1-t)x)) \, dx
\]

as \( n \to \infty \) where \( C \) is the weak limit of the empirical copula process, as given in Theorem 2.1.4.

**Proof.** Let \( C_n = \sqrt{n}(\hat{C}_n - C) \) be the empirical copula process and \( k_n = 2\psi^{-1}((n+1)^{-1}) \). Using Lemma 4.1.1 one obtains

\[
B_n^P(t) = \frac{1}{E(Z)} \int_0^\infty C_n(\psi(tx),\psi((1-t)x)) \, dx
\]

\[
= \frac{1}{E(Z)} \left\{ \int_{k_n}^{\infty} C_n(\psi(tx),\psi((1-t)x)) \, dx + \int_0^{k_n} C_n(\psi(tx),\psi((1-t)x)) \, dx \right\}
\]

\[
= \frac{1}{E(Z)} \left\{ I_{1,n}(t) + I_{2,n}(t) \right\}.
\]

The following Lemma shows that \( I_{1,n} \) is asymptotically negligible.

**Lemma 4.1.2.** Suppose that \( \psi(x) \in RV_\rho \) with \( \rho < -2 \), and \( k_n = 2\psi^{-1}((n+1)^{-1}) \). Then for any \( t \in [0,1] \)

\[
I_{1,n}(t) = \int_{k_n}^{\infty} C_n(\psi(tx),\psi((1-t)x)) \, dx \to 0
\]
4.1. Pickands type estimator for Archimax Copulas

almost surely as \( n \to \infty \).

**Proof.** If \( x > k_n, \hat{C}_n(\psi(tx),\psi((1-t)x)) = 0 \). Indeed, assume without loss of generality that \( t \geq 1/2 \) (if \( t < 1/2 \), replace \( t \) by \( 1-t \)), then

\[
\begin{align*}
x &> 2\psi^{-1}(1/(n+1)) \\
tx &> 2t\psi^{-1}(1/(n+1)) \geq \psi^{-1}(1/(n+1)) \\
\psi(tx) &< 1/(n+1).
\end{align*}
\]

Recall that \( \hat{C}_n \) is the empirical cumulative distribution function of the scaled ranks. No scaled rank \( \hat{U}_i \) or \( \hat{V}_i \) can be strictly smaller than \( 1/(n+1) \), therefore \( \hat{C}_n(\psi(tx),\psi((1-t)x)) = 0 \). Now,

\[
|I_{1,n}(t)| = \left| \int_{k_n}^{\infty} -n^{1/2} C(\psi(tx),\psi((1-t)x)) \, dx \right| = \left| n^{1/2} \int_{k_n}^{\infty} -\psi(A(t)x) \, dx \right| \leq n^{1/2} \int_{k_n}^{\infty} \psi(A(t)x) \, dx = \frac{n^{1/2}}{A(t)} \int_{2A(t)\psi^{-1}(1/(n+1))}^{\infty} \psi(x) \, dx \leq \frac{n^{1/2}}{A(t)} \int_{\psi^{-1}(1/(n+1))}^{\infty} \psi(x) \, dx.
\]

With the appropriate choice of a slowly varying function \( L_1 \),

\[
\psi(x) = L_1(x)x^{\rho},
\]

Also

\[
\lim_{x \to \infty} \frac{1/\psi(tx)}{1/\psi(x)} = \lim_{x \to \infty} \frac{\psi(x)}{\psi(tx)} = 1/x^{\rho}
\]

therefore \( \frac{1}{\psi(x)} \in RV_{-\rho} \).

We also have that \(-\rho > 0\) and \( \frac{1}{\psi(x)} \) is non-decreasing and \( \frac{1}{\psi(x)} = \infty \). Thus we can apply
Chapter 4. Estimating the Pickands dependence function in Archimax Copulas

Theorem 0.8 (v) in Resnick (1987) (see Theorem 2.4.1) and find that

$$\psi^{-1}(1/x) \in RV_{\frac{-1}{\rho}}.$$  \hfill (4.8)

Since $\rho < -1$, we apply Karamata's characterization theorem which states that $\int_{t}^{\infty} \psi(s) ds = t^{\rho+1} L(t)$ for some slowly varying function $L$ (see Theorem 2.4.3). Hence we obtain

$$\frac{n^{\frac{1}{2}}}{A(t)} \int_{\psi^{-1}((n+1)^{-1})}^{\infty} \psi(x) dx = \left(\frac{n}{n+1}\right)^{\frac{1}{2}} \frac{1}{A(t)} \frac{1}{(n+1)^{1/2+1/\rho}} L_{1}(\psi^{-1}((n+1)^{-1})) .$$

Therefore,

$$|I_{1,n}(t)| \leq \left(\frac{n}{n+1}\right)^{\frac{1}{2}} \frac{1}{A(t)} \frac{1}{(n+1)^{1/2+1/\rho}} L_{1}(\psi^{-1}((n+1)^{-1})) \to 0 \quad \text{almost surely as } n \to \infty .$$

To handle $I_{2,n}(t)$, we decompose it as per Stute's representation:

$$I_{2,n}(t) = J_{1,n}(t) + J_{2,n}(t) + J_{3,n}(t) + J_{4,n}(t) ,$$

where

$$J_{1,n}(t) = \int_{0}^{k_{n}} \alpha_{n}(\psi(tx), \psi((1-t)x)) dx$$

$$J_{2,n}(t) = -\int_{0}^{k_{n}} \alpha_{n}(\psi(tx), 1) \hat{C}_{1}(\psi(tx), \psi((1-t)x)) dx$$

$$J_{3,n}(t) = -\int_{0}^{k_{n}} \alpha_{n}(1, \psi((1-t)x)) \hat{C}_{2}(\psi(tx), \psi((1-t)x)) dx$$

$$J_{4,n}(t) = \int_{0}^{k_{n}} R_{n}(\psi(tx), \psi((1-t)x)) dx .$$

In the above, $\alpha_{n}$ and $R_{n}$ are as in Equation (2.5) of Subsection 2.1.2. In the sequel, we show that $J_{4,n}$ is asymptotically negligible.
Lemma 4.1.3. Suppose that $\psi \in RV_\rho$ where $\rho < -4$. Then

$$J_{4,n} = \int_0^{k_n} R_n(\psi(t x), \psi((1-t)x)) dx \to 0$$

almost surely as $n \to \infty$.

Proof. Recall that $k_n = 2\psi^{-1}((n+1)^{-1})$. According to Tsukahara (2005),

$$\sup_{(u,v) \in [0,1]^2} |R_n(u,v)| = O\left\{ n^{-1/4} \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\}.$$ 

Thus

$$\sup_{t \in [0,1]} |J_{4,n}(t)| = O\left\{ n^{-1/4} k_n \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\}.$$ 

and since $\psi \in RV_\rho$ where $\rho < -4$, $\psi^{-1} \left( \frac{1}{n} \right) \in RV_\beta$ where $\beta = -\frac{1}{\rho} < \frac{1}{4}$, by the same argument as in (4.8) in the proof of Lemma 4.1.2. Therefore for a suitable slowly varying function $L$,

$$\psi^{-1} \left( \frac{1}{n+1} \right) = L\left\{ \frac{1}{n+1} \right\} \left( \frac{1}{n+1} \right)^\beta.$$ 

and hence

$$\sup_{t \in [0,1]} |J_{4,n}(t)| = O\left\{ n^{-1/4} 2L \left\{ \frac{1}{n+1} \right\} \left( \frac{1}{n+1} \right)^\beta \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\}$$

converges to 0 almost surely as $n \to \infty$, as claimed. \[\square\]

To handle the terms $J_{1,n}(t)$, $J_{2,n}(t)$ and $J_{3,n}(t)$, note first that the convergence of the integrands is well established because $\alpha_n$ converges weakly to $\alpha$ as seen in Theorem 2.1.3. To show that the integrals themselves have a suitable limit, fix $\omega \in \left( \frac{-1}{\rho}, \frac{1}{2} \right)$ and let $q_\omega(t) = t^\omega (1-t)^\omega$ for all $t \in [0,1]$. Let

$$\mathbb{C}_{n,\omega}(u,v) = \alpha_n(u,v) / q_\omega(u \wedge v).$$
We can then rewrite our first three integrals as follows:

\[
J_{1,n}(t) = \int_0^{k_n} G_{n,\omega}(\psi(t x), \psi((1 - t) x)) K_{1,\omega}(x, t) d x
\]

\[
J_{2,n}(t) = -\int_0^{k_n} G_{n,\omega}(\psi(t x), 1) K_{2,\omega}(x, t) d x
\]

\[
J_{3,n}(t) = -\int_0^{k_n} G_{n,\omega}(1, \psi((1 - t) x)) K_{3,\omega}(x, t) d x,
\]

where

\[
K_{1,\omega}(x, t) = q_\omega(\psi(t x) \wedge \psi((1 - t) x))
\]

\[
K_{2,\omega}(x, t) = \dot{C}_1(\psi(t x), \psi((1 - t) x)) q_\omega(\psi(t x))
\]

\[
K_{3,\omega}(x, t) = \dot{C}_2(\psi(t x), \psi((1 - t) x)) q_\omega(\psi((1 - t) x))
\]

We will now show that for each \(i \in \{1, 2, 3\}\) and \(t \in (0, 1)\) there exists an integrable function \(K_i^*: (0, \infty) \to \mathbb{R}\) such that \(K_{i,\omega}(x, t) \leq K_i^*(x, t)\) for all \(x \in (0, k_n)\). First, note that

\[
K_{1,\omega}(x, t) \leq \min(\psi(t x)^\omega, \psi((1 - t) x)^\omega) \leq \left\{ \psi\left(\frac{x}{2}\right) \right\}^\omega.
\]

Since \(\psi(x)^\omega \in RV_{\rho\omega}\) and \(\omega\rho < -1\), \(\{\psi(x)\}^\omega\) is integrable by Karamata’s theorem.

For \(K_{2,\omega}(x, t)\), we have that \(q_\omega(\psi(t x)) \leq \psi(t x)^\omega\) so that

\[
K_{2,\omega}(x, t) \leq \dot{C}_1(\psi(t x), \psi((1 - t) x)) [\psi(t x)]^\omega.
\]

However,

\[
\dot{C}_1(\psi(t x), \psi((1 - t) x)) = \frac{(\psi^{-1})'(\psi(t x))}{(\psi^{-1})'(\psi(x A(t)))} \left( A(t) + (1 - t) A'(t) \right)
\]

(4.9)
4.1. Pickands type estimator for Archimax Copulas

and because $\psi$ is decreasing and $(\psi^{-1})'$ is increasing and negative,

$$\frac{(\psi^{-1})'(\psi(tx))}{(\psi^{-1})'(\psi(xA(t)))} \leq 1.$$ 

Put together, we have the following bound:

$$K_{2,\omega}(x, t) \leq \left( A(t) + (1 - t)A'(t) \right) \{\psi(tx)\}^\omega = K^*_2(x, t).$$

The function $K_{3,\omega}(x, t)$ can be bounded above analogously:

$$K_{3,\omega}(x, t) \leq \left( A(t) - tA'(t) \right) \{\psi((1 - t)x)\}^\omega = K^*_3(x, t).$$

Both bounds $K^*_2(x, t)$ and $K^*_3(x, t)$ are integrable in $x \in (0, k_n)$ for all $t \in (0, 1)$.

We can now apply the Theorem G.1 in Genest & Segers (2009) which states that the weighted bivariate empirical process $G_{n,\omega}(u_1, u_2) = \alpha_n(u, v)/q_\omega(u \land v)$ converges to a centered Gaussian process $G_\omega(u, v)$ pinned down at zero along $u = 0, v = 0$ and at $(u, v) = (1, 1)$ with covariance

$$\text{cov}\{G_\omega(u, v), G_\omega(u', v')\} = \frac{C(u, v)C(u', v') - C(u \land u', v \land v')}{q_\omega(u \land u')q_\omega(v \land v')}.$$

Using the Continuous Mapping Theorem (see van der Vaart & Wellner (1996), Chapter 1) we can deduce that the integrals $J_{1,n}, J_{2,n}, J_{3,n}$ jointly converge as follows:

$$J_{1,n}(t) \to w \int_0^\infty \alpha \left( \psi(tx), \psi((1 - t)x) \right) dx,$$

$$J_{2,n}(t) \to w - \int_0^\infty \alpha \left( \psi(tx), 1 \right) \hat{C}_1 \left( \psi(tx), \psi((1 - t)x) \right) dx,$$

$$J_{3,n}(t) \to w - \int_0^\infty \alpha \left( 1, \psi((1 - t)x) \right) \hat{C}_2 \left( \psi(tx), \psi((1 - t)x) \right) dx,$$

as $n \to \infty$. Since both $J_{4,n}(t)$ and $J_{1,n}(t)$ converge to zero almost surely for all $t \in (0, 1)$ as
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\( n \) tends to infinity,

\[
\mathbb{B}_n^P(t) \to_w \frac{1}{E(Z)} \int_0^\infty C(\psi(tx), \psi((1-t)x)) \, dx
\]

for all \( t \in (0, 1) \) as \( n \to \infty \). Next,

\[-A^2(t)\mathbb{B}_n^P(t) \to_w -A^2(t) \frac{1}{E(Z)} \int_0^\infty C(\psi(tx), \psi((1-t)x)) \, dx \]

as \( n \to \infty \), and

\[1 + n^{-1/2} \mathbb{B}_n^PA \to 1\]

in probability as \( n \to \infty \). We use Slutsky’s theorem (van der Vaart & Wellner (1996), Page 32) to conclude that

\[
\mathbb{A}_n^P(t) = \frac{-A^2(t)\mathbb{B}_n^P(t)}{1 + n^{-1/2}\mathbb{B}_n^PA(t)} \to_w -A^2(t) \frac{1}{E(Z)} \int_0^\infty C(\psi(tx), \psi((1-t)x)) \, dx
\]

as \( n \to \infty \), as claimed. \( \square \)

4.1.2 Generators of Finite Support

**Theorem 4.1.2.** Suppose that \( \phi = \psi^{-1} \) is an Archimedean generator such that \( \psi^{-1}(0) = \psi^{-1}(0) := \lim_{t \to 0} \psi^{-1}(t) < \infty \). Let \( \dot{C}_1 \) and \( \dot{C}_2 \) be the first order partial derivatives of \( C_{\psi,A} \). Then

\[
\mathbb{A}_n^P \to_w -A^2 \frac{1}{E(Z)} \int_0^\infty C(\psi(tx), \psi((1-t)x)) \, dx
\]

as \( n \to \infty \) where \( C \) is the weak limit of the empirical copula process, as given in Theorem 2.1.4.

**Proof.** The proof is very similar to the proof of Theorem 4.1.1. There is no need to split the integral

\[
\mathbb{B}_n^P(t) = \frac{1}{E(Z)} \int_0^\infty C_n(\psi(tx), \psi((1-t)x)) \, dx
\]
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\[
\frac{1}{E(Z)} \int_0^{2\psi^{-1}(0)} C_n(\psi(tx), \psi((1-t)x)) \, dx
\]
as we did before. We also use Stute's representation:

\[
\frac{1}{E(Z)} B_n(t) = J_{1,n}(t) + J_{2,n}(t) + J_{3,n}(t) + J_{4,n}(t),
\]
where

\[
J_{1,n}(t) = \int_0^{2\psi^{-1}(0)} \alpha_n(\psi(tx), \psi((1-t)x)) \, dx
\]
\[
J_{2,n}(t) = -\int_0^{2\psi^{-1}(0)} \alpha_n(\psi(tx), 1) \dot{C}_1(\psi(tx), \psi((1-t)x)) \, dx
\]
\[
J_{3,n}(t) = -\int_0^{2\psi^{-1}(0)} \alpha_n(1, \psi((1-t)x)) \dot{C}_2(\psi(tx), \psi((1-t)x)) \, dx
\]
\[
J_{4,n}(t) = \int_0^{2\psi^{-1}(0)} R_n(\psi(tx), \psi((1-t)x)) \, dx.
\]

We have that

\[
\sup_{t \in [0,1]} |J_{4,n}(t)| = O\left\{ n^{-1/4} \left\{ \log(n) \right\}^{1/2} \left\{ \log\log(n) \right\}^{1/4} \right\},
\]
so that \(J_{4,n}(t)\) is asymptotically negligible as before.

The rest of the proof remains unchanged from that of the regular variation case.

4.2 CFG type estimator for Archimax Copulas

Recall that if \(E(\log Z)\) exists,

\[
E(\log Z) = E(\log \xi(t)) + E(\log A(t)) = E(\log \xi(t)) + \log(A(t)),
\]

and

\[
A(t) = \exp\{E(\log Z) - E(\log \xi(t))\}.
\]

From this, we extend the Capéraà-Fougères-Genest (CFG) estimator for extreme-value copulas of the dependence function \(A\). The CFG type estimator for Archimax copulas is
defined as follows:

\[ \log A_{n}^{CFG}(t) = E (\log Z) - \frac{1}{n} \sum_{i=1}^{n} \log (\hat{\xi}_{i}(t)) , \]

provided that \( E (\log Z) \) exists.

In order to investigate the limiting behaviour of the empirical process \( \hat{A}_{n}^{CFG} = n^{\frac{1}{2}} \left( A_{n}^{CFG} - A \right) \), we can again first rewrite the CFG type estimator as an integral.

**Lemma 4.2.1.** Provided \( E (\log Z) \) exists,

\[ \log A_{n}^{CFG}(t) = E (\log Z) - \int_{0}^{\infty} \{ \hat{C}_{n}(\psi(tx),\psi((1-t)x)) - \mathbf{1}(x \leq 1) \} \frac{dx}{x}. \]

**Proof.** Note that

\[ \log(z) = \int_{1}^{\infty} \mathbf{1}(x \leq z) \frac{dx}{x} - \int_{0}^{1} \mathbf{1}(x > z) \frac{dx}{x} = \int_{0}^{\infty} \{ \mathbf{1}(x \leq z) - \mathbf{1}(x \leq 1) \} \frac{dx}{x}. \]

Therefore,

\[ E (\log Z) - \log A_{n}^{CFG}(t) = \frac{1}{n} \sum_{i=1}^{n} \log (\hat{\xi}_{i}(t)) \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \int_{0}^{\infty} \{ \mathbf{1}(x \leq \hat{\xi}_{i}(t)) - \mathbf{1}(x \leq 1) \} \frac{dx}{x} \]

\[ = \int_{0}^{\infty} \{ \hat{C}_{n}(\psi(tx),\psi((1-t)x)) - \mathbf{1}(x \leq 1) \} \frac{dx}{x}. \]

Next, define the following process

\[ \mathbb{B}_{n}^{CFG}(t) = n^{1/2} \{ \log A_{n}^{CFG}(t) - \log A(t) \} . \]

As in the case of the Pickands type estimator, we will first focus on the limit \( \mathbb{B}_{n}^{CFG} \) because it is easier to handle. Once its limit has been established, the limiting process of \( \hat{A}_{n}^{CFG} \) will follow from it by Slutsky’s lemma.
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4.2.1 Regularly Varying Generators

**Theorem 4.2.1.** Suppose that \( \psi(x) \in RV_{\rho} \) with \( \rho < 0 \) and \( 1 - \psi\left(\frac{1}{x}\right) \in RV_{-\gamma} \) with \( \gamma > 0 \) and that \( E[\log Z] \) exists. Let \( \hat{C}_1 \) and \( \hat{C}_2 \) be the first order partial derivatives of \( C_{\psi,A} \). Then

\[
\mathbb{A}_n^{CFG} \rightarrow w - A \int_0^{\infty} \mathbb{C}(\psi(tx), \psi((1-t)x)) \frac{dx}{x}
\]

as \( n \to \infty \) where \( \mathbb{C} \) is the weak limit of the empirical copula process, as given in Theorem 2.1.4.

**Proof.** In the following lemma we represent \( \mathbb{B}_n^{CFG} \) as an integral.

**Lemma 4.2.2.** Suppose that \( E[\log Z] \) exists and that \( \psi(x) \in RV_{\rho} \) with \( \rho < 0 \) and \( 1 - \psi\left(\frac{1}{x}\right) \in RV_{-\gamma} \) with \( \gamma > 0 \). Then

\[
\mathbb{B}_n^{CFG}(t) = n^{1/2} \left\{ \log A_n^{CFG}(t) - \log A(t) \right\} = -\int_0^{\infty} \mathbb{C}_n(\psi(tx), \psi((1-t)x)) \frac{dx}{x}.
\]

**Proof.** First note that

\[
\int_0^{\infty} \mathbb{C}_n(\psi(tx), \psi((1-t)x)) \frac{dx}{x}
\]

\[= n^{1/2} \int_0^{\infty} \left\{ C(\psi(tx), \psi((1-t)x)) - \hat{C}_n(\psi(tx), \psi((1-t)x)) \right\} \frac{dx}{x}
\]

\[= n^{1/2} \int_0^{\infty} \left\{ \psi(xA(t)) - \hat{C}_n(\psi(tx), \psi((1-t)x)) \right\} \frac{dx}{x}
\]

\[= n^{1/2} \int_0^{\infty} \left\{ \psi(xA(t)) - 1(x \leq 1) - \left( \hat{C}_n(\psi(tx), \psi((1-t)x)) - 1(x \leq 1) \right) \right\} \frac{dx}{x}
\]

\[= n^{1/2} \left[ \int_0^{\infty} \left\{ \psi(xA(t)) - 1(x \leq 1) \right\} \frac{dx}{x} + \log \left( A_n^{CFG}(t) \right) - E[\log(Z)] \right],
\]

where the last step follows from Lemma 4.2.1.

Now using the fact that \( \psi(x) \in RV_{\rho} \) with \( \rho < 0 \) and \( 1 - \psi\left(\frac{1}{x}\right) \in RV_{-\gamma} \) with \( \gamma > 0 \),

\[
\int_0^{\infty} \left\{ \psi(xA(t)) - 1(x \leq 1) \right\} \frac{dx}{x} = \int_0^{A(t)} \left\{ \psi(x) - 1 \right\} \frac{dx}{x} + \int_{A(t)}^{\infty} \left\{ \psi(x) \right\} \frac{dx}{x}.
\]
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\[
\begin{align*}
&= \left[ \log(x) (\psi(x) - 1) \right]_0^{A(t)} - \int_0^{A(t)} \log(x) (\psi(x))' \, dx \\
&\quad + \left[ \log(x) \psi(x) \right]_0^{A(t)} - \int_{A(t)}^{\infty} \log(x) (\psi(x))' \, dx \\
&= -\log(A(t)) + E \left( \log Z \right).
\end{align*}
\]

Here the regular variation of \( \psi \) ensured that \( \lim_{x \to 0^+} \log(x) (1 - \psi(x)) \), \( \lim_{x \to \infty} \log(x) (\psi(x)) \) = 0. This is due to Bingham et al. (1989), Proposition 1.5.1. This completes this proof. \( \Box \)

Similarly to before, let \( C_n = \sqrt{n} (\hat{C}_n - C) \), \( k_n = 2 \psi^{-1} ((n + 1)^{-1}) \) and \( l_n = \frac{1}{n^\delta} \), where \( \delta > 1/\gamma \). Then

\[
\mathbb{P}_n^{CFG} (t) = -\left\{ \int_0^{l_n} C_n (\psi(t x), \psi((1 - t) x)) \frac{dx}{x} \right. \\
+ \int_{l_n}^{k_n} C_n (\psi(t x), \psi((1 - t) x)) \frac{dx}{x} \\
+ \int_{k_n}^{\infty} C_n (\psi(t x), \psi((1 - t) x)) \frac{dx}{x} \bigg\}
= - \left\{ I_{3, n} (t) + I_{2, n} (t) + I_{1, n} (t) \right\}.
\]

In the following two lemmas, we show that \( I_{1, n} \) and \( I_{3, n} \) are asymptotically negligible.

**Lemma 4.2.3.** Suppose that \( \psi(x) \in RV_\rho \) with \( \rho < 0 \), then for \( t \in (0, 1) \)

\[
I_{1, n} (t) = \int_{k_n}^{\infty} C_n (\psi(t x), \psi((1 - t) x)) \frac{dx}{x} \to 0
\]

almost surely as \( n \to \infty \).

**Proof.** As in the proof of Lemma 4.1.2, if \( x > k_n \), \( \hat{C}_n (\psi(t x), \psi((1 - t) x)) = 0 \) so that

\[
|I_{1, n} (t)| = \left| \int_{k_n}^{\infty} -n^{\frac{1}{2}} C (\psi(t x), \psi((1 - t) x)) \frac{dx}{x} \right| \\
= n^{\frac{1}{2}} \int_{k_n}^{\infty} -\psi (A(t) x) \frac{dx}{x} \\
\leq n^{\frac{1}{2}} \int_{k_n}^{\infty} \psi (A(t) x) \frac{dx}{x}
\]

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\[
\begin{align*}
&= \frac{n^2}{A(t)} \int_{2A(t)\psi^{-1}((n+1)^{-1})}^{\infty} \psi(x) \frac{dx}{x} \\
&\leq \frac{n^2}{A(t)} \int_{\psi^{-1}((n+1)^{-1})}^{\infty} \psi(x) \frac{dx}{x}
\end{align*}
\]

With the appropriate choice of a slowly varying function \( L_1 \), \( \psi(x) = L_1(x)x^\rho \) so that

\[
\psi(x)/x = L_1(x)x^{\rho-1}
\]

and as argued in the proof of Lemma 4.1.2, \( \psi^{-1}(1/x) \in RV_{\rho}^{-1} \). We apply Karamata’s theorem, and obtain:

\[
\frac{n^2}{A(t)} \int_{\psi^{-1}((n+1)^{-1})}^{\infty} \psi(x) \frac{dx}{x} = n^2 \frac{1}{A(t)} \frac{1}{(n+1)} L_1(\psi^{-1}((n+1)^{-1}))
\]

Therefore,

\[
|I_{1,n}(t)| \leq n^2 \frac{1}{A(t)} \frac{1}{(n+1)} L_1(\psi^{-1}((n+1)^{-1})) \to 0
\]

almost surely as \( n \to \infty \).

**Lemma 4.2.4.** Suppose \( 1 - \psi(\frac{1}{x}) \in RV_{-\gamma} \) with \( \gamma > 0 \). Then

\[
I_{3,n}(t) = \int_{0}^{\infty} C_n(\psi(t\delta), \psi((1-t)\delta)) \frac{dx}{x} \to 0
\]

almost surely as \( n \to \infty \).

**Proof.** By Theorem 2.4.3, since \( 1 - \psi(1/x) \in RV_{-\gamma} \) with \( \gamma > 0 \), there exists a slowly varying function \( L_1 \) such that for \( x > 0 \),

\[
1 - \psi(1/x) = L_1(x)x^{-\gamma}.
\]

Then for \( t > 0 \),

\[
1 - \psi(1/t^\delta) = L_1(t^\delta)t^{-\delta\gamma}.
\]
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In other words, if a function \( f(t) \) is regularly varying with index \( a \), then \( f(t^b) \) is regularly varying with index \( a^b \) (see Proposition 1.5.7. in Bingham et al. (1989)). Also, \((t + 1)\) is regularly varying function with index 1. Combining the results,

\[
\lim_{s \to \infty} \frac{L_1((ts)^\delta)(ts)^{-\delta \gamma}(ts+1)}{L_1(s^\delta)s^{-\delta \gamma}(s+1)} = \lim_{s \to \infty} \frac{L_1((ts)^\delta)(ts)^{-\delta \gamma}}{L_1(s^\delta)s^{-\delta \gamma}} \lim_{s \to \infty} \frac{1 + ts}{1 + s} = t^{-\delta \gamma} = t^{1-\delta \gamma}.
\]

Since \( \gamma \delta > 1 \), we can use Proposition 1.5.1. from Bingham et al. (1989) which states that any regularly varying function with strictly negative index converges to zero.

Since \( L_1(t^\delta) t^{-\delta \gamma}(t + 1) \to 0 \) as \( t \to \infty \), there exists a \( T > 0 \) such that for \( t > T \),

\[
1 \geq L_1\left(t^\delta\right) t^{-\delta \gamma}(t + 1) > L_1\left(t^\delta\right) t^{1-\delta \gamma}
\]

\[
\frac{1}{t + 1} > L_1\left(t^\delta\right) t^{-\gamma \delta},
\]

\[
1 - \frac{t}{t + 1} > 1 - \psi\left(\frac{1}{t^\alpha}\right).
\]

So for \( n \) large enough, since \( x < l_n \),

\[
\frac{n}{n + 1} < \psi(l_n) \leq \psi(x) \leq \psi(t x) \wedge \psi((1 - t)x).
\]

Therefore, \( \hat{C}_n(\psi(tx), \psi((1 - t)x)) = 1 \), and

\[
|I_{3,n}(t)| \leq \int_{0}^{l_n} \left|C_n(\psi(tx), \psi((1 - t)x))\right| \frac{1}{x} |dx|
\]

\[
= n^{1/2} \int_{0}^{l_n} \left|(1 - \psi(x A(t)))\right| \frac{1}{x} |dx|
\]

\[
= n^{1/2} \int_{0}^{l_n} \left|(1 - \psi(x A(t)))\right| \frac{1}{x} |dx|
\]

\[
= n^{1/2} \int_{n^{\delta/A(t)}}^{\infty} \left|(1 - \psi(1/x))\right| \frac{1}{x} |dx|.
\]

Since \( (1 - \psi(1/x)) \frac{1}{x} \in RV_{-\gamma-1} \) and \(-\gamma - 1 < -1\), we can use Karamata’s theorem to find that

\[
|I_{3,n}(t)| \leq n^{1/2} \int_{n^{\delta/A(t)}}^{\infty} \left|(1 - \psi(1/x))\right| \frac{1}{x} |dx|
\]
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\[ n^{1/2} \left( \frac{n^\delta}{A(t)} \right)^{\gamma - 1 + 1} L_1 \left( \frac{n^\delta}{A(t)} \right) = n^{1/2 - \gamma \delta} A(t)^{\gamma} L_1 \left( \frac{n^\delta}{A(t)} \right) \]

which vanishes as \( n \to \infty \) since \( 1/2 - \gamma \delta < 0 \).

To determine the limit of \( I_{2,n}(t) \), we decompose it using Stute's representation:

\[ I_{2,n}(t) = J_{1,n}(t) + J_{2,n}(t) + J_{3,n}(t) + J_{4,n}(t) , \]

where

\[
\begin{align*}
J_{1,n}(t) &= \int_{l_n}^{k_n} \alpha_n \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} \\
J_{2,n}(t) &= -\int_{l_n}^{k_n} \alpha_n \left( \psi(tx), 1 \right) \dot{C}_1 \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} \\
J_{3,n}(t) &= -\int_{l_n}^{k_n} \alpha_n \left( 1, \psi((1-t)x) \right) \dot{C}_2 \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} \\
J_{4,n}(t) &= \int_{l_n}^{k_n} R_n \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} .
\end{align*}
\]

The following lemma shows that \( J_{4,n}(t) \) is asymptotically negligible.

**Lemma 4.2.5.** \( J_{4,n}(t) = \int_{l_n}^{k_n} R_n \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} \to 0 \) almost surely as \( n \to \infty \), for all \( t \in (0, 1) \).

**Proof.** According to Tsukahara (2005),

\[
\sup_{(u, v) \in [0,1]^2} |R_n(u, v)| = O \left\{ n^{-1/4} \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\}
\]

Thus

\[
\sup_{t \in [0,1]} |J_{4,n}(t)| = \log \left( \frac{k_n}{l_n} \right) O \left\{ n^{-1/4} \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\}
\]

Recall that \( k_n = 2\psi^{-1} \left( (n + 1)^{-1} \right) \). Since \( \psi \in RV_\rho \) where \( \rho < 0 \), \( \psi^{-1} \left( \frac{1}{n} \right) \in RV_\beta \) where \( \beta = -\frac{1}{\rho} \), by the same argument as in (4.8) in the proof of Lemma 4.1.2. Therefore for a suitable
slowly regularly varying function $L$,

$$k_n = 2L \left\{ \frac{1}{n+1} \right\} \left( \frac{1}{n+1} \right)^\beta.$$ 

Therefore,

$$\sup_{t \in [0,1]} |J_{4,n}(t)| = O \left\{ n^{-1/4} \log \left( n^\delta \right) \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\}$$

$$= O \left\{ n^{-1/4} \log \left( n^\delta 2L \left\{ \frac{1}{n+1} \right\} \left( \frac{1}{n+1} \right)^\beta \right) \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\} \to 0$$

almost surely as $n \to \infty$ as needed.

To compute the limits of $I_{i,n}(t)$, $i \in \{1,2,3\}$, fix $\omega \in (0, 1/2)$ and let $q_\omega(t) = t^\omega (1 - t)^\omega$ for all $t \in [0,1]$ and let $\mathbb{G}_{n,\omega}(u,v) = \alpha_n(u,v)/q_\omega(u \wedge v)$. We can then rewrite our first three integrals as follows:

$$J_{1,n}(t) = \int_{I_n}^{k_n} \mathbb{G}_{n,\omega} \left( \psi(tx), \psi((1-t)x) \right) K_{1,\omega}(x,t) \, dx$$

$$J_{2,n}(t) = -\int_{I_n}^{k_n} \mathbb{G}_{n,\omega} \left( \psi(tx), 1 \right) K_{2,\omega}(x,t) \, dx$$

$$J_{3,n}(t) = -\int_{I_n}^{k_n} \mathbb{G}_{n,\omega} \left( 1, \psi((1-t)x) \right) K_{3,\omega}(x,t) \, dx ,$$

where

$$K_{1,\omega}(x,t) = q_\omega \left( \psi(tx) \wedge \psi((1-t)x) \right) / x$$

$$K_{2,\omega}(x,t) = \dot{C}_1 \left( \psi(tx), \psi((1-t)x) \right) q_\omega \left( \psi(tx) \right) / x$$

$$K_{3,\omega}(x,t) = \dot{C}_2 \left( \psi(tx), \psi((1-t)x) \right) q_\omega \left( \psi((1-t)x) \right) / x .$$

As for the Pickands type estimator in Section 4.1, we need to find integrable upper bounds to the functions $K_{i,\omega}$.

For $x \geq 1$, the same bounds as for the Pickands type estimator work here. However, we
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need to find a suitable bound for $x \in (l_n, 1)$. First note that the bounds for the partial derivatives $\dot{C}_1$ and $\dot{C}_2$ remain as argued in the proof of Theorem 4.1.1 for the Pickands type estimator (see Equation (4.9) Page 46):

$$\dot{C}_1 (\psi(t x), \psi((1 - t)x)) \leq \left(A(t) + (1 - t) A'(t)\right)$$
$$\dot{C}_3 (\psi(t x), \psi((1 - t)x)) \leq \left(A(t) - t A'(t)\right).$$

Furthermore,

$$q_\omega (\psi(t x) \land \psi((1 - t)x)) = (\psi(t x) \land \psi((1 - t)x))^\omega \left(1 - \psi(t x) \land \psi((1 - t)x)\right)^\omega$$
$$\leq (1 - \psi(t x) \land \psi((1 - t)x))^\omega$$
$$\leq (1 - \psi(x))^\omega$$
$$\sim x^{\gamma \omega} \text{ as } x \to 0$$

Therefore an upper bound for $K_{1,\omega}$ is $\frac{(1 - \psi^{-1}(x))^\omega}{x}$ which behaves like $x^{\gamma \omega - 1}$ in a neighborhood of zero. Since $\gamma \omega - 1 > -1$, the upper bound is integrable on $(0, 1)$.

For $K_{2,\omega}$ and $K_{3,\omega}$ we find the bounds

$$K_{2,\omega} \leq \frac{(1 - \psi(t x))^\omega}{x} \left(A(t) + (1 - t) A'(t)\right)$$
$$K_{3,\omega} \leq \frac{(1 - \psi((1 - t)x))^\omega}{x} \left(A(t) - t A'(t)\right),$$

which are also integrable on $(0, 1)$.

Next, we apply theorem G.1 in Genest & Segers (2009) as we did for the Pickands type estimator in Theorem 4.1.1.

Using the continuous mapping theorem (van der Vaart & Wellner (1996), Page 20) we can
deduce that the integrals $J_{1,n}, J_{2,n}, J_{3,n}$ jointly converge as follows:

\[
J_{1,n}(t) \rightarrow_w \int_0^\infty \frac{\alpha \left( \psi(t x), \psi((1-t)x) \right)}{x} dx
\]

\[
J_{2,n}(t) \rightarrow_w -\int_0^\infty \frac{\alpha \left( \psi(t x), \psi((1-t)x) \right) C_1 \left( \psi(t x), \psi((1-t)x) \right)}{x} dx
\]

\[
J_{3,n}(t) \rightarrow_w -\int_0^\infty \frac{\alpha \left( 1, \psi((1-t)x) \right) C_2 \left( \psi(t x), \psi((1-t)x) \right)}{x} dx
\]

as $n \rightarrow \infty$. Since both $J_{4,n}(t)$ and $I_{1,n}(t)$ converge to zero almost surely as $n$ tends to infinity for all $t \in (0, 1)$,

\[
E_n^{CFG}(t) \rightarrow_w \int_0^\infty C \left( \psi(t x), \psi((1-t)x) \right) \frac{dx}{x}
\]

as $n \rightarrow \infty$. Since $E_n^{CFG} = n^{1/2} \log \left( \frac{A_n^{CFG}}{A} \right)$,

\[
A e^{n^{-1/2}E_n^{CFG}} = A_n^{CFG}.
\]

Expanding with a Taylor series about 0, we obtain

\[
e^{n^{-1/2}E_n^{CFG}(t)} = 1 + n^{-1/2}E_n^{CFG}(t) + \frac{n^{-1}E_n^{CFG^2}(t)}{2} + \frac{n^{-3/2}E_n^{CFG^3}(t)}{3!} + \ldots.
\]

Therefore,

\[
A_n^{CFG}(t) = A(t) + n^{-1/2}A(t)E_n^{CFG}(t) + o_p\left\{ n^{-1}E_n^{CFG^2}(t) \right\}
\]

and

\[
A_n^{CFG}(t) = n^{1/2} \left( A_n^{CFG} - A \right)(t) = A(t)E_n^{CFG}(t) + o_p\left\{ n^{-1/2}E_n^{CFG^2}(t) \right\}.
\]

Since $n^{-1/2}E_n^{CFG^2}(t)$ converges to zero in probability, Slutsky’s theorem implies that

\[
A_n^{CFG}(t) \rightarrow_w A(t) \int_0^\infty C \left( \psi(t x), \psi((1-t)x) \right) \frac{dx}{x}
\]

as $n \rightarrow \infty$. 

\[\Box\]
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4.2.2 Generators of Finite Support

Theorem 4.2.2. Suppose that \( \phi = \psi^{-1} \) is an Archimedean generator such that \( \lim_{t \to 0} \psi^{-1}(t) < \infty \) and \( 1 - \psi(1/x) \in RV_\gamma \) with \( \gamma > 0 \). Let \( C_1 \) and \( C_2 \) be the first order partial derivatives of \( C_{\psi,A} \). Then

\[
A_n^{CFG} \to w - A \int_0^\infty C\left(\psi(tx), \psi((1-t)x)\right) \frac{dx}{x}
\]

as \( n \to \infty \) where \( C \) is the weak limit of the empirical copula process, as given in Theorem 2.1.4.

Proof. Here, we decompose our process as follows, where \( l_n = n^{-\delta} \) for some \( \delta > 1/\gamma \):

\[
B_n^{CFG}(t) = -\int_0^\infty C_n\left(\psi(tx), \psi((1-t)x)\right) \frac{dx}{x}
\]

\[
= -\int_0^{2\psi^{-1}(0)} C_n\left(\psi(tx), \psi((1-t)x)\right) \frac{dx}{x}
\]

\[
= -\left\{ \int_0^{l_n} C_n\left(\psi(tx), \psi((1-t)x)\right) \frac{dx}{x} + \int_{l_n}^{2\psi^{-1}(0)} C_n\left(\psi(tx), \psi((1-t)x)\right) \frac{dx}{x} \right\}
\]

= \{-I_{3,n}(t) + I_{2,n}(t)\}.

Proving that \( I_{3,n}(t) \) vanishes almost surely as \( n \) tends to infinity is done as before in Lemma 4.2.4.

Similarly to the regular variation case (Theorem 4.2.1), we use Stute’s representation to obtain

\[
I_{2,n}(t) = I_{1,n}(t) + I_{2,n}(t) + I_{3,n}(t) + I_{4,n}(t),
\]

where

\[
I_{1,n}(t) = \int_{l_n}^{2\psi^{-1}(0)} \alpha_n\left(\psi(tx), \psi((1-t)x)\right) \frac{dx}{x}
\]
Chapter 4. Estimating the Pickands dependence function in Archimax Copulas

\[ J_{2,n}(t) = -\int_{l_n}^{2} \alpha_n \left( \psi(tx), 1 \right) \dot{C}_1 \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} \]
\[ J_{3,n}(t) = -\int_{l_n}^{2} \alpha_n \left( 1, \psi((1-t)x) \right) \dot{C}_2 \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} \]
\[ J_{4,n}(t) = \int_{l_n}^{2} R_n \left( \psi(tx), \psi((1-t)x) \right) \frac{dx}{x} . \]

\[ J_{4,n}(t) \] is asymptotically negligible since

\[ \sup_{t \in [0,1]} |J_{4,n}(t)| = O \left\{ n^{-1/4} \log(2\psi^{-1}(0)n^\delta) \left( \log(n) \right)^{1/2} \left( \log \log(n) \right)^{1/4} \right\} \to 0 \text{ as } n \to \infty . \]

The rest of the proof follows as in the proof of Theorem 4.2.1. \[ \Box \]

4.3 Summary

Recall the form of the Pickands type (P) and Capéraà-Fougères-Genest type (CFG) estimators for the Pickands dependence function of a bivariate Archimax copula \( C_{\psi,A} \), viz.

\[ A_{n,P}^P(t) = \frac{E(Z)}{\frac{1}{n} \sum_{i=1}^{n} \hat{\xi}_i(t)} , \]
which requires \( E(Z) \) to be finite, and

\[ A_{n,CFG}^{CFG}(t) = \exp \left\{ E(\log Z) \frac{1}{n} \sum_{i=1}^{n} \log(\hat{\xi}_i(t)) \right\} , \]
which requires \( E(\log Z) \) to be finite.

In Theorem 4.1.1, we required \( \psi(x) \) to be regularly varying (at infinity) with index \( \rho < -4 \) in order to establish the limiting process of \( A_{n,P}^P = n^{\frac{1}{2}} \left( A_{n,P}^P - A \right) \). Regarding \( A_{n,CFG}^{CFG} = n^{\frac{1}{2}} \left( A_{n,CFG}^{CFG} - A \right) \), conditions of Theorem 4.2.1 include regular variation of \( \psi \) at infinity with index \( \rho < 0 \) as well as the regular variation of \( 1 - \psi \) at zero with index \( -\gamma < 0 \). Note that as shown in Larsson & Nešlehová (2011), \( \gamma \leq 1 \). Indeed because \( \psi \) is \( d \)-monotone, \( \gamma \) cannot be greater than 1. Additionally, both estimators require \( A \) to be twice continuously
differentiable as well as the existence of the first order partial derivatives \( \hat{C}_{1, \psi, A} \) and \( \hat{C}_{2, \psi, A} \) of the copula \( C_{\psi, A} \). It is important to note that the established limits of \( \hat{A}_n \) and \( \hat{A}_n^{CFG} \) imply that the estimators are consistent when the conditions are fulfilled.

The conditions of regular variation can be linked to the maximum and minimum domains of attraction of \( C_{\psi, A} \). Indeed, recall Theorems 3.0.4 and 3.0.5 in Chapter 3. Theorem 3.0.4 requires that \( \phi(1 - 1/t) \in RV_m \) for some \( m \geq 1 \), or equivalently \( 1/\phi(1 - 1/t) \in RV_m \). We can then use Theorem 0.8(v) from Resnick (1987) to show that this is equivalent to saying that \( 1 - \psi(1/x) \in RV_{-1/m} \). This is the same condition that we use for the convergence of the Capéraà-Fougères-Genest type estimator. The condition of regular variation in Theorem 3.0.5 is that \( \phi(1/t) \in RV_{1/m} \) for some \( m \geq 0 \), which is equivalent to saying \( \psi(t) \in RV_{-m} \).

If a bivariate Archimax copula \( C_{\psi, A} \) fulfills the conditions of Theorem 4.1.1, then the conditions of Theorem 3.0.5 are met and the minimum domain of attraction of \( C_{\psi, A} \) is known. If \( C_{\psi, A} \) fulfills those of Theorem 4.2.1, then the conditions of both Theorems 3.0.4 and 3.0.5 are satisfied, meaning that the maximum and minimum domains of attraction \( C_{\psi, A} \) are known.

The existence of \( E(Z) \) and \( E(\log Z) \) is also linked to the regular variation of \( \psi \). If \( \psi \in RV_\rho \) where \( \rho < -1, E(Z) < \infty \). See, for example, Section 3.3.1 in Embrechts et al. (1997).

Theorems 4.1.2 and 4.2.2 establish the limits of the processes \( \hat{A}_n^P \) and \( \hat{A}_n^{CFG} \) for the special case for which the Archimedean generator \( \psi \) has a finite support.
In this Chapter, we conduct a Monte-Carlo simulation study to compare the effectiveness of the two estimators in finite samples, using different Archimedean generators and different Pickands dependence functions. The main quantity of interest is the Mean Integrated Squared Error (MISE):

\[
\text{MISE}(\hat{A}, A) = \int_0^1 \{\hat{A}(t) - A(t)\}^2 \, dt.
\]

We estimate this error by averaging squared errors on an evenly spaced grid \(t_1, \ldots, t_m \in [0, 1]\):

\[
\overline{\text{MISE}}(\hat{A}, A) = \frac{1}{m} \sum_{i=1}^{m} \{\hat{A}(t_i) - A(t_i)\}^2.
\]

In this experiment, a grid of resolution 0.01 was chosen, and each reported value is averaged on \(N_{MC} = 1000\) Monte Carlo replicates. Graphs will also be presented for ease of interpretation. It is important to note that some chosen generators do not satisfy the conditions of regular variation that were found to be sufficient for asymptotic unbiasedness in the previous chapter. Four families of Archimedean generators were chosen for this study, namely the Clayton, Pareto & Inverse Pareto Mixture, Inverse Pareto and Gumbel families. We used the asymmetric Gumbel stable tail dependence function with \(\beta = 3\) and \((\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}\) to study the effect of asymmetry and strength of tail dependence on the estimators. We expect to see positive bias in smaller samples for the
Pickands estimator due to the form of the estimator. The overestimation is attributed to Jensen's inequality when margins are known, viz.

\[ E(\overline{A}_n^p) \geq \frac{E(Z)}{E\left(\frac{1}{n} \sum_{i=1}^{n} \xi_i(t)\right)} = \frac{E(Z)}{E(Z)/A(t)} = A(t). \]

### 5.1 Clayton Family

The first family of Archimedean generators used is the Clayton family. Recall that \( A_n^p(t) \) requires the computation of \( E(Z) \) while \( A_n^{CFG}(t) \) requires computing \( E(\log(Z)) \). For the Clayton generator, \( E(Z) \) only exists for \( \theta \in (0, 1) \) and is given by:

\[ E(Z) = \int_0^{\infty} (1 + \theta t)^{-1/\theta} \, dt = \frac{1}{1 - \theta}. \]

\( E(\log(Z)) \) is finite for any \( \theta > 0 \) but not tractable so numerical integration will be used.

Next we need to verify the conditions for regular variation. First, note that

\[ \lim_{x \to \infty} \frac{\psi(tx)}{\psi(x)} = \lim_{x \to \infty} \frac{(1 + \theta tx)^{-1/\theta}}{(1 + \theta x)^{-1/\theta}} = t^{-1/\theta}, \]

and hence that \( \psi \in RV_{-1/\theta} \). Next

\[ \lim_{x \to \infty} \frac{1 - \psi(\frac{1}{x})}{1 - \psi(\frac{1}{tx})} = \lim_{x \to \infty} \frac{1 - (1 + \theta tx)^{-1/\theta}}{1 - (1 + \theta x)^{-1/\theta}} = \lim_{x \to \infty} \frac{1/t (1 + \theta tx)^{-1/\theta - 1}}{(1 + \theta x)^{-1/\theta - 1}} = \frac{1}{t}, \]

which shows that \( 1 - \psi(1/x) \in RV_{-1} \) regardless of \( \theta \). Theorem 4.1.1 requires \( \psi \in RV_\rho \) with \( \rho < -4 \), so the Pickands type estimator has only been proven to have a weak limit for this generator when \( \theta \in (0, 1/4) \). The Capéraà-Fougères-Genest type estimator on the other hand has a weak limit for all \( \theta > 0 \). As an illustration, Figures 5.1 and 5.2 each plot one realization for a relatively small sample size \( n = 100 \). One can see in particular that asymmetry affects the quality of estimation. The Pickands type estimator visibly deteriorates as \( \theta \) grows larger.
Chapter 5. Simulation Study

The Pickands type estimator is significantly more biased for larger $\theta$. In Figure 5.3, it is clear that the CFG type estimator is favorable over the Pickands type estimator. Note that the range of $\theta$’s taken includes values higher than $1/4$, for which the Pickands type estimator was not proven to be consistent. We can also observe the detrimental effect of asymmetry on the MISE for both estimators. Tables 5.1, 5.2 and 5.3 present the detailed results of this simulation experiment. The value of $\theta$ only affects the CFG type estimators’ performance for values exceeding 1, as shown by Table 5.4.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\theta$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P $n=100$</td>
<td>2.02</td>
<td>2.97</td>
<td>4.50</td>
<td>8.09</td>
<td>18.24</td>
<td>44.24</td>
<td>125.60</td>
<td>451.85</td>
<td>2828.48</td>
<td></td>
</tr>
<tr>
<td>CFG</td>
<td>1.10</td>
<td>1.17</td>
<td>1.13</td>
<td>1.13</td>
<td>1.11</td>
<td>1.19</td>
<td>1.17</td>
<td>1.17</td>
<td>1.10</td>
<td>1.07</td>
</tr>
<tr>
<td>P $n=250$</td>
<td>0.74</td>
<td>1.04</td>
<td>1.69</td>
<td>3.09</td>
<td>7.34</td>
<td>19.32</td>
<td>62.45</td>
<td>246.68</td>
<td>1740.23</td>
<td></td>
</tr>
<tr>
<td>CFG</td>
<td>0.41</td>
<td>0.43</td>
<td>0.43</td>
<td>0.43</td>
<td>0.44</td>
<td>0.42</td>
<td>0.47</td>
<td>0.47</td>
<td>0.48</td>
<td></td>
</tr>
<tr>
<td>P $n=500$</td>
<td>0.38</td>
<td>0.49</td>
<td>0.72</td>
<td>1.50</td>
<td>3.43</td>
<td>10.21</td>
<td>35.99</td>
<td>164.87</td>
<td>1250.10</td>
<td></td>
</tr>
<tr>
<td>CFG</td>
<td>0.23</td>
<td>0.23</td>
<td>0.21</td>
<td>0.23</td>
<td>0.24</td>
<td>0.23</td>
<td>0.25</td>
<td>0.24</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Clayton generator $\psi_\theta$ and an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.5, 0.5)$ (symmetric case). Number of Monte Carlo replicates $N_{MC} = 1000$.

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\theta$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P $n=100$</td>
<td>1.81</td>
<td>2.55</td>
<td>3.95</td>
<td>7.63</td>
<td>17.02</td>
<td>41.28</td>
<td>122.56</td>
<td>441.56</td>
<td>2814.12</td>
<td></td>
</tr>
<tr>
<td>CFG</td>
<td>1.44</td>
<td>1.38</td>
<td>1.48</td>
<td>1.49</td>
<td>1.38</td>
<td>1.30</td>
<td>1.41</td>
<td>1.35</td>
<td>1.27</td>
<td></td>
</tr>
<tr>
<td>P $n=250$</td>
<td>0.71</td>
<td>0.90</td>
<td>1.51</td>
<td>2.56</td>
<td>6.15</td>
<td>17.90</td>
<td>58.07</td>
<td>240.42</td>
<td>1743.16</td>
<td></td>
</tr>
<tr>
<td>CFG</td>
<td>0.59</td>
<td>0.57</td>
<td>0.60</td>
<td>0.58</td>
<td>0.57</td>
<td>0.60</td>
<td>0.56</td>
<td>0.56</td>
<td>0.60</td>
<td></td>
</tr>
<tr>
<td>P $n=500$</td>
<td>0.38</td>
<td>0.48</td>
<td>0.67</td>
<td>1.19</td>
<td>2.94</td>
<td>9.01</td>
<td>34.11</td>
<td>159.23</td>
<td>1250.06</td>
<td></td>
</tr>
<tr>
<td>CFG</td>
<td>0.34</td>
<td>0.34</td>
<td>0.35</td>
<td>0.33</td>
<td>0.32</td>
<td>0.31</td>
<td>0.31</td>
<td>0.33</td>
<td>0.33</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.2: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Clayton generator $\psi_\theta$ and an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.3, 0.7)$ (mild asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$. 

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Figure 5.1: Rank-based Pickands (left) and Capéraà-Fougères-Genest (right) type estimates (in black) of the Pickands dependence function (in red) for a sample of size $n = 100$ from a bivariate Archimax copula with asymmetric Gumbel Pickands dependence function with parameters $β = 3$ and $(κ, λ) = (0.9, 0.9)$, and a Clayton generator $ψ_θ$ with parameters $θ = \{0.1, 0.2, 0.5\}$ (top to bottom).
Figure 5.2: Rank-based Pickands (left) and Capéraà-Fougères-Genest (right) type estimates (in black) of the Pickands dependence function (in red) for a sample of size $n = 100$ from a bivariate Archimax copula with an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.5, 0.2)$, and a Clayton generator $\psi_\theta$ with parameters $\theta = \{0.1, 0.2, 0.5\}$ (top to bottom).
Figure 5.3: Mean Integrated Squared Error (MISE) for rank-based Pickands (in orange) and Capéraà-Fougères-Genest (in blue) type estimators for samples of various sizes from a bivariate Archimax copula with an asymmetric Gumbel Pickands dependence function with $\beta = 3$, $(\kappa, \lambda) = \{(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)\}$ (left to right) and a Clayton generator $\psi_\theta$ with parameters $\theta \in (0.1, 0.4)$. Sample sizes $n = 100$ (full lines), $n = 250$ (dashed lines), $n = 500$ (dotted lines). Number of Monte Carlo replicates $N_{MC} = 1000$.

Table 5.3: Mean Integrated Squared Error (MISE) ($x1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Clayton generator $\psi_\theta$ and an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.1, 0.9)$ (strong asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$. 

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\theta$</th>
<th>0.1</th>
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<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
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<tbody>
<tr>
<td>P $n=100$</td>
<td></td>
<td>2.43</td>
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<td>4.86</td>
<td>8.67</td>
<td>19.69</td>
<td>46.39</td>
<td>133.97</td>
<td>493.81</td>
<td>3159.63</td>
</tr>
<tr>
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<td>2.46</td>
<td>2.43</td>
<td>2.31</td>
<td>2.41</td>
<td>2.23</td>
<td>2.39</td>
<td>2.16</td>
<td>2.16</td>
<td>2.11</td>
</tr>
<tr>
<td>P $n=250$</td>
<td></td>
<td>1.14</td>
<td>1.34</td>
<td>1.86</td>
<td>3.19</td>
<td>7.23</td>
<td>20.30</td>
<td>64.31</td>
<td>269.25</td>
<td>1972.27</td>
</tr>
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<td>1.08</td>
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<td>1.03</td>
<td>1.03</td>
<td>1.01</td>
<td>0.98</td>
<td>0.99</td>
<td>0.92</td>
</tr>
<tr>
<td>P $n=500$</td>
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<td>0.99</td>
<td>1.65</td>
<td>3.57</td>
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<td>0.65</td>
<td>0.59</td>
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</table>
Chapter 5. Simulation Study

Table 5.4: Mean Integrated Squared Error (MISE) ($\times 1000$) for the rank-based Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Clayton generator $\psi_\theta$ with high parameter values $\theta \in \{1, \ldots, 9\}$ and an asymmetric Gumbel Pickands dependence function with parameter $\beta = 3$. Number of Monte Carlo replicates $N_{MC} = 1000$.  

<table>
<thead>
<tr>
<th>$(\kappa, \lambda)$</th>
<th>$n$</th>
<th>$\theta = 1$</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
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<td>3.47</td>
<td>6.58</td>
<td>11.29</td>
<td>16.51</td>
<td>24.97</td>
<td>34.00</td>
<td>45.72</td>
<td>59.30</td>
</tr>
<tr>
<td></td>
<td>250</td>
<td>0.67</td>
<td>1.13</td>
<td>1.88</td>
<td>2.94</td>
<td>4.12</td>
<td>5.64</td>
<td>7.36</td>
<td>9.88</td>
<td>12.22</td>
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<td>0.49</td>
<td>0.73</td>
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<td>1.47</td>
<td>1.91</td>
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<td>3.83</td>
</tr>
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<td>1.90</td>
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<td>4.23</td>
<td>5.44</td>
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<tr>
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<td>500</td>
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<td>7.00</td>
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<td>18.03</td>
<td>26.90</td>
<td>40.20</td>
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<td>1.91</td>
<td>2.88</td>
<td>4.57</td>
<td>6.15</td>
<td>9.08</td>
<td>10.69</td>
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<tr>
<td></td>
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<td>0.56</td>
<td>0.72</td>
<td>1.02</td>
<td>1.54</td>
<td>1.99</td>
<td>2.81</td>
<td>3.50</td>
<td>4.41</td>
</tr>
</tbody>
</table>
5.2 Inverse Pareto-Simplex Family

In Chapter 4, weak limits of the Pickands and Capéraà-Fougères-Genest type estimators are also established for generators on a finite support. As calculated in Subsection 2.3.2, the corresponding Archimedean generator for a radial component whose reciprocal has Pareto distribution is given by

\[ \psi_{\gamma,2}(x) = 1 - x^{\frac{\gamma}{\gamma - 1}} + \frac{x^{\gamma}}{\gamma - 1}, \text{ for } x \in [0, 1]. \]

The index of regular variation for \(1 - \psi(1/x)\) is simply \(-\min\{\gamma, 1\}\), therefore both the Pickands and the CFG type estimator can be used for all \(\gamma > 0\). We can compute \(E(Z)\) and \(E(\log(Z))\) as follows:

\[ E(Z) = \int_0^1 \left( 1 - x^{\frac{\gamma}{\gamma - 1}} + \frac{x^{\gamma}}{\gamma - 1} \right) dx = 1 - \frac{\gamma}{2(\gamma - 1)} + \frac{1}{\gamma^2 - 1} \]

\[ E(\log(Z)) = \int_0^1 \log(x) \frac{\gamma}{\gamma - 1} \left( 1 - x^{\gamma - 1} \right) dx = -\frac{1}{\gamma} - 1. \]

See Tables 5.5, 5.6 and 5.7 for results of the simulation study. It is clear that both estimators perform worse for smaller \(\theta\). There is also a noticeable effect of asymmetry in the Pickands dependence function on both estimators mean integrated squared error. Interestingly, the Pickands type estimator has a better performance than the CFG type estimator, although the difference is less noticeable for larger \(\gamma\) and for larger sample sizes \(n\).
Figure 5.4: Mean Integrated Squared Error (MISE) for rank-based Pickands (in orange) and Capéraà-Fougères-Genest (in blue) type estimators for samples of various sizes from a bivariate Archimax copula with Gumbel Pickands dependence function with $\beta = 3$ and $(\kappa, \lambda) = [(0.5, 0.5), (0.3, 0.7), (0.1, 0.9)]$ (left to right) and an Inverse Pareto-Simplex generator $\psi_\gamma$ with parameters $\gamma \in (0.1, 0.9)$. Sample sizes $n = 100$ (full lines), $n = 250$ (dashed lines), $n = 500$ (dotted lines). Number of Monte Carlo replicates $N_{MC} = 1000$.

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Table 5.5: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with an Inverse Pareto-Simplex generator $\psi_\gamma$ with parameters $\gamma \in [0.1, \ldots, 0.9]$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.5, 0.5)$ (symmetric case). Number of Monte Carlo replicates $N_{MC} = 1000$. 

70
5.2. Inverse Pareto-Simplex Family

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Table 5.6: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraâ-Fougères-Genest (CFG) type estimators for samples of various sizes from bivariate Archimax copulas with an Inverse Pareto-Simplex generator $\psi_\gamma$ with parameters $\gamma \in \{0.1, \ldots, 0.9\}$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.3, 0.7)$ (mild asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$.

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Table 5.7: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraâ-Fougères-Genest (CFG) type estimators for samples of various sizes from bivariate Archimax copulas with an Inverse Pareto-Simplex generator $\psi_\gamma$ with parameters $\gamma \in \{0.1, \ldots, 0.9\}$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.1, 0.9)$ (strong asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$. 
5.3 Mixture Family

In this section the Archimedean generator of interest is the Pareto & Inverse Pareto-Simplex generator, as derived in Equation (2.16):

\[
\psi_{\alpha,\gamma,2}(x) = \begin{cases} 
1 - \frac{x}{2} \gamma^{-1} + \frac{1}{2} \gamma^{-1} - \frac{xa}{2(a+1)} & \text{for } x \leq 1 \\
\frac{1}{2} x^{-a} & \text{for } x > 1,
\end{cases}
\]  

(5.1)

where the parameters \(\gamma\) and \(\alpha\) are strictly positive. \(1 - \psi(1/x)\) is regularly varying with index \(-\min\{\gamma, 1\}\) and \(\psi(x)\) is regularly varying with index \(-\alpha\). This means that in Chapter 4 the established weak limit for the Pickands type estimator is valid for all \(\gamma\) and for \(\alpha > 4\).

For the Capéraà-Fougères-Genest type estimator, all parameter values are suitable. Next we need to compute \(E(Z)\) and \(E(\log(Z))\):

\[
E(Z) = \int_0^\infty \psi_{\alpha,\gamma,2}(x) dx = \frac{\alpha + 2\alpha\gamma + \gamma}{4\alpha + 4\alpha\gamma + 4\gamma + 4} + \frac{\alpha}{2\alpha^2 - 2}
\]

\[
E(\log(Z)) = \int_0^\infty \log(x) \frac{d}{dx} \left(1 - \psi_{\alpha,\gamma,2}(x)\right) dx = \frac{1}{2\alpha + 2} - \frac{2\gamma + 1}{2\gamma} + \frac{1}{2\alpha(\alpha + 1)}.
\]

Note that \(E(Z)\) is only finite for \(\alpha > 1\) while \(E(\log Z)\) is finite for all parameter values.

Tables 5.8, 5.9 and 5.10 present the results for \(\alpha = 4\), for which the Pickands type estimator was not shown to be consistent by Theorem 4.1.1. Tables 5.11, 5.12 and 5.13 present the results of the same simulation study except that \(\alpha = 6\), for which Theorem 4.1.1 does hold. There is a slight decrease in mean integrated squared error overall.

As was the case for the inverse Pareto-simplex generator in Section 5.2, the Pickands type estimator exhibits less error than the Capéraà-Fougères-Genest type estimator for small \(\gamma\). However as \(\gamma\) gets closer to 1, the Capéraà-Fougères-Genest type estimator has similar if not smaller MISE. We also notice decreasing mean integrated squared error for larger parameter values of \(\gamma\). Like all previous experiments, asymmetry increases error for both estimators.
5.3. Mixture Family

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Table 5.8: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Pareto & Inverse Pareto-Simplex generator $\psi_{\alpha,\gamma}$ with parameters $\alpha = 4$ and $\gamma \in \{0.1, \ldots, 0.9\}$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.5, 0.5)$ (symmetric case). Number of Monte Carlo replicates $N_{MC} = 1000$.

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Table 5.9: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from bivariate Archimax copulas with a Pareto & Inverse Pareto-Simplex generator $\psi_{\alpha,\gamma}$ with parameters $\alpha = 4$ and $\gamma \in \{0.1, \ldots, 0.9\}$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.3, 0.7)$ (mild asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$. 73
Chapter 5. Simulation Study

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Table 5.10: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from bivariate Archimax copulas with an Inverse Pareto-Simplex generator $\psi_{a,\gamma}$ with parameters $\alpha = 4$ and $\gamma \in \{0.1, \ldots, 0.9\}$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.1, 0.9)$ (strong asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$.

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<td>0.43</td>
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<tr>
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<td>0.31</td>
<td>0.30</td>
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Table 5.11: Mean Integrated Squared Error (MISE) ($\times 1000$) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Pareto & Inverse Pareto-Simplex generator $\psi_{a,\gamma}$ with parameters $\alpha = 6$ and $\gamma \in \{0.1, \ldots, 0.9\}$ and an asymmetric Gumbel dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.5, 0.5)$ (symmetric case). Number of Monte Carlo replicates $N_{MC} = 1000$. 
5.3. Mixture Family

<table>
<thead>
<tr>
<th>Estimator</th>
<th>( \gamma )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
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<td>P ( n = 100 )</td>
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<td>2.79</td>
<td>2.36</td>
<td>2.04</td>
<td>1.85</td>
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<td>0.98</td>
<td>0.88</td>
<td>0.81</td>
<td>0.77</td>
<td>0.73</td>
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<td>CFG</td>
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<td>0.88</td>
<td>0.76</td>
<td>0.78</td>
<td>0.70</td>
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<tr>
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<td>0.63</td>
<td>0.53</td>
<td>0.46</td>
<td>0.48</td>
<td>0.44</td>
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</tr>
<tr>
<td>CFG</td>
<td>2.18</td>
<td>0.97</td>
<td>0.63</td>
<td>0.57</td>
<td>0.52</td>
<td>0.49</td>
<td>0.43</td>
<td>0.46</td>
<td>0.46</td>
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</tr>
</tbody>
</table>

Table 5.12: Mean Integrated Squared Error (MISE) \((\times 1000)\) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from bivariate Archimax copulas with a Pareto & Inverse Pareto-Simplex generator \( \psi_{\alpha,\gamma} \) with parameters \( \alpha = 6 \) and \( \gamma \in \{0.1, \ldots, 0.9\} \) and an asymmetric Gumbel dependence function with parameters \( \beta = 3 \) and \((\kappa, \lambda) = (0.3, 0.7)\) (mild asymmetry). Number of Monte Carlo replicates \( N_{MC} = 1000 \).

<table>
<thead>
<tr>
<th>Estimator</th>
<th>( \gamma )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>P ( n = 100 )</td>
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<td>5.60</td>
<td>4.05</td>
<td>3.39</td>
<td>2.92</td>
<td>2.62</td>
<td>2.26</td>
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<td>CFG</td>
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<td>6.81</td>
<td>5.16</td>
<td>4.07</td>
<td>3.55</td>
<td>3.20</td>
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</tr>
<tr>
<td>P ( n = 250 )</td>
<td>4.04</td>
<td>2.55</td>
<td>1.84</td>
<td>1.65</td>
<td>1.40</td>
<td>1.25</td>
<td>1.11</td>
<td>1.13</td>
<td>1.05</td>
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<td>CFG</td>
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<td>1.23</td>
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<td>1.12</td>
<td></td>
</tr>
<tr>
<td>P ( n = 500 )</td>
<td>2.13</td>
<td>1.40</td>
<td>1.10</td>
<td>0.99</td>
<td>0.87</td>
<td>0.82</td>
<td>0.79</td>
<td>0.76</td>
<td>0.76</td>
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</tr>
<tr>
<td>CFG</td>
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<td>0.81</td>
<td>0.65</td>
<td>0.62</td>
<td>0.61</td>
<td>0.58</td>
<td>0.60</td>
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</tr>
</tbody>
</table>

Table 5.13: Mean Integrated Squared Error (MISE) \((\times 1000)\) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from bivariate Archimax copulas with an Inverse Pareto-Simplex generator \( \psi_{\alpha,\gamma} \) with parameters \( \alpha = 6 \) and \( \gamma \in \{0.1, \ldots, 0.9\} \) and an asymmetric Gumbel dependence function with parameters \( \beta = 3 \) and \((\kappa, \lambda) = (0.1, 0.9)\) (strong asymmetry). Number of Monte Carlo replicates \( N_{MC} = 1000 \).
5.4 Gumbel Family

The final family of Archimedean generators studied is the Gumbel family. As seen in Table 2.1, \( \psi_\theta(t) = \exp\left(-t^{1/\theta}\right) \) for \( t \geq 0 \) and parameter value \( \theta > 1 \). The Pickands type estimator \( A_n^P(t) \) requires the computation of \( E(Z) \) while the Capéraâ-Fougères-Genest type estimator \( A_n^{CFG}(t) \) requires computing \( E(\log(Z)) \). For the Gumbel generator, these two expectations are given, for all \( \theta > 1 \), by

\[
E(Z) = \int_0^\infty \exp\left(-t^{1/\theta}\right) dt = \int_0^\infty \theta x^{\theta-1} \exp(-x) dx = \theta \Gamma(\theta) = \Gamma(\theta + 1)
\]

\[
E(\log(Z)) = \int_0^\infty \log(t) \frac{d}{dt} \left(1 - \psi_\theta(t)\right) dt = \int_0^\infty \log(t) t^{1/\theta-1} \exp\left(-t^{1/\theta}\right) \frac{1}{\theta} dt
\]

\[
= \theta \int_0^\infty \log(x) \exp(-x) dx = -\theta \gamma,
\]

where \( \gamma \) is the Euler-Mascheroni constant.

It is important to note that the Gumbel generator \( \psi_\theta(t) \) has an infinite domain and is not regularly varying (nor is \( 1 - \psi_\theta(1/t) \)). Therefore none of the theorems in Chapter 4 apply to this example.

Results of this simulation experiment are presented in Tables 5.14, 5.15 and 5.16. The Pickands type estimator is strongly biased in all cases, except for large \( n \) and small values of \( \theta \). There is an important reduction in MISE for both estimators when increasing the sample size \( n \). We can also observe the detrimental effect of asymmetry on the MISE for both estimators. The Capéraâ-Fougères-Genest (CFG) type estimator clearly performs better, although its bias also increases as \( \theta \) gets larger.

These simulation results suggest that the theorems in Chapter 4 could possibly be extended to Archimedean families with rapidly varying generators such as the Gumbel generator.
### Table 5.14: Mean Integrated Squared Error (MISE) (×1000) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Gumbel generator $\psi_\theta$ and an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.5, 0.5)$ (symmetric case). Number of Monte Carlo replicates $N_{MC} = 1000$.

<table>
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<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
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<td>362.17</td>
<td>779.48</td>
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<td>100.25</td>
<td>221.69</td>
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<td>0.89</td>
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<td>1.16</td>
<td>1.34</td>
<td>1.45</td>
<td>1.54</td>
</tr>
<tr>
<td>P</td>
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<td>42.77</td>
<td>89.91</td>
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### Table 5.15: Mean Integrated Squared Error (MISE) (×1000) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Gumbel generator $\psi_\theta$ and an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.3, 0.7)$ (mild asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$.

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<th>2.5</th>
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<td>4.55</td>
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<td>5.90</td>
</tr>
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<td>7.29</td>
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<td>1.19</td>
<td>1.32</td>
<td>1.40</td>
<td>1.53</td>
<td>1.73</td>
</tr>
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<td>0.54</td>
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<td>0.64</td>
<td>0.72</td>
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Table 5.16: Mean Integrated Squared Error (MISE) (x1000) for rank-based Pickands (P) and Capéraà-Fougères-Genest (CFG) type estimators for samples of various sizes from a bivariate Archimax copulas with a Gumbel generator $\psi_{\theta}$ and an asymmetric Gumbel Pickands dependence function with parameters $\beta = 3$ and $(\kappa, \lambda) = (0.1, 0.9)$ (strong asymmetry). Number of Monte Carlo replicates $N_{MC} = 1000$. 

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<th>2.5</th>
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<td>5.97</td>
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<td>1.87</td>
<td>2.00</td>
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<tr>
<td>P</td>
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<td>1.66</td>
<td>3.66</td>
<td>8.59</td>
<td>20.24</td>
<td>43.13</td>
<td>98.02</td>
<td>192.28</td>
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<tr>
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<td>0.65</td>
<td>0.72</td>
<td>0.73</td>
<td>0.83</td>
<td>0.87</td>
<td>0.93</td>
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</table>


6 Conclusion

6.1 Findings

This thesis contributes to the modeling of bivariate multivariate extremes through the use of a relatively new family of copulas called Archimax copulas. This class of copulas is more flexible than extreme-value copulas, meaning it could be a preferable choice over the latter class which it generalizes. However, there are two functions that need to be estimated. This thesis focuses on the estimation of the Pickands dependence function.

Two rank-based estimators for this function are proposed, neither of them requiring margins to be known. In order to establish the consistency of these estimators, additional assumptions of regular variation on the Archimedean generator \( \psi \) were necessary. The Pickands type estimator imposes a relatively strong bound on the index of regular variation of the Archimedean generator at infinity. On the other hand, the Caperaa-Fougeres-Genest (CFG) type estimator does not require such a restrictive bound on the index at infinity, but instead requires regular variation at zero. Under these conditions of regular variation and smoothness assumptions on the copula and Pickands dependence function, the limiting process of \( n^{1/2} (A_n - A) \) for both estimators is established in Chapter 4 and unsurprisingly relates to the limit of the empirical copula process.

The simulation study in Chapter 5 demonstrates the strength and weaknesses of these estimators. In cases where the CFG type estimator can be computed, it seems to be a
preferable choice over the Pickands type estimator. Indeed the Pickands type estimator suffers from significant bias, especially in smaller samples. Both estimators’ performance decreases significantly for strongly asymmetric Pickands dependence functions.

6.2 Future work

The estimation of Archimax copulas being a relatively unexplored subject, many questions arise from the work done in this thesis.

As seen in the simulation study, the behavior of the estimators for Archimax copulas with rapidly varying Archimedean generators is not yet understood. It is of immediate interest to address this issue, since such generators such as the Gumbel, Frank and Joe generators are quite commonly used. Recall that the independence generator leads to the special case of the extreme-value copula and was studied in Genest & Segers (2009).

It is of high interest to further to extend the results of this thesis towards dimensions higher than two, meaning we would attempt to estimate the stable tail dependence function $l(x_1, \ldots, x_d)$ of an Archimax copula $C_{\psi,l}(u_1, \ldots, u_d) = \psi \circ l \{\psi(u_1), \ldots, \psi(u_d)\}$.

As a side issue, one could adapt the theorems in Chapter 4 to the case where the margins $F$ and $G$ are known, in which case $C$ is replaced by $\alpha$. The proofs would remain identical except that $\hat{C}_n$ is replaced by the empirical distribution of the copula sample $((F(X_i), G(Y_i)))_{i \in \{1, \ldots, n\}}$ and the there is no need for Stute’s representation of the integral $I_{2,n}$. However to follow the same reasoning, the integral bound $k_n$ needs to be replaced by $2\phi^{-1}(m_n)$, where $m_n = \min \{F(X_1), \ldots, F(X_n), G(Y_1), \ldots, G(Y_n)\}$. This makes $k_n$ a random variable. It seems that this would require a slightly different theorem than Theorem G.1 from Genest & Segers (2009) where the weighted bivariate empirical process is multiplied by an indicator function $1(x < k_n)$.

The simulation study could be extended to more cases, with different Pickands dependence functions and Archimedean generators. More importantly, it would be of great interest to study the effect of misspecification of the Archimedean generator on the performance of the two estimators.
Most importantly, in practice it is not realistic to assume knowledge of the Archimedean generator, so a joint estimation method for both functions defining Archimax copulas needs to be developed.
Bibliography


