CONTROL OF AEROELASTIC OSCILLATIONS OF WING STRUCTURES USING BONDED PIEZOELECTRIC STRIPS

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This thesis is dedicated to my parents,

for their endless love, support and encouragement.
ABSTRACT

The objective of this project is the analysis and control of the aeroelastic oscillations of a wing structure with bonded piezoelectric strips subject to unsteady subsonic aerodynamic loads. Active control of aeroelastic oscillations and the use of piezoelectric materials in vibration analysis and control of structures have been the subjects of many researches in air vehicle design. However, most of the research in these areas has been restricted to simple models such as two-dimensional or quasi-steady aerodynamic models. Hence, in order to obtain accurate and valid results, considering the effects of unsteadiness and three-dimensionality in the modeling is necessary.

This complex problem requires time-dependent simultaneous solution of the dynamics equations of the elastic structure with the piezoelectric strips, coupled with the equations of the unsteady flow past oscillating wings. In the present thesis, the unsteady subsonic aerodynamic loading of a trapezoidal wing structure is calculated using numerical panel methods for two- and three-dimensional flows. The developed models are validated with the existing literature and the results show good agreement.

Piezoelectric strips are employed as sensors and actuators bonded to the surface of the wing. The finite element formulation of the combined structural model for the wing and the piezoelectric strips is presented. The structural model is coupled with the aerodynamic model using an interactive computer model to transfer the data at each time step and solve the equations simultaneously.

The transient analysis is used to simulate the aeroelastic oscillations and an active PID feedback controller is proposed and applied to suppress the oscillations. The numerical results for the various cases of the control of oscillations caused by the vertical gust loads are presented. The results show that the PID control is effective in reducing the amplitude of the oscillations in
relatively short time and with relatively small gains, hence low cost. A systematic approach is presented to calculate the gains of the feedback controller using the system matrices. An analysis is performed as well to investigate the effects of the actuator placement on the performance of the controller in suppression of the oscillation for various scenarios. It was demonstrated that the actuators placed close to the wing root are more effective in reducing the amplitude of the oscillations in less time.
RESUME

L'objectif de ce projet est l'analyse et le contrôle des oscillations aéroélastiques d'une structure de l'aile avec des bandes collées piézoélectriques soumises à des charges aérodynamiques subsoniques instables. Le contrôle actif des oscillations aéroélastiques et l'utilisation de matériaux piézoélectriques dans l'analyse des vibrations et le contrôle des structures a fait l'objet de nombreuses recherches dans la conception des véhicules aériens. Cependant, la plupart des recherches dans ces domaines ont été limitées à des modèles simples tels que les modèles aérodynamiques à deux dimensions ou quasi-stables. Par conséquent, afin d'obtenir des résultats précis et valables, il est nécessaire de considérer les effets de l'instabilité et de la tridimensionnalité dans la modélisation.

Ce problème complexe nécessite de résoudre de manière simultanée en fonction du temps l'équation de la dynamique de la structure élastique avec des bandes piézoélectriques couplée avec les équations du flux instable autour des ailes oscillantes. Dans cette thèse, la charge aérodynamique subsonique instable d'une structure d'aile trapézoïdale est calculée en utilisant des méthodes de panneaux numériques pour des flux à deux et trois dimensions. Les modèles développés sont validés avec la littérature existante et les résultats démontrent un bon accord.

Des bandes piézoélectriques sont utilisées comme capteurs et des actionneurs liés à la surface de l'aile. La formulation des éléments finis du modèle structurel de l'aile combiné avec les bandes piézoélectriques est présentée. Le modèle structurel est couplé avec le modèle aérodynamique à l'aide d'un modèle informatique interactif pour transférer les données à chaque pas de temps, et résoudre les équations simultanément.

L'analyse transitoire est utilisée pour simuler les vibrations aéroélastiques et un contrôleur de retour PID actif est proposé et appliqué pour supprimer les
oscillations. Les résultats numériques pour les différents cas de contrôle des oscillations provoquées par les charges des rafales de vents verticales sont présentés. Les résultats démontrent que le contrôle PID est efficace pour réduire l'amplitude des oscillations en relativement peu de temps et avec relativement peu de gains, donc à faible coût. Une approche systématique est présentée pour calculer les gains du contrôleur de rétroaction en utilisant les matrices du système. L'analyse est aussi effectuée pour étudier les effets de la mise en place de l'actionneur sur la performance du dispositif de commande dans la suppression de l'oscillation de divers scénarios. Il a été démontré que les actionneurs placés à proximité de l'emplanture de l'aile sont plus efficaces dans la réduction de l'amplitude des oscillations en moins de temps.
STATEMENT OF ORIGINALITY

In this work, the following can be considered as a contribution to the existing knowledge:

1- A three-dimensional numerical model based on panel methods was developed for the unsteady subsonic flows past flexible wings executing flexural and torsional oscillations. A FORTRAN program was developed by the author to calculate the unsteady pressure distribution and the unsteady aerodynamic forces acting on the oscillating wing. The effects of three-dimensionality and the unsteadiness were included in the model to simulate the real behaviour of the flow and the vortex shedding.

2- A finite element model of the cantilever wing structure combined with the surface bonded piezoelectric strips was developed. Appropriate 3-D coupled-field solid elements were used in modeling the piezoelectric sensors and actuators using the finite element software ANSYS.

3- An interactive computer model was developed to simultaneously solve the unsteady fluid flow equations and the structural dynamics equations by transferring the data between two aerodynamic and structural models at each time step. In order to this, the two programs were connected utilizing the User Programmable Features in the ANSYS package. More precisely, the FORTRAN code was implemented as a function in ANSYS program to be called at each time step for updated aerodynamic loads.

4- A PID controller was developed to suppress the oscillations of the wing using bonded piezoelectric actuators. An APDL code was written by the author to calculate the updated actuator voltages at each time step and apply them to the pertinent actuator strips on the combined aero-structural model.
5- A systematic analytical approach was developed to calculate the feedback gains of the PID controller by using the system matrices such as mass, damping and stiffness matrices.

6- An analysis was performed to investigate the effect of the actuator placement on the wing surface in control of the aeroelastic oscillations. Various scenarios were studied to evaluate the effect of the locations of the actuators in reducing the amplitude of the oscillations.
I would like to express my appreciation to all those who made it possible for me to complete this dissertation. First of all, I wish to extend my sincere gratitude to my supervisors, Prof. Dan Mateescu and Prof. Arun K. Misra for their guidance, encouragement and support throughout this research and the time spent in reviewing this thesis. I am also grateful to them for their financial support which enabled me to pursue this degree.

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NOMENCLATURE

ROMAN SCRIPTS

\( b \) \hspace{1cm} \text{wing span}

\( \Delta b_{ij} \) \hspace{1cm} \text{length of the panel along span (y axis)}

\( B_i \) \hspace{1cm} \text{elements of right hand side matrix (in equations (3.74) and (4.62))}

\( [B_u], [B_\phi] \) \hspace{1cm} \text{derivatives of the shape functions}

\( c \) \hspace{1cm} \text{wing chord}

\( \Delta c_{ij} \) \hspace{1cm} \text{length of the panel along chord (x axis)}

\( [c_E] \) \hspace{1cm} \text{stiffness matrix under constant electric field}

\( C_f \) \hspace{1cm} \text{charge amplifier gain}

\( C_L \) \hspace{1cm} \text{non-dimensional lift coefficient}

\( C_{ij} \) \hspace{1cm} \text{influence coefficient}

\( C_{m_0} \) \hspace{1cm} \text{non-dimensional moment coefficient}

\( C_{i\omega t} \) \hspace{1cm} \text{influence coefficient of the latest wake}

\( [C] \) \hspace{1cm} \text{structural damping matrix}

\( [C_w] \) \hspace{1cm} \text{elemental damping matrix of wing structure}

\( [C_\phi] \) \hspace{1cm} \text{dielectric damping matrix}

\( [C_{w_0}] \) \hspace{1cm} \text{global damping matrix of wing structure}

\( d_{31}, d_{32}, d_{33} \) \hspace{1cm} \text{piezoelectric constants}

\( [d] \) \hspace{1cm} \text{piezoelectric coupling matrix}

\{D\} \hspace{1cm} \text{electric displacement vector}
\([\mathbf{D}]\) derivation operator
\([\mathbf{e}]\) piezoelectric stress matrix
\(E_b\) Young’s modulus of the beam
\(E_p\) Young’s modulus of the piezoelectric strips
\(\{\mathbf{E}\}\) electric field vector
\(f\) frequency
\(\{\mathbf{f}_i\}\) external mechanical force on the \(i\)th element
\(\{\mathbf{f}_w\}\) elemental external mechanical loads on the wing
\(\{\mathbf{F}\}\) external mechanical force
\(\mathbf{F}_b, \mathbf{F}_s, \mathbf{F}_p\) vectors of body, surface, and point forces
\(F_s\) surface shape function
\(\{\mathbf{g}_i\}\) applied electric charge
\(G\) total number of panels on the wing
\(h_0\) normal distance (in the Biot-Savart method)
\(H\) electrical enthalpy
\(i_s\) electrical current
\(I\) moment of inertia
\(\mathbf{I}\) identity matrix
\(\mathbf{i}, \mathbf{j}, \mathbf{k}\) Cartesian unit vectors along the \(x, y, z\) directions
\(k\) reduced frequency
\(K\) kinetic energy
\(K_c, K_f\) constants (in equations (5.10) and (5.13))
\(K_p, K_I, K_D\) proportional, integral and derivative gains
\([\mathbf{K}]\) structural stiffness matrix
xx
\([K_w]\) \quad \text{elemental stiffness matrix of wing structure}

\([K_{uu}]\) \quad \text{stiffness matrix}

\([K_{u\phi}]\) \quad \text{piezoelectric coupling matrix}

\([K_{\phi\phi}]\) \quad \text{capacitance matrix}

\([K_w^*]\) \quad \text{global stiffness matrix of wing structure}

\([K_{uu}^*]\) \quad \text{assembled stiffness matrix}

\([K_{u\phi}^*]\) \quad \text{assembled piezoelectric coupling matrix}

\([K_{\phi\phi}^*]\) \quad \text{assembled capacitance matrix}

\(l_b\) \quad \text{length (span) of the wing along the } y \text{ axis}

\(l_p\) \quad \text{length of the piezoelectric strip along the } y \text{ axis}

\(\Delta l\) \quad \text{panel length}

\(L\) \quad \text{total lift}

\(\Delta L_j\) \quad \text{aerodynamic lift on panel } j

\(\mathcal{L}\) \quad \text{Lagrangian}

\(M\) \quad \text{Mach number}

\(M_0\) \quad \text{total moment about the leading edge}

\([M]\) \quad \text{structural mass matrix}

\([M_w]\) \quad \text{elemental mass matrix of wing structure}

\([M^*]\) \quad \text{assembled mass matrix}

\([M_w^*]\) \quad \text{global mass matrix of wing structure}

\(n\) \quad \text{unit normal vector}

\(n_x, n_y, n_z\) \quad \text{normal vector components along } xyz \text{ directions}

\(n_w\) \quad \text{normal vector on the wake}
\[ [N_u] \] \quad \text{matrix of shape functions for the displacement field}

\[ [N_{\phi}] \] \quad \text{matrix of shape functions for the electric potential}

\( p \) \quad \text{local fluid pressure}

\( p_{\infty} \) \quad \text{far field reference pressure}

\( \Delta p \) \quad \text{pressure difference between the upper and the lower surfaces}

\( \Delta p_j \) \quad \text{pressure difference on panel } j

\( P \) \quad \text{arbitrary point in the fluid domain}

\( q \) \quad \text{perturbation velocity vector}

\( q_b \) \quad \text{perturbation velocity due to body panels}

\( q_w \) \quad \text{perturbation velocity due to wake panels}

\( Q \) \quad \text{electric charge}

\( r \) \quad \text{position vector}

\( r_1, r_2 \) \quad \text{position vector of the edges of the vortex line element}

\( r_0 \) \quad \text{position vector connecting the edges of vortex line element}

\( R_f \) \quad \text{current amplifier constant}

\([s]\) \quad \text{compliance matrix of the material}

\([s_{E}]\) \quad \text{compliance matrix under constant electric field}

\{S\} \quad \text{elastic strain vector}

\( S_b \) \quad \text{body surface enclosing the fluid domain}

\( S_{\infty} \) \quad \text{sphere with a large radius tending to infinity}

\( \Delta S_{ij} \) \quad \text{area of the panel } ij

\( t \) \quad \text{time}

\( t_1, t_2 \) \quad \text{limits of time}

\( t_b \) \quad \text{thickness of the wing}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_p$</td>
<td>thickness of the piezoelectric strip</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>time step</td>
</tr>
<tr>
<td>$T$</td>
<td>period of oscillation</td>
</tr>
<tr>
<td>${T}$</td>
<td>stress vector</td>
</tr>
<tr>
<td>$T_x, T_y, T_z$</td>
<td>rotation matrices with respect to the $x, y, z$ axes</td>
</tr>
<tr>
<td>${u}$</td>
<td>nodal displacement vector</td>
</tr>
<tr>
<td>${\delta u}$</td>
<td>arbitrary variation of the displacement field</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>perturbation velocity components along the $x, y, z$ directions</td>
</tr>
<tr>
<td>$u_b, v_b, w_b$</td>
<td>body induced velocity components</td>
</tr>
<tr>
<td>$u_w, v_w, w_w$</td>
<td>wake induced velocity components</td>
</tr>
<tr>
<td>$u_{ij}, v_{ij}, w_{ij}$</td>
<td>induced velocity components at point $i$ due to the vortex $j$</td>
</tr>
<tr>
<td>$u^<em>_{ij}, v^</em><em>{ij}, w^*</em>{ij}$</td>
<td>induced velocity components at point $i$ due to the unit vortex $\Gamma_j = 1$</td>
</tr>
<tr>
<td>$U$</td>
<td>strain energy density of the piezoelectric continuum</td>
</tr>
<tr>
<td>$U_b, V_b, W_b$</td>
<td>body surface velocity components</td>
</tr>
<tr>
<td>$U_\infty$</td>
<td>free-stream velocity along $x$ axis</td>
</tr>
<tr>
<td>$U'_X, U'_Y, U'_Z$</td>
<td>components of the global degrees of freedom of the structure</td>
</tr>
<tr>
<td>$\mathbf{v}_{rel}$</td>
<td>relative velocity of wing with respect to the air</td>
</tr>
<tr>
<td>$\mathbf{V}$</td>
<td>fluid velocity vector</td>
</tr>
<tr>
<td>$V_0$</td>
<td>translational velocity vector of the system’s origin</td>
</tr>
<tr>
<td>$V_b$</td>
<td>body surface velocity vector</td>
</tr>
<tr>
<td>$V_\infty$</td>
<td>free-stream velocity</td>
</tr>
<tr>
<td>$V_r, V_\theta$</td>
<td>radial and tangential velocity components</td>
</tr>
<tr>
<td>$V_{12}$</td>
<td>induced velocity vector due to the vortex element 12</td>
</tr>
<tr>
<td>$V_{KL}$</td>
<td>induced velocity on point $K$ due to the vortex ring $L$</td>
</tr>
</tbody>
</table>
\( \mathcal{V}_a \) actuator voltage

\( \mathcal{V}_d \) difference between the sensor and the reference voltages

\( \mathcal{V}_s \) sensor voltage

\( \mathcal{V}_{s_0} \) reference sensor voltage

\( \Psi \) fluid domain

\( \Psi_\xi \) infinitesimally small sphere around the singularity point

\( w_b \) width (chord) of wing along \( x \) axis

\( w_p \) width of the piezoelectric strip along \( x \) axis

\( w \) transverse deformation

\( w' \) slope at any point of the piezoelectric strip

\( \delta \mathcal{W} \) virtual work

\( x, y, z \) body-fixed Cartesian coordinates

\( x_0 \) distance between the leading edge and first piezoelectric strip along \( x \) axis

\( x_1, x_2, x_3 \) axes equivalent to \( x, y, z \) axes

\( X, Y, Z \) inertial Cartesian coordinates

\( X_0, Y_0, Z_0 \) location of the origin of the body-fixed frame

\( \Delta x_0 \) distance between two consequent piezoelectric strips along \( x \) axis

\( \Delta x_i, \Delta z_i \) translation of the \( i \)th wake vortex along \( x, z \) directions

\( \Delta x_w \) distance between the first shedding vortex and the airfoil TE

\( \Delta x_{ij}, \Delta y_{ij} \) lengths of the panel \( ij \) along the \( x \) and \( y \) axes

\( y_0 \) distance between the wing root and first piezoelectric strip along \( y \) axis

\( \Delta y_0 \) distance between two consequent piezoelectric strips along \( y \) axis
GREEK SCRIPTS

$\alpha$  angle of attack

$\alpha_0, \alpha_A$  magnitude of angle of attack oscillations

$\alpha_D, \beta_D$  Rayleigh damping parameters

$\beta_j$  incidence angle of the panel $j$

$\gamma$  circulation along length $l$

$\Gamma$  airfoil circulation

$\Gamma_c$  circulation around a closed material contour

$\Gamma_j$  panel vortex strengths ($j = 1, \ldots, N$)

$\Gamma_w$  wake circulation

$\Gamma_w_T$  vortex strength of the latest wake

$\Gamma_w_T$  vortex strength of the wake at trailing edge

$\bar{\Gamma}$  non-dimensional panel circulation

$\delta$  variational operator

$\Delta$  Laplace operator

$\eta$  wing’s surface shape

$\varepsilon$  dielectric constant

$\varepsilon_0$  vacuum permittivity

$\varepsilon_p$  piezoelectric strain in the $x$ direction

$\varepsilon_{11}$  axial mechanical strain

$[\varepsilon_T]$  dielectric matrix under constant stress

$[\varepsilon_S]$  dielectric matrix under constant strain

$\theta$  angular coordinate

$\Delta \theta$  difference of slopes at the ends of the sensor
\( \Theta \)  \quad \text{instantaneous orientation of body frame}

\( \mu_w \)  \quad \text{wake doublet strength}

\( \rho \)  \quad \text{mass density of air}

\( \rho_\infty \)  \quad \text{far field reference density}

\( \sigma, \mu \)  \quad \text{source and doublet strengths}

\( \sigma \)  \quad \text{electrical charge density}

\( \sigma_b, \sigma_s, q_E \)  \quad \text{body, surface, and point charges}

\( \sigma_b \)  \quad \text{bending stress distribution}

\( \sigma_p \)  \quad \text{stress in the piezoelectric strip}

\( \tau \)  \quad \text{tangential unit vector}

\( \phi, \alpha, \psi \)  \quad \text{rotation angles about the } x, y, z \text{ axes}

\( \delta \phi_E \)  \quad \text{arbitrary variation of electrical potential}

\( \varphi \)  \quad \text{perturbation velocity potential}

\( \Phi \)  \quad \text{total fluid velocity potential}

\( \Phi_w, \Phi'_w \)  \quad \text{velocity potentials on the two sides of the wake}

\( \Phi_\infty \)  \quad \text{free-stream velocity potential}

\( \omega \)  \quad \text{angular velocity of the body frame}

\( \zeta_m, \omega_m \)  \quad \text{modal damping ratio and natural frequency}

**SUBSCRIPTS**

\( p \)  \quad \text{piezoelectric strip}

\( u, l \)  \quad \text{upper and lower surfaces}

\( w \)  \quad \text{wing structure}
CHAPTER 1

INTRODUCTION

1.1 Overview

The past few decades have witnessed a growing interest in better understanding of aeroelastic phenomena which has been believed to have major influence on advances of air vehicle design. Aeroelastic oscillations, which are due to the interaction between aerodynamic and structural loads, may lead to catastrophic structural failure or reduce the fatigue life of the air vehicle.

There are passive methods of avoiding the unwanted aeroelastic oscillations such as increasing the structural stiffness by increasing the thickness of the structure, but these solutions result in extra structural weight which reduces the overall performance of the flight vehicle. Hence, as an important alternative, active control methods have been developed to improve the aircraft performance with lower stiffness characteristics. Over the past two decades, various active
control schemes, including classical and modern control methods, have been proposed and improved by numerous researchers. A historical review on the research related to analysis and control of aeroelastic systems is provided in an excellent survey by Mukhopadhyay (2003).

One of the important elements in any control system is the proper choice of sensors and actuators. Sensors as components to obtain the required measurements from the physical systems, and actuators as devices to apply the control command to the system, are used in control systems to change the system response to a desired one. The proper choice of the types of the sensor and actuator is dependent on the desired control approach and the requirements of the system. Conventional control approach is based on the use of aerodynamic control surfaces to change the aerodynamic loads distribution around the flexible wing. These control surfaces such as flaps or ailerons are usually operated by servo-hydraulic, electrical, and electromechanical actuators. Despite the popularity of this approach in active control of aeroelastic oscillations, there are several disadvantages, such as additional weight due to numerous parts, vulnerability and potential failure, corrosion and leakage, hydraulic lag, limited bandwidth, to name a few.

Recently, application of smart materials has expanded to various fields of aerospace and mechanical systems such as active vibration and noise control. Smart materials have the capability to respond to a changing external environment such as loads or shape change, as well as to a changing internal environment such as damage or failure (Chopra 2002) and include many types of material such as piezoelectric materials, shape memory alloys, magneto-strictive materials, electro-strictive materials, thermoelectric materials, etc. Among all these materials and alloys, piezoelectric materials are the most popular and have been widely used in active vibration control. This is mainly due to their excellent frequency response characteristics and capability in reciprocal conversion between the electrical and mechanical strain energy. Also, piezoelectric materials are compact, cheap and light weight, and can be easily bonded to the surface of the structure which makes them a good alternative to other control methods. For more information, the
reader is referred to an extensive survey by Chopra (2002) which has reviewed the state-of-the-art of smart structures and integrated systems as well as an excellent review paper by Giurgiutiu (2000) on the smart-materials actuation solutions for aeroelastic and vibration control. It is worth noting that the mass of the piezoelectric sensors and actuators adds to the mass of the structure; however due to the fact that the thickness of the piezoelectric sensors and actuators is very small, the increase in weight is not expected to be large compared to those for the passive methods of control.

Detailed modeling of aeroelastic oscillations is of great importance in designing the proper control method to suppress the oscillations. Various methods of control of aeroelastic oscillations have been studied in recent years and some of the conventional approaches have been able to model the elastic structure and the unsteady loads, but mainly with simplified models. Classic structural modeling simplified their models to two-dimensional structures. Many researchers also limited their aerodynamic models to quasi-steady subsonic or supersonic aerodynamic models. Hence, study of the aeroelastic oscillations for a more complex three-dimensional structure under three-dimensional unsteady flow is of significant interest.

1.2 Literature Review

The literature review in this thesis has two sections. The first section aims to review the recent studies on the active control of aeroelastic oscillations. The second section focuses mainly on the recent studies related to analysis and control of oscillations using piezoelectric materials.

1.2.1 Active Control of Aeroelastic Oscillations

In the recent decades, there has been considerable research effort towards the analysis and control of aeroelastic oscillations and many researchers have dedicated their time and effort to analytical techniques and wind-tunnel
INTRODUCTION


There are various methods to control the aeroelastic oscillations and suppress the limit cycle oscillations and flutter. One of the popular methods is to use a control surface of the wing as an actuator to modify the distribution of aerodynamic loads around the lifting surface. Most of the preliminary studies actuated the trailing-edge control surface such as a flap and employed its deflection in the active control methodology to suppress the aeroelastic oscillations. Later on, use of multiple control surfaces including the leading-edge control surface was investigated by some researchers. This section of the thesis begins with the studies related to the single control surface actuation and discusses the modeling and control aspects of the recent studies. Then, the studies related to the multiple control surfaces are discussed.

A great deal of research work have modeled the structure as a classical two-dimensional one and considered linear aerodynamic theories such as quasi-steady theory or Theodorsen’s method which have been thoroughly explained in most well-known textbooks on aeroelasticity such as Fung (1955) and Bisplinghoff and Ashley (1975). Some examples of recent studies on control of two-dimensional aeroelastic oscillations using single control surface are the works done by De Marqui Jr et al. (2005), Librescu et al. (2005b), Na et al. (2006), Ardelean et al. (2006), and McEver et al. (2007). De Marqui Jr et al. (2005) considered the flutter as a two degree-of-freedom (2DoF) pitching-plunging motion and employed a low-order approximation to the Theodorsen’s function to calculate the aerodynamics loads. They performed wind tunnel experiments on a rectangular rigid wing with a NACA 0012 airfoil section connected to a flexible mount system and were able to successfully control the pitch and plunge displacements by applying a state feedback controller. Librescu et al. (2005b) investigated the active control of a three degree-of-freedom airfoil operating in an incompressible flow field and exposed to blast/gust loads. The aerodynamic forces for the wing-
aileron system were derived from the Theodorsen’s equations. They implemented various control methodologies such as optimal control and compared their performances in flutter suppression. Na et al. (2006) employed a similar structural and aerodynamic modeling and designed a linear quadratic Gaussian (LQG) control strategy using a sliding mode observer.

Ardelean et al. (2006) and McEver et al. (2007) performed some wind tunnel experiments on a two-dimensional wing model to suppress the flutter using a trailing-edge flap powered by a V-stack piezoelectric actuator. V-stack piezoelectric actuator consists of two piezoelectric stacks placed in the middle of an airfoil which can cause a small differential motion due to the contraction and expansion of stacks in the presence of the applied voltages. This small motion can be transferred to flap via a slider-crank lever mechanism. Ardelean et al. (2006) used a positive position feedback control to add damping to the unstable flutter mode and eventually increase the flutter speed. McEver et al. (2007) used the same modeling and implemented an adaptive control technique involving Q parameterization to stabilize the wing flutter. It should be noted that the flap deflection, rather than the actuator voltage, was commanded to control the flutter and the piezoelectric actuator was employed to only actuate the flap.

The early considerations in structural and aerodynamic modeling were addressed to linear modeling; however, recent models have introduced the effects of the structural or aerodynamic nonlinearities. In an excellent survey paper, Dowell et al. (2003a) presented the state-of-the-art studies and ideas on nonlinear aeroelasticity and described the importance of considering the nonlinear effects in the analysis and active control of aeroelastic systems. As an example, one of the recent studies on the control of the nonlinear two-dimensional aeroelasticity using single control surface has been conducted by Behal et al. (2006a) which was based on the earlier works of Ko et al. (1997), Block and Strganac (1998), and Ko et al. (1998, 1999) studied the classical two degree-of-freedom pitching-plunging motion and considered continuous nonlinear restoring moment in the pitch degree of freedom. A quasi-steady aerodynamic model for an incompressible flowfield was chosen and an output feedback control to regulate
the pitch displacement to a setpoint was proposed and the efficacy of the control method in suppressing the limit cycle oscillations in the subcritical flight speed regime was demonstrated.

There have been many research studies investigating the active control of the two-dimensional aeroelastic oscillations in the supersonic speeds as well. Some examples of the more recent ones which have employed single control surface as an actuator are the research papers by Rao et al. (2006), Na et al. (2007), Lee et al. (2010), and Cao and Zhao (2011). The aeroelastic vibrations of a nonlinear two-dimensional wing-flap system operating in supersonic/hypersonic flight speed regimes was examined by Rao et al. (2006). They modified the nonlinear piston theory aerodynamics to account for flap deflections and implemented an output feedback control law to suppress flutter. Na et al. (2007) and Lee et al. (2010) investigated robust control strategies such as linear quadratic Gaussian (LQG) and sliding mode control for such an aeroelastic system exposed to blast/sonic-boom pressure pulses. In a similar modeling, Cao and Zhao (2011) applied a linear quadratic regulator (LQR) and a cubic nonlinear feedback controller to suppress the limit cycle oscillations.

Recently, utilization of multiple control surfaces in active control of aeroelasticity has caught the attention of many researchers. Dowell et al. (2003b) discussed the benefits of using a leading-edge control surface to counteract the tendency of a trailing-edge control surface to undergo the unfavourable aeroelastic effects such as control surface reversal. They suggested a control strategy to use a combination of the leading- and trailing-edge control surface rotations to enhance the performance of flutter suppression. Another good investigation on the active control of aeroelasticity using leading- and trailing-edge control surfaces is the study by Platanitis and Strganac (2004) which involved some analytical studies and wind tunnel experiments. They modeled the limit cycle oscillations by applying the quasi-steady aerodynamic loads on a two degree-of-freedom nonlinear wing section model and implemented a control method based on adaptive full feedback linearization via Lie algebraic methods to suppress the oscillations. They compared these results with those when a single trailing-edge
control surface is actuated and stated that using multiple control surfaces is more advantageous in suppressing the limit cycle oscillations. Other researches adapted the model published by Platanitis and Strganac (2004) and proposed different control schemes such as adaptive and neural network (Gujjula et al. 2005), full-state feedback (Behal et al. 2006b), adaptive output feedback (Reddy et al. 2007), state-feedback linear-parameter-varying controller (Prime et al. 2010), modular output feedback (Wang et al. 2011), and continuous robust feedback (Wang et al. 2012). All these studies provided modifications to the dynamic model or control algorithms to improve the performance of suppressing the aeroelastic oscillations.

One of the important issues in control of aeroelastic oscillations is the unavoidable presence of time delays between the act of sensors and actuators in the closed loop systems. Although time delays exist in real systems, their effect was not included in the traditional feedback control models. However, recent studies show that time delays can change the behaviour of the system stability in closed loop systems, and control designers can apply intended time delays to control the aeroelastic instabilities such as flutter (Yuan et al. (2004), Librescu and Marzocca (2005), etc.). Many researchers have investigated the time-delayed feedback control of the aeroelastic systems, such as Yuan et al. (2004), Librescu et al. (2005a), Marzocca et al. (2005), Yu et al. (2007), and Zhao (2009a, 2009b). Most of the structural models were considered as two degree-of-freedom pitching-plunging systems.

Yuan et al. (2004) and Yu et al. (2007) studied the time-delayed feedback control for supersonic speeds and obtained the aerodynamic loads using third order approximation of piston theory aerodynamics. Yuan et al. (2004) discussed the effect of the time-delayed proportional feedback control on the flutter instability boundary and its character (benign/catastrophic). They demonstrated that the time delay improves the stabilization of the flutter instability and incorporation of small time delay in both linear and nonlinear feedback control is beneficial in postponing the occurrence of flutter. Their work was extended by Yu et al. (2007) for full-state proportional and velocity feedback controls.
In some related studies, Librescu et al. (2005a) and Marzocca et al. (2005) investigated the time-delay effects of two-dimensional lifting surfaces in an incompressible flight speed regime using the concept of Volterra series in conjunction with the indicial aerodynamic functions. A stability analysis of a two-dimensional airfoil with time-delayed feedback control was provided by Zhao (2009a) in an incompressible flow with quasi-steady aerodynamic loads. The flutter boundaries of the controlled aeroelastic system were predicted for various time delays and the effectiveness of the method was proven by numerical simulations. In another study, Zhao (2009b) investigated the time-delay effects on the stability of a high aspect-ratio wing with multiple control surfaces and proposed an optimal control to suppress the flutter where the aerodynamic loads were obtained by doublet lattice method.

It can be observed that most of the existing literature has restricted their modeling to quasi-steady subsonic or supersonic aerodynamic models and although the study of the aeroelastic oscillations under 3-D unsteady subsonic flow is of significant interest, it has not been addressed properly by the prior literature. Therefore, Chapters 3 and 4 of this thesis have been dedicated to present the unsteady aerodynamic modeling used in the present work using the numerical panel methods. It is worth mentioning that more sophisticated CFD methods can be used for unsteady aerodynamic calculations but they are very computationally intensive and not convenient for studying the fluid-structure interaction problems.

1.2.2 Analysis and Control of Oscillations Using Piezoelectric Materials

Numerous studies on the piezoelectric materials during last few decades show great interest towards the usage of these materials in analysis and control of mechanical and aerospace vibrations. Some examples of the earlier control studies are the papers by Bailey and Hubbard Jr (1985), Crawley and de Luis (1987), Crawley et al. (1988), etc. The reader is referred to a useful textbook by Fuller et
al. (1996) for more information on piezoelectric materials and active control of vibrations. Giurgiutiu (2000) reviewed the application of smart materials actuation for aeroelastic and vibration control in fixed-wing and rotary-wing air vehicles. Many earlier studies in this area excluded the effects of aerodynamic loads and confined their analyses to elastic oscillations, which may be because of the complexity of the aeroelastic modeling. The present survey firstly reviews the studies related to the vibrations of the structures and then those related to aeroelastic oscillations, focusing mostly on the recent studies.

A great number of studies have investigated the analysis and control of elastic vibrations in beams, plates, including composite beams or plates. Tang and Wang (2001) investigated the performance of some basic active-passive hybrid piezoelectric networks and compared the passive damping and active control of these networks. The piezoelectric patch was bonded to the surface of the beam and was connected to a shunt circuit. Lee et al. (2003) studied the arrangement of matched pairs of piezoelectric transducers bonded on a cantilever beam for vibration control. They used the PVDF type of transducers and discussed the stability and performance of the matched pairs for direct velocity feedback control.

Yang et al. (2005) proposed an optimization design using genetic algorithms to find the optimal placement and size of the PZT patches for a beam model bonded with piezoelectric sensors and actuators. They also formulated a feedback control law and found the control gains using genetic algorithm. Ramesh Kumar and Narayanan (2008) employed a model-based LQR controller to obtain the optimal locations of collocated piezoelectric sensor–actuator pairs on flexible beams. They also examined the active vibration control performance using classical control methods such as proportional and velocity feedback, and also using optimal control law based on LQR.

There has been some research specifically on composite structures, where the effects of the anisotropic material properties of the host structure should be considered in the modeling (Librescu et al. (1997), Song et al. (2002), Sun and
In studies by Librescu et al. (1997) and Song et al. (2002), the anisotropy properties of the composite materials exploited in structural tailoring was combined with actuating capabilities of piezoelectric materials in order to suppress the vibrations. They modeled a beam host structure where PZT patches were bonded or embedded into the host structure and applied control strategies such as proportional and velocity feedback to control the vibrations. They found that the combination of structural tailoring and control by means of adaptive materials is effective in suppressing the structural vibrations.

Pai et al. (1993) also presented a geometrically nonlinear plate theory for the analysis and control of elastic laminated composite plates with integrated piezoelectric actuators/sensors. In a recent Ph.D. thesis, Wang (2012) studied the vibration and buckling control of piezo-laminated composite structures with surface bonded or embedded piezoelectric sensors and actuators. They developed the finite element formulations for the beam and plate structures and applied an active control to suppress the elastic oscillations.

The control stability analysis of a controlled beam with partially debonded piezoelectric patch was investigated by Sun and Tong (2002). They studied the effects of edge debondings of the actuator layer on the control stability of a cantilevered beam controlled by a proportional and derivative (PD) controller. Na et al. (2004) performed some simulations on the thin-walled beam cantilevers bonded with piezoactuators and proposed feedback control to minimize the dynamic response of the beam subject to a constraint on the input voltage applied to the piezoactuators. In a related work, Librescu and Na (2005) compared the performance of the various feedback controllers for similar thin-walled beam. In their work, the closed-loop dynamic response time-histories were obtained by using the piezoelectrically induced moment control.

Many researchers explored the analysis and control of elastic vibrations of plate structures. One example is the experiments performed by Yang and Bian (1996) on two composite laminated plates embedded with piezoelectric sensors and actuators. They investigated the effectiveness of vibration suppression by the
velocity feedback from embedded piezoelectric sensors and actuators and showed that both the bending and torsional vibration can be effectively reduced. Behrens et al. (2003) proposed a piezoelectric shunt-damping controller as an alternative method for reducing structural vibrations. Piezoelectric shunt-damping refers to placing an electrical impedance across the terminals of a piezoelectric transducer that acts as a medium to dissipate the mechanical energy of the host structure. They analyzed and experimentally validated the effect of current flowing shunt controller for two piezoelectric laminated structures.

Xu and Koko (2004) proposed a method to control smart structures using finite element code with piezoelectric elements and state/output feedback control law. They performed simulations using a commercial finite element software for beam and plate structures with perfectly bonded PZT patches and used the LQR method to obtain feedback gains for output feedback controller. In another study by Qiu and Haraguchi (2006), the application of self-sensing actuator was investigated to control the vibrations of a two-dimensional plate. A self-sensing actuator is a piezoelectric element which works both as a sensor and an actuator. An adaptive feedback controller using a finite impulse response (FIR) filter was also included in the system and results were verified by experiments. They observed that the vibration level was decreased at almost all the resonance peaks.

In a recent study by Zhao (2010), active/passive hybrid piezoelectric circuits were employed to suppress random vibrations of a quadrilateral plate. They established the finite element model of the plate bonded with piezoelectric actuators in commercial software ANSYS and imported the results to MATLAB platform to control the vibrations. They demonstrated that hybrid piezoelectric configuration in vibration suppression is more effective than a system with passive or purely active control.

Some researches exploited optimization methods such as genetic algorithms to find optimal locations of piezoelectric sensors/actuators on plates (for example, Bruant et al. (2010), Julai and Tokhi (2010)). Bruant et al. (2010) discussed the optimal position and orientation of the piezoelectric actuators and sensors for a simply supported elastic plate. They reported that the influence of the patches’
orientation is not really significant for a simply supported plate; however for more complex structures this effect would be considerable. There is another perspective towards the vibration of plates equipped with piezoelectric patches which is the concept of converting waste vibration energy available in their environment to usable electrical energy or energy-harvesting concept. De Marqu¡ Jr et al. (2009) developed an electromechanically coupled finite element model for piezoelectric energy harvesting from base excitations. They employed the classical plate theory for cantilevered beams and plates with piezoceramic layers and predicted the electrical power output of piezoelectric energy harvester plates.

Of particular relevance to the work presented in this thesis, are the studies which have included the effects of aerodynamic loads along with piezoelectric sensors/actuators in analysis and control of aeroelastic oscillations. Shrivastava et al. (2000) and Mateescu et al. (2010) studied the feasibility of active control of aeroelastic oscillations by using piezoelectric actuator strips bonded to the surface of a delta wing that is modeled as a cantilevered triangular plate. The dynamics of the wing structure was studied under the combined effects of the unsteady supersonic aerodynamic loading (determined using a hybrid analytical-numerical method) and of the oscillatory voltage excitation applied to the piezoelectric actuator strips. It was found that the amplitude of aeroelastic oscillations could be effectively reduced by choosing particular combinations of excitation voltages on a small number of piezoelectric strips bonded on the wing.

Tuzcu and Meirovitch (2006) investigated the possibility of controlling vibrations of a flexible aircraft using piezoelectric actuators. The components of the flexible aircraft were modeled as beams undergoing both bending and torsion. Aerodynamic loads were calculated by means of strip theory using aerodynamic stability derivatives, and a feedback controller was designed employing the LQR method. It was found that by providing sufficient power sources, piezoelectric actuators can be effective in vibration suppression.

Song and Li (2011), studied the active aeroelastic flutter characteristics and vibration control of supersonic beams with piezoelectric actuators/sensors. The
aerodynamic pressure was obtained by supersonic piston theory and the active damping effect of the structural system was gained by applying a negative velocity feedback. It was observed that, within a certain value of the feedback control gain, using piezoelectric patches can lead to significant improvement in the flutter characteristics and decrease in the amplitudes of oscillations.

Regarding the studies on plate structures, there has been some work by Richard et al. (2001), Rule et al. (2001), and Richard and Clark (2003) on the optimal placement of piezoelectric sensor/actuator pairs to control the flutter of a delta wing. Richard et al. (2001) performed computational and experimental studies on active control of a delta wing with PZT/PVDF sensors and actuators. A structural model based on the Ritz method was coupled to a reduced-order potential-flow aerodynamic model through feedback, to simulate the aeroelastic structure and was verified by wind tunnel tests. The results showed that a single sensor/actuator pair can be designed to significantly increase the damping of the flutter mode above the uncontrolled flutter velocity. Similar findings were reported by Rule et al. (2001) and Richard and Clark (2003).

Agneni et al. (2003) presented a modal-based FE modeling of shunted piezoelectric devices bonded on a host structure in order to analyze their effectiveness in increasing the passive damping of elastic and aeroelastic systems. They verified the results of applying the proposed technique on simple structures such as a cantilever beam and a simply supported plate with experimental test and also performed experiments on a wing of a fully composite glider. The results of the glider showed a weak capability of the passive devices in improving the flutter margin, but demonstrated a significant performance in reduction of the gust response amplitude at flight speeds close to flutter, where the effects of aerodynamic damping becomes less important.

Suleman and Costa (2004) conducted wind tunnel experiments on a flexible aircraft to examine the feasibility of applying piezoelectric actuators to suppress the aeroelastic vibrations. The sensor digital signal was feedback in the control law to suppress flutter and buffeting of the structure. The performance of the wing
equipped with piezoelectric actuators was compared to the more conventional aileron controlled wing. The obtained results showed that piezoelectric actuators were always effective in flutter suppression of the stand alone wing. In another work, Rocha et al. (2007) studied the flutter control of a wing with active piezoelectric actuators distributed on the wing skin and performed the wind tunnel experiments. They compared the results with the passive wing and showed that the active wing is able to decrease the wing displacement and increase the amount of damping.

Li et al. (2006) investigated the active flutter suppression of a wing model using a single piezoceramic element (PZT) in a wind tunnel. Finite element analysis was employed for modeling the wing as a flat plate followed by a Doublet Point Method for aerodynamic analysis. A control law based on LQG was designed which was able to suppress the flutter oscillations. Harash et al. (2005) modeled a medium-aspect-ratio elastic wing with a slender body attached to its tip and studied the effects of geometric structural nonlinear behavior and aerodynamic stall on both flutter instability boundary and on nonlinear response. They also performed closed loop fuzzy logic based control tests to suppress flutter/ limit cycle oscillations.

There has been also some research regarding the application of piezoelectric materials on control of plate oscillations in supersonic flow, such as Abdel-Motagaly et al. (2005), Kim et al. (2008), Fazelzadeh and Jafari (2008), to name a few. Abdel-Motagaly et al. (2005) examined a coupled structural-electrical modal finite element formulation for composite panels where piezoelectric sensors and actuators were embedded on the top and bottom surfaces of the panels. They used the modeling to suppress the nonlinear supersonic panel flutter in the presence of an airflow yaw angle. It was observed that the flow yaw angle could greatly affect the limit-cycle behaviour and panel deflection shape. In a similar study, Kim et al. (2008) used an output feedback controller of linear quadratic regulator form to suppress the nonlinear panel flutter at supersonic speeds. They applied quasi-steady first order piston theory to model the aerodynamic pressure around thin flat composite plates with embedded piezoelectric actuators.
Fazelzadeh and Jafari (2008) proposed an active optimal feedforward/integral control for a supersonic panel under gust disturbance effects with piezoelectric actuators. They modelled aerodynamic loading using first-order piston theory with gust velocity effects and applied a linear quadratic regulator controller to minimize the panel deflection. Parametric studies were also performed for various piezoelectric actuator configurations and demonstrated the effectiveness of the controller for flutter suppression and gust alleviation for various piezoelectric configurations.

1.3 Objectives of the Present Research

Based on the above literature review, it becomes clear that the most of the research work in the area of the control of aeroelastic oscillations simplified their models to two-dimensional structural or quasi-steady aerodynamic models and a rigorous model to include the effects of unsteadiness and three-dimensionality in the aerodynamic and structural modeling has not been addressed, or addressed insufficiently, by the prior literature.

The general objective of this research is the analysis and control of the aeroelastic oscillations of a trapezoidal wing structure with bonded piezoelectric strips subject to three-dimensional unsteady subsonic aerodynamic loads. The bonded piezoelectric strips will be used either as sensors to measure the dynamic response of the wing or as actuators to control the aeroelastic oscillations. This complex problem requires the time-dependent simultaneous solution of the dynamics equations of the elastic structure and the piezoelectric strips coupled with the equations of the unsteady flow past oscillating wings.

The goal is to develop a numerical model for the three-dimensional unsteady subsonic flows past the trapezoidal wings executing the flexural oscillations using panel methods. In addition, a finite element model will be developed to solve the dynamic equations of aircraft structures, using appropriate 3-D finite element discretizations of the elastic structure. Also, the structural model will be combined with a finite element model of the piezoelectric strips bonded to the wing
structure. For the combined finite element model, the constitutive equations of the piezoelectric strips will be used.

An interactive computer model will be developed to simultaneously solve the 3-D unsteady fluid flow equations and the dynamic equations of the structure with bonded piezoelectric strips, by combining the 3-D unsteady aerodynamic panel method and the structural finite element method. An active control method will be developed to suppress the oscillations of the wing using bonded piezoelectric actuators excited by oscillatory voltages. Also, an analysis will be performed to obtain the effective arrangement of sensor/actuators required to control the aeroelastic oscillations.

1.4 Thesis Outline

A general introduction to the active control of aeroelastic oscillations as well as the use of piezoelectric materials has been presented in this chapter, Chapter 1. In addition, an extensive literature review is presented on the recent control methods for suppression of aeroelastic oscillations as well as the use of piezoelectric materials in analysis and control of oscillations.

Chapter 2 presents a description of the aeroelastic problem considered in this thesis, including the assumptions and choice of methodologies. It also briefly reviews the background theory for various components of the study such as the aerodynamic, structural and control models.

Chapter 3 and 4 introduce the aerodynamic modeling for two- and three-dimensional unsteady flows, respectively. The details of the aerodynamic model for both steady and unsteady flows are discussed and the models are validated by comparing the results with the existing literature.

In Chapter 5, the structural dynamics model and the finite element formulation of the wing structure as well as the piezoelectric sensors/actuators is presented.

An active control model to suppress the aeroelastic oscillations is proposed in Chapter 6. The coupling between the structural and aerodynamic models is
discussed and a systematic method to obtain the gains of the feedback controller is introduced.

In Chapter 7, the results of the numerical simulations for control of 2-D and 3-D aeroelastic oscillations using piezoelectric sensors and actuators is presented and discussed.

Finally, conclusions, comments, and recommendations for future work are presented in Chapter 8.
CHAPTER 2

PROBLEM DESCRIPTION

2.1 Introduction

In this chapter, a description of the aeroelastic problem considered in this thesis, including the assumptions and choice of methodologies, is presented. The background theory behind various parts of the problem solved in this thesis, is briefly reviewed, which can be used as an introduction to better understand the more thorough modeling presented in the subsequent chapters.

It first starts with a description of the kinematics of the system and the aerodynamic modeling and an introduction to the 2-D and 3-D panel methods. Then the structural modeling, characterization of piezoelectric materials and related basic equations are discussed. At the end, a description of the chosen control technique and the required formulation are presented.
2.2 Problem Description

Control of aeroelastic oscillations involves the interaction between structural dynamics, unsteady aerodynamics, and flight control system. Figure 2.1 shows a schematic of this interaction as a form of an aeroservoelastic pyramid which is an extension of the famous Collar aeroelastic triangle (Wright and Cooper 2008). In an aeroelasticity problem, there is a mutual interaction between the structural and aerodynamic forces; hence, a coupling exists between aerodynamic loads and the structural deformations.

In general, when deformable structures are subject to an unsteady airflow, the aerodynamic loads cause the structure to deform. These structural deformations change the pressure distribution on the structure, consequently induce modified aerodynamic loads which again produce additional structural deformations. By employing a control system to suppress the amplitude of the aeroelastic oscillations, now there are forces due to the control system which have influence on other forces.

An important issue in computational aeroelasticity is the exact description of the mathematical models for each loading in order to better simulate the real flight condition. As explained in the literature review section of Chapter 1, most of the research on the control of aeroelastic oscillations has been restricted to simple
models such as two-dimensional structural or quasi-steady aerodynamic models. These models are able to provide preliminary results with some degree of accuracy, but to obtain accurate and valid results, it is necessary to include the effects of unsteadiness and three-dimensionality.

In this study, a control method has been applied to suppress the aeroelastic oscillations of a trapezoidal wing with bonded piezoelectric strips caused by unsteady subsonic aerodynamic loading. The trapezoidal wing is modeled as a cantilever plate undergoing small transverse oscillations. The aerodynamic loading is determined using a numerical panel method for the unsteady incompressible subsonic regime to calculate the unsteady pressure distribution and the generalized aerodynamic forces acting on the oscillating wing. Figure 2.2 illustrates a schematic of the trapezoidal wing including the wing and the wake panels. The kinematics of the system is presented in section 2.3 and the panel method and the aerodynamic model are described briefly in section 2.4.

Figure 2.2 Trapezoidal wing in the unsteady three-dimensional incompressible flow
A finite element formulation is used to model the structural dynamics of the three-dimensional wing, a brief description of which is given in section 2.5. The piezoelectric strips are considered to be PZT patches bonded to the wing surface. In section 2.6, the difference between PZT and other forms of piezoelectric material and the reason for choosing PZT patches are described. Also, the equations describing the constitutive law of the piezoelectric materials are presented.

PZT strips are considered either as actuators placed on the top of the wing surface or as sensors placed on the bottom surface as shown in Figure 2.3. The thickness of the PZT strips is considered to be very small in order to avoid the disturbance of the airflow around the wing. The output voltage of the piezoelectric sensors is used as a feedback to be applied to the piezoelectric actuators.

In this thesis, coupling between the structural finite element model and the three-dimensional unsteady aerodynamic model is taken into account. More precisely, at each time step, the results of the structural model in the form of the structural deformations should be transferred to the aerodynamic model as the

![Figure 2.3 A section cut from the wing structure with piezoelectric sensors and actuators](image-url)
input to calculate the pressure distribution and aerodynamic loading. Then, these computed aerodynamic forces should be introduced to the structural model for transient analysis.

After connecting the two models, the combined model is used to obtain the oscillatory structural deformations under various flight conditions. An active control methodology is proposed to suppress the oscillations predicted by this combined model. Also, to study the influence of actuator locations on the control effectiveness, different scenarios of the actuator placement are compared.

In the following sections, a brief description of the employed method for each part of the modeling and some theoretical background are presented.

2.3 Kinematics of the System

The trapezoidal wing can execute low frequency flexural oscillations in unsteady subsonic incompressible flow. The flexural oscillations consist of the rigid-body translational and oscillatory rotational motions of the wing center of mass, as well as the flexural structural deformations.

The flexural structural deformations of the wing can be obtained by simultaneous solution of the unsteady flow and the structural dynamics equations at each time step and the rigid-body translational and rotational motions can be defined by introducing the appropriate coordinate systems and transformation, as discussed further below.

The unsteady flow past the oscillating wing is referred to a Cartesian system of coordinates. Two parallel coordinate systems can be defined to describe the unsteady motion of the wing: one is the inertial frame of reference (XYZ) and the other is the body-fixed frame (xyz) which is attached to the wing. At time $t = 0$ these two frames coincide, but as wing starts moving ($t > 0$), the body-fixed frame moves away from the inertial frame.
Figure 2.4 illustrates the two coordinate systems for a 3-D wing moving in an unsteady flow. The absolute motion of the origin of the frame \(xyz\) is described by its position vector:

\[
R_0(t) = X_0\hat{i} + Y_0\hat{j} + Z_0\hat{k}
\]  

(2.1)

and the instantaneous orientation of this frame is expressed as:

\[
\Theta(t) = (\phi, \alpha, \psi)^T
\]  

(2.2)

where \(\phi, \alpha, \psi\) are the rotation angles about \(x, y, z\) axes, respectively. The transformation between the two coordinate systems based on the flight path can be written as:

\[
\begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix} = T_x(t)T_y(t)T_z(t)\begin{bmatrix}
X - X_0 \\
Y - Y_0 \\
Z - Z_0
\end{bmatrix}
\]  

(2.3)

where \(T_x(t), T_y(t), T_z(t)\) are the rotation matrices associated with rotations about the \(x, y, z\) axes, respectively. In this thesis, it is assumed that the rotation is mainly due to the pitch oscillation, hence:
where $I$ is the identity matrix. The transformation matrix for the pitch oscillation can be written as follows:

\[
T_x(t) = T_z(t) = \begin{bmatrix}
\cos \alpha(t) & 0 & -\sin \alpha(t) \\
0 & 1 & 0 \\
\sin \alpha(t) & 0 & \cos \alpha(t)
\end{bmatrix}
\]  \hspace{1cm} (2.5)

The pitch oscillation is assumed to be caused by changes of the flow angle of attack $\alpha(t)$ due to the low-frequency gusts in the cosine form as:

\[
\alpha(t) = \alpha_0 + \alpha_A \cos 2\pi ft
\]  \hspace{1cm} (2.6)

where $f$ is the frequency.

The body surface velocity $V_b$ can be written as the rate of change of the $xyz$ coordinates which has three components of translational, rotational and relative velocities. Hence, the velocity of the wing surface in the inertial coordinate system ($XYZ$) can be written in the general form by:

\[
(V_b)_{XYZ} = V_0 + \omega \times r + v_{rel}
\]  \hspace{1cm} (2.7)

where $V_0$ is the translational velocity of the system’s origin and $\omega \times r$ is the velocity due to rotation of the wing, where $\omega$ is the angular velocity of the body frame and $r = [x, y, z]^T$ is the position vector. Since it is assumed that the rotation is mainly due to the pitch oscillation, then $\dot{\phi} = \dot{\psi} = 0$ and the angular velocity can be written as:

\[
\omega = \dot{\alpha} \hat{j}
\]  \hspace{1cm} (2.8)

In addition, $v_{rel}$ is the velocity due to the wing’s relative motion in the $xyz$ frame (for instance, due to the change in the wing’s shape or structural deformations).

In this thesis, it is assumed that the free-stream velocity $U_\infty$ is along the positive direction of the $x$ axis. Equivalently, it can be assumed that the wing is moving along the negative direction of the $x$ axis with velocity of $U_\infty$. Hence, it
can be assumed that the origin of the body-fixed frame has a translational velocity of \( V_0 = -U_\infty \) (as shown in Figure 2.4). Equation (2.7) can be written in the body-fixed frame \((xyz)\) as:

\[
(V_b)_{xyz} = \omega \times r + v_{rel}
\]

(2.9)

More details on the calculation of the unsteady velocity field are presented in Chapters 3 and 4.

### 2.4 Aerodynamic Modeling

A classical flow model which can provide reliable flow field predictions for a wide variety of flying bodies is the incompressible potential flow model. In such a model, the flow is subsonic and incompressible over the entire flow field and the viscous effects are assumed to be negligible.

While there are several CFD methods, such as finite volume method or finite element method, which can solve more complicated fluid mechanics problems, albeit at relatively lower speed and higher cost, for the subsonic flow applications when the viscous effects are negligible, the panel method is a good inexpensive option.

In general, the solution of the potential flow model can be found by dividing the surface into a finite number of parts called panels and then distributing “singularities” of unknown strength over these panels. Hence, the velocity field is obtained by solving a set of linear algebraic equations to determine the unknown strengths of the singularities. These types of methods are called panel methods in general which are useful and flexible numerical methods and do not require intensive computation.

The panel methods are flexible in the sense that they address a wide range of simple to complex geometries and can vary based on the type of singularity. The flexibility of the panel methods and their relative efficiency are important in practice and have made them to be more practical choices compared to the exact
analytical methods, which generally are not capable of handling similar range of geometries as the panel methods can do. In addition, one of the main advantages in using the panel methods is the fact that there is no need to define a grid throughout the flow field. Also, panel methods can be used in three-dimensional applications.

In this thesis, it is assumed that the flow is unsteady incompressible subsonic and the effects of viscosity are negligible. Hence the flow is incompressible potential flow and the governing equation is Laplace’s equation as follows:

$$\Delta \Phi \equiv \nabla^2 \Phi = 0$$ \hspace{1cm} (2.10)

where \( \Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \) is the Laplace operator and \( \Phi \) is the fluid velocity potential as defined below:

$$\Phi = \Phi_\infty + \varphi$$ \hspace{1cm} (2.11)

where \( \Phi_\infty \) is the free-stream velocity potential and \( \varphi \) is the perturbation velocity potential. The Laplace’s equation can be written as follows:

$$\nabla^2 \varphi = 0$$ \hspace{1cm} (2.12)

In the panel method, a set of algebraic equations should be solved which can be obtained by solving the Laplace’s equation along with the fulfilment of the boundary condition. The boundary condition here is the zero normal flow on the solid surface which can be stated as:

$$\mathbf{V} \cdot \mathbf{n} = \mathbf{V}_b \cdot \mathbf{n}$$ \hspace{1cm} (2.13)

where \( \mathbf{V} = \nabla \Phi \) is the fluid velocity and \( \mathbf{V}_b \) is the velocity of the wing surface in the body-fixed frame defined in equation (2.9). Here \( \mathbf{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k} \) is the unit vector normal to the surface. The boundary condition in equation (2.13) can result in:

$$\nabla \Phi \cdot \mathbf{n} = (\mathbf{\omega} \times \mathbf{r} + \mathbf{v}_{rel}) \cdot \mathbf{n}$$ \hspace{1cm} (2.14)
or by substituting from equation (2.11):

\[
\nabla \Phi_{\infty} \cdot \mathbf{n} + \nabla \varphi \cdot \mathbf{n} = (\omega \times \mathbf{r} + \mathbf{v}_{rel}) \cdot \mathbf{n}
\]  

(2.15)

where \( \mathbf{V}_{\infty} = \nabla \Phi_{\infty} \) is the free-stream velocity. It is worth mentioning that the flight path is known at each time step, so the right hand side of equation (2.15) is known. The free-stream velocity \( \mathbf{V}_{\infty} \) is also known which can go to the right hand side of the equation, resulting in:

\[
\nabla \varphi \cdot \mathbf{n} = (\omega \times \mathbf{r} + \mathbf{v}_{rel}) \cdot \mathbf{n} - \mathbf{V}_{\infty} \cdot \mathbf{n}
\]  

(2.16)

The unit normal vector \( \mathbf{n} \) can be determined by introducing the surface shape. If the surface shape is defined as:

\[
F_{s} = z - \eta(x, y, t) = 0
\]  

(2.17)

the surface normal vector can be obtained in the general form from:

\[
\mathbf{n} = \frac{\nabla F_{s}}{|\nabla F_{s}|}
\]  

(2.18)

where: \( \nabla F_{s} = -\frac{\partial \eta}{\partial x} \hat{i} - \frac{\partial \eta}{\partial y} \hat{j} + \hat{k} \), therefore:

\[
\mathbf{n} = \frac{1}{\sqrt{\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2 + 1}} \left(-\frac{\partial \eta}{\partial x} \hat{i} - \frac{\partial \eta}{\partial y} \hat{j} + \hat{k}\right)
\]  

(2.19)

The left hand side of equation (2.16) can be written as a linear combination of the singularity strengths (unknown at this point) which results in a linear set of equations. The unknown singularity strengths can be obtained by solving this set of equations, and consequently, the aerodynamic loading can be calculated using the Bernoulli equation. The details of this procedure are explained in Chapters 3 and 4 for two- and three-dimensional flow, respectively.

Various types of singularities (source, vortex, doublet) can be selected in panel methods. If the thickness of the wing airfoil is considered negligible, the
source singularities are not required and only the vortex singularities can be used. In this thesis, first a two-dimensional unsteady aerodynamic model is considered in order to validate the results with the existing literature, and after the validation the more complicated case of three-dimensional flow is modeled. For each case, the aerodynamic loading is calculated firstly for steady flow conditions and secondly for unsteady flow by introducing the required modifications and considering the effects of time-dependant terms.

For 2-D model, the Discrete Vortex Method is selected which places the point vortices on the surface of each panel. Then, the Laplace equation should be solved and the zero-normal-flow boundary condition should be satisfied which results in obtaining the vortex strengths. The aerodynamic loads for the steady flow can be calculated using the Kutta-Joukowski theorem. For the unsteady flow, the Bernoulli equation can be used to obtain the pressure distribution and the aerodynamic loads.

The 3-D panel method used in the present thesis is similar to the 2-D one, except that the vortex ring method is used, where the surface of the wing is divided into a finite number of surface panels with vortex ring elements. In order to apply the panel methods, a FORTRAN program is used which solves the relevant set of equations and calculates the generalized aerodynamic loads. The details of the aerodynamic formulation of the 2-D and 3-D models are explained in Chapters 3 and 4, respectively.

2.5 Structural Modeling

In order to achieve a practical and effective design of a controller to suppress the aeroelastic oscillations, an accurate and reliable model for analysis of structural and aerodynamic characteristics is needed. Various methods, such as classical closed form solutions, finite difference method, finite element method and boundary element method, have been extensively used to provide appropriate models for various types of structures.
Closed form solutions are available only for structures with very simple geometric shape. Hence, for more complex structures, the finite element method (FEM) can be employed in the absence of closed form solutions. In particular, when the structure includes smart materials such as piezoelectric sensors and actuators with coupling between mechanical and electrical equations, it is necessary to use more sophisticated FEM in order to be able to get any result.

In this thesis, the finite element software ANSYS is used to study the aeroelastic oscillations. The wing structure is modeled as a cantilever plate with brick elements and piezoelectric strips as small patches bonded on the wing surface with coupled-field elements. The finite element formulation of the cantilever plate yields:

\[
[M]\ddot{\{u\}} + [C]\dot{\{u\}} + [K]\{u\} = \{F\} \tag{2.20}
\]

where \([M]\), \([C]\), and \([K]\) are the structural mass matrix, structural damping matrix, and structural stiffness matrix, respectively. \(\{u\}\) is the nodal displacement vector, and the symbol dot denotes differentiation with respect to time. \(\{F\}\) is the applied load vector, which in an aeroelastic problem is the aerodynamic load obtained from the aerodynamic model in the FORTRAN program.

In the following section, a brief introduction to piezoelectric materials and some basic equations are given; the details of the finite element modeling for both the host structure and the piezoelectric sensors and actuators are presented in Chapter 5.

2.6 Piezoelectric Materials

2.6.1 Overview

The phenomenon of piezoelectricity is the ability of certain crystalline materials such as ceramics to generate an electrical voltage when subjected to an externally applied mechanical load, and vice versa. Even though the discovery of the piezoelectric materials goes back to 1880, their application as sensors and
actuators for reduction of vibrations, noise control and other areas is relatively new. Piezoelectric materials have been manufactured in many different forms based on their intended applications. The most popular ones are Lead-Zirconate-Titanate (PZT) which is a ceramic, and Polyvinylidene Fluoride (PVDF) which is a polymer. Properties of typical piezoelectric materials are presented in Table 2.1 (Preumont 2002).

**Table 2.1 Typical properties of piezoelectric materials (Preumont 2002)**

<table>
<thead>
<tr>
<th>Material properties*</th>
<th>PZT</th>
<th>PVDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piezoelectric constants:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d_{33}$ (10^{-12} C/N or m/V)</td>
<td>300</td>
<td>-25</td>
</tr>
<tr>
<td>$d_{31}$ (10^{-12} C/N or m/V)</td>
<td>-150</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_{31} = 15$, $d_{32} = 3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>bi-axial:</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$d_{31} = d_{32} = 3$</td>
</tr>
<tr>
<td>$e_{31} = d_{31} / s^E (C/m^2)$</td>
<td>-7.5</td>
<td>0.025</td>
</tr>
<tr>
<td>Dielectric constant $\varepsilon_T / \varepsilon_0$</td>
<td>1800</td>
<td>10</td>
</tr>
<tr>
<td>(Vacuum permittivity: $\varepsilon_0 = 8.85 \times 10^{-12} F/m$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Young’s modulus (GPa)</td>
<td>50</td>
<td>2.5</td>
</tr>
<tr>
<td>Maximum stress (MPa):</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tension</td>
<td>80</td>
<td>200</td>
</tr>
<tr>
<td>Compression</td>
<td>600</td>
<td>200</td>
</tr>
<tr>
<td>Max. Operating Temp. ($^{\circ}C$)</td>
<td>80$^{\circ} - 150^\circ$</td>
<td>90$^\circ$</td>
</tr>
<tr>
<td>Max. Electric field (V/mm)</td>
<td>2000</td>
<td>5 $\times 10^5$</td>
</tr>
<tr>
<td>Density ($kg/m^3$)</td>
<td>7600</td>
<td>1800</td>
</tr>
</tbody>
</table>

* The variables are defined in the following section.
Piezopolymers (PVDF) are more robust against damage, but they lack sufficient stiffness compared to PZT. Piezoceramics (PZT) are stiffer and their high stiffness provides sufficient energy density, and their fast response times provide wide operating bandwidth. Also, PZTs are widely used due to their high piezoelectric, dielectric and elasticity coefficients.

It is worth noting that in Table 2.1, the value of piezoelectric coefficient \( e_{31} \) for PZT is 300 times larger than that for PVDF. This shows that piezoceramics have greater actuation capability compared to piezopolymers (Preumont 2002). Based on the above mentioned reasons, PZT patches are used in the present study both as sensors and actuators bonded to the surface of the wing.

2.6.2 Constitutive Equations of Piezoelectric Materials

Piezoelectricity is described mathematically by coupling two constitutive equations for mechanical and electrical problems. For mechanical systems, mechanical constitutive equation is the Hooke’s law which defines the strain of an elastic body when it is subjected to a mechanical stress. Using the notations of the IEEE standard on piezoelectricity (1988), the Hooke’s law for a one-dimensional elastic body can be written as follows:

\[
S = sT
\]

(2.21)

where \( S \) is the elastic strain, \( T \) is the stress \( (N/m^2) \), and \( s \) is the compliance of the material which is the inverse of the Young’s modulus. Here, it is assumed that the elastic body is placed in a zero electric field.

For electrical problems, electrical constitutive equation describes the relation between charge and electric field in a dielectric material. For instance, in an unstressed one-dimensional dielectric medium with the dielectric constant of \( \varepsilon \), charge per unit area can be defined as follows:

\[
D = \varepsilon E
\]

(2.22)
where $D$ is the electric displacement or charge per unit area ($C/m^2$), and $E$ is the electric field ($V/m$). For a piezoelectric material, the mechanical and electrical constitutive equations can be combined, resulting in (Preumont 2002):

$$S = s_e T + dE$$
$$D = dT + \varepsilon_f E$$

(2.23)

For multidimensional media, one can obtain the constitutive law for piezoelectric materials as follows (Piefort 2001):

$$\{S\} = [s_e] \{T\} + [d]^T \{E\}$$
$$\{D\} = [d] \{T\} + [\varepsilon_f] \{E\}$$

(2.24)

where $\{S\}$ is the elastic strain vector, $\{T\}$ is the stress vector, and $\{E\}$ is the electric field vector. $\{D\}$ is the electric displacement vector which is also called the electric flux density vector. $[s_e]$ is the compliance matrix when the electric field is constant and $[\varepsilon_f]$ is the dielectric matrix under constant stress.

Here, $[d]$ is the piezoelectric coupling matrix which relates the strain to the electric field in the absence of mechanical stress or relates the electric flux density vector $D$ to the stress under a zero electric field (short circuit teeth electrodes). The superscript $T$ stands for the transpose of the matrix. It is obvious that if $d = 0$, there is no coupling between mechanical and electric fields and the two parts of equation (2.23) will be uncoupled to produce equations (2.21) and (2.22).

Equation (2.24) is in Strain-Charge form, but can be transformed into Stress-Charge form as follows:

$$\{T\} = [c_e] \{S\} - [e]^T \{E\}$$
$$\{D\} = [e] \{S\} + [\varepsilon_S] \{E\}$$

(2.25)

or equivalently:

$$\begin{bmatrix} \{T\} \\ \{D\} \end{bmatrix} = \begin{bmatrix} [c_e] & -[e]^T \\ [e] & [\varepsilon_S] \end{bmatrix} \begin{bmatrix} \{S\} \\ \{E\} \end{bmatrix}$$

(2.26)
where \([c_E]\) is the stiffness matrix under constant electric field, \([\varepsilon]\) is the piezoelectric stress matrix which is the constant matrix relating the electric flux density vector to the strain for short-circuited electrodes and \([\varepsilon_s]\) is the dielectric matrix under constant strain.

The piezoelectric materials are dielectric materials which, although do not conduct current, can be polarized under the influence of an externally applied electric field. They are usually polarized along one direction, conventionally taken to be along the \(x_3\) (or \(z\)) axis.

The direction of polarization is determined during the manufacturing process when the piezoelectric material is under a high electrical field in the specified direction. If the direction of polarization is along the \(x_3\) (or \(z\)) axis as illustrated in Figure 2.5, the constitutive equations can be written in the following form:

**Actuation:**

\[
\begin{pmatrix}
S_{11} \\
S_{22} \\
S_{33} \\
2S_{23} \\
2S_{13} \\
2S_{12}
\end{pmatrix} = \begin{pmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & s_{44} & 0 & 0 \\
0 & 0 & 0 & s_{55} & 0 & s_{66} \\
0 & 0 & 0 & 0 & s_{66}
\end{pmatrix} \begin{pmatrix}
T_{11} \\
T_{22} \\
T_{33} \\
T_{23} \\
T_{13} \\
T_{12}
\end{pmatrix} + \begin{pmatrix}
0 & 0 & d_{31} \\
0 & 0 & d_{32} \\
0 & 0 & d_{33} \\
0 & d_{24} & 0 \\
d_{15} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
E_1 \\
E_2 \\
E_3
\end{pmatrix}
\] (2.27)

**Sensing:**

\[
\begin{pmatrix}
D_1 \\
D_2 \\
D_3
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & d_{15} & 0 \\
0 & 0 & 0 & d_{24} & 0 \\
d_{31} & d_{32} & d_{33} & 0 & 0
\end{pmatrix} \begin{pmatrix}
T_{11} \\
T_{22} \\
T_{33} \\
T_{23} \\
T_{13} \\
T_{12}
\end{pmatrix} + \begin{pmatrix}
\varepsilon_{11} & 0 & 0 \\
\varepsilon_{22} & 0 & 0 \\
0 & \varepsilon_{33} & 0
\end{pmatrix} \begin{pmatrix}
E_1 \\
E_2 \\
E_3
\end{pmatrix}
\] (2.28)

The double subscript indicates the direction of the applied electric field and the direction of the material expansion/contraction.
2.7 Control Methodology

Many active control strategies have been used to control aeroelastic vibrations. The most common ones are based on the principle of feedback, where the control signal is compared to a reference signal and the error is used to calculate the control command. There are many modern control methods which can provide good control performance such as pole placement, LQR, LQG, etc., but they are too complicated and expensive to implement in a high order system.

The LQR and LQG controllers are not very practical, since they have the same state dimension as the plant system, and their implementation for the high-order systems can be problematic. On the other hand, the LQR controller needs the full knowledge of the system states, which is not possible in most of the practical aeroelastic control problems.

Also, there are situations of possible sensor failure which makes full state unavailable. Among various feedback control strategies, proportional-integral-derivative (PID) compensator is a simple, conventional and by far the most common control algorithm which can control a large set of systems. The PID compensator is usually sufficiently robust against aerodynamic forces and unmodeled disturbances.
In the PID control method, the difference or error between the measured output and the setpoint value is calculated and the feedback is determined by using this error after calculating proportional (P), integral (I), and derivative (D) components. If the PID controller parameters (the gains of the P, I, D terms) are chosen incorrectly, the controlled response can be unstable.

There are various empirical methods that can be used to set the gain values, but often tuning of the gains is necessary in order to get good performance. In the present work, a method of determining the gains for the PID controller is developed which is discussed later in Chapter 6.

In any control technique, choice of sensors and actuators has a crucial role. In this study, piezoelectric sensors and actuators are selected due to their practical and advantageous characteristics such as light weight, flexible design, fast electro-mechanical response characteristics, and low cost. The piezoelectric patches are placed on the top and bottom surfaces of the wing and considered to be perfectly bonded to the surface.

The sensors and actuators can have two arrangements: collocated or non-collocated. Non-collocated arrangements are used in some cases, for instance

![Figure 2.6 Control schematic using piezoelectric strips](image-url)
where the areas of structure having desirable sensing do not coincide with the ones having desirable actuation. However, collocated arrangement has several advantages such as favourable closed loop stability. In the context of root locus technique, this property is a consequence of alternating pole and zero along the imaginary axis. Preumont (2002) showed that this property guarantees the asymptotic stability of a wide class of single-input single-output control systems since the root locus remains entirely within the left half plane. Such a collocated control system is said to guarantee robustness.

In this thesis, the collocated arrangement is used for piezoelectric sensors and actuators. Both sensing and actuating parameters are the voltages in the piezoelectric strips. Control schematic using the piezoelectric strips is shown in Figure 2.6. If the sensing voltage is denoted as $V_s$, it can be feedback to generate the actuator voltage $V_a$ using the following classical PID law:

$$V_a(t) = K_P V_d(t) + K_I \int_0^t V_d(\tau) d\tau + K_D V_d(t)$$  \hspace{1cm} (2.29)

where the voltage difference $V_d$ is the difference between the sensor voltage and the setpoint voltage:

$$V_d(t) = V_s(t) - V_{s0}$$  \hspace{1cm} (2.30)

where the setpoint voltage, $V_{s0}$, is the reference sensor voltage and $K_P$, $K_I$, and $K_D$ are the proportional, integral and derivative gains, respectively.

For the case of aeroelastic oscillations, the reference situation corresponds to the initiation of the oscillations where the aerodynamic load is considered to be steady. For numerical purposes, equation (2.29) should be discretized and the details are discussed in Chapter 6.
CHAPTER 3

AERODYNAMIC MODEL FOR STEADY AND UNSTEADY 2-D FLOWS

3.1 Introduction

This chapter discusses the details of the aerodynamic modeling for two-dimensional unsteady flows. Although the main goal of the aerodynamic modeling in the present work is to calculate the three-dimensional unsteady aerodynamic loading, in this chapter the modeling of a two-dimensional flow is considered firstly, in order to validate the results with the present literature and also to design a control methodology to suppress the aeroelastic oscillations. In Chapter 4, the aerodynamic modeling for the more complicated case of three-dimensional unsteady flow is presented.
In Chapter 2, a brief discussion was held about the present numerical methods and the choice of panel method for the present work. In this chapter, first as an introduction to the employed panel method, the vortex solution of the potential flows is introduced and then the details of the aerodynamic model for two-dimensional flow for both steady and unsteady cases are explained. It is worth mentioning that first the formulation for aerodynamic modeling of the steady flow is reviewed to have a better understanding of the present approach, and then the required modifications due to the unsteady flow is described to include the effects of the time-dependant terms.

3.2 Theoretical Basis for the Potential Flow Problem

As indicated initially in Chapter 2, in the present thesis, it is assumed that the flow is unsteady incompressible subsonic and the effects of viscosity are considered negligible. Hence the flow is incompressible potential flow and the governing equation is Laplace’s equation as follows:

$$\nabla^2 \Phi = 0$$

(3.1)

where $\Phi$ is the fluid velocity potential in a fluid domain $V$ enclosed by surface $S_b$ as shown in Figure 3.1. The goal of the aerodynamic analysis is to determine the velocity potential $\Phi$ and consequently velocity field in the entire flow-field. The potential of a unit strength source placed at an arbitrary point $S$ on the boundary for 2-D case can be denoted as:

$$f_s(R) = \frac{1}{2\pi}\ln R$$

(3.2)

where $R = (x - \xi)i + (y - \eta)j + (z - \zeta)k$ and the point $P(x,y,z)$ is an arbitrary point in the volume $V$.

By introducing a vector $\vec{W}$ as follows:

$$\vec{W} = \Phi\nabla f_s(R) - f_s(R)\nabla \Phi$$

(3.3)
where $\Phi$ is chosen as a harmonic function and is the solution of Laplace’s equation. By substituting this vector in the Gauss divergence theorem:

$$\int_{\mathbb{V}} \nabla \cdot \mathbf{W} \, dV = \int_{S} \mathbf{W} \cdot \mathbf{n} \, dA$$

one can obtain (Mateescu 2008b):

$$\int_{S} \left[ f_{s}(R) \nabla \Phi - \Phi \nabla f_{s}(R) \right] \cdot \mathbf{n} \, dA = \int_{S} \left[ (\mathbf{n} \cdot \nabla \Phi) f_{s} - \Phi (\mathbf{n} \cdot \nabla f_{s}) \right] \, dA = 0$$  \hfill (3.5)

In order to obtain the velocity potential at the point $P$, this point should be removed from the domain $\mathbb{V}$, otherwise when $R \to 0$, the velocity potential becomes infinity. For this reason, a small sphere $\mathbb{V}_\varepsilon$ with very small radius $\varepsilon$ is considered and the integral surface is stated as:

$$S = S_b + S_c + S_\varepsilon$$  \hfill (3.6)

where $S_c$ is the branch cut which is an infinitesimally thin two-sided surface, and $S_\varepsilon$ is the surface of the sphere $\mathbb{V}_\varepsilon$. Hence, equation (3.5) can be restated as:
\[
\int_{S_b+S_c+S_e} \left[ f_\nu(R) \nabla \Phi - \Phi \nabla f_\nu(R) \right] \cdot n \, dA = 0
\]  
(3.7)

Since the normal vectors for the branch cut \( S_c \) are in opposite direction, their contribution in the integral term can be cancelled, and one can obtain:

\[
\int_{S_b} \left[ f_\nu(R) \nabla \Phi - \Phi \nabla f_\nu(R) \right] \cdot n \, dA = -\int_{S_e} \left[ f_\nu(R) \nabla \Phi - \Phi \nabla f_\nu(R) \right] \cdot n \, dA
\]  
(3.8)

As the radius of the sphere \( V_\varepsilon \) tends to zero \( (R|_{S_\varepsilon} = \varepsilon \to 0) \):

\[
[n \cdot \nabla f_\nu]_{S_\varepsilon} = \left[ n \cdot f_\nu'(R) \frac{\nabla R}{R} \right]_{S_\varepsilon} = f_\nu'(\varepsilon) = \frac{1}{2\pi \varepsilon}
\]  
(3.9)

and:

\[
\int_{S_\varepsilon} dA = A_\varepsilon = 2\pi \varepsilon
\]  
(3.10)

In equation (3.8), the first part of the right hand side term can be expressed as:

\[
-\int_{S_\varepsilon} \left[ f_\nu(R) \nabla \Phi \right] \cdot n \, dA = -f_\nu(\varepsilon) \int_{S_\varepsilon} \nabla \Phi \cdot n \, dA =
\]

\[
-f_\nu(\varepsilon) \int_{V_\varepsilon} \nabla \cdot (\nabla \Phi) \, dV = 0
\]  
(3.11)

It can be assumed that as \( \varepsilon \to 0 \), the velocity potential \( \Phi \) is constant within the volume \( V_\varepsilon \). Therefore, the second part of equation (3.8) is given by:

\[
\int_{S_\varepsilon} \left[ \Phi \nabla f_\nu(R) \right] \cdot n \, dA = \Phi_P f_\nu'(\varepsilon) \int_{S_\varepsilon} dA = \Phi_P = \Phi(x, y, z)
\]  
(3.12)

As a result, the velocity potential of point \( P \) can be shown

\[
\Phi(x, y, z) = \Phi_P = \int_{S_b} f_\nu(R) \nabla \Phi \cdot n \, dA - \int_{S_b} \Phi \nabla f_\nu(R) \cdot n \, dA
\]  
(3.13)
In equation (3.13), the first term in the right hand side represents the single layer potential corresponding to a source distribution over the bounding surface $S_b$, while the second term is the double layer potential corresponding to a doublet distribution over the bounding surface $S_b$.

In general, the velocity potential can be stated as:

$$K \Phi(x, y, z) = \int_{S_b} f_s(R) \nabla \Phi \cdot n_s \, dA - \int_{S_b} \Phi \nabla f_s(R) \cdot n_s \, dA$$

(3.14)

where:

$$K = \begin{cases} 
1 & P \in \mathbb{V} \\
0.5 & P \in S_b \\
0 & P \notin \mathbb{V} 
\end{cases} \quad (3.15)$$

For the case where a solid lifting body is placed in a uniform stream as illustrated in Figure 3.2, the governing equation is Laplace’s equation [equation (3.1)], but the total velocity potential $\Phi$ is the summation of the free-stream velocity potential $\Phi_\infty$ and the perturbation velocity potential $\varphi$, as follows:

$$\Phi = \Phi_\infty + \varphi$$

(3.16)

If the total velocity vector is denoted by $\mathbf{V}$, one can write:

$$\nabla \Phi = \mathbf{V} = \mathbf{V}_\infty + \mathbf{q}$$

(3.17)

where $\mathbf{V}_\infty = \nabla \Phi_\infty$ is the free-stream velocity and $\mathbf{q} = \nabla \varphi$ is the perturbation velocity vector.

When there is a solid body in the fluid domain, the integral surface $S$ consists of the solid body and the wake surfaces as shown in Figure 3.2, and is written as:

$$S = S_B + S_w + S_w^* + S_\infty$$

(3.18)

where $S_B$ is the solid body surface, $S_w, S_w^*$ are two thin surfaces on both sides of the body wake. It is worth mentioning that since the flow inside the wake is rotational, the interior of the body wake is excluded from the fluid domain.
In equation (3.18), $S_\infty$ is a sphere with a very large radius tending to infinity. Hence, equation (3.14) can be expanded as follows (Mateescu 2008b):

$$K\Phi(x, y, z) = \Phi_\infty + \int_{S_B} \left[ (n_B \cdot \nabla \Phi)_f(R) - \Phi(n_B \cdot \nabla f) \right] dA +$$

$$\int_{S_w} \{ [n_w \cdot \nabla (\Phi_w - \Phi_w^*)]f_s(R) - (\Phi_w - \Phi_w^*)[n_w \cdot \nabla f_s(R)] \} dA$$

(3.19)

It should be noted that the normal velocity does not change across the wake. Thus:

$$n_w \cdot \nabla (\Phi_w - \Phi_w^*) = 0$$

(3.20)

where $\Phi_w, \Phi_w^*$ are the velocity potentials on the two side of the wake. Hence, equation (3.19) is restated as:

$$K\Phi(x, y, z) = \Phi_\infty + \int_{S_B} \left[ (n_B \cdot \nabla \Phi)_f(R) - \Phi(n_B \cdot \nabla f) \right] dA -$$

$$\int_{S_w} (\Phi_w - \Phi_w^*)[n_w \cdot \nabla f_s(R)] dA$$

(3.21)

This equation can be extended for two-dimensional airfoils with circulation around the airfoils by considering a camberline surface $S_c$ in the airfoil:
\[ \Phi = \Phi_\infty + \int_{S_B} \left\{ [n_B \cdot \nabla (\Delta \Phi_B)]f_s(R) - \Delta \Phi_B(n_B \cdot \nabla f_s) \right\} dA + \]
\[ \int_{S_C} \left\{ [n_C \cdot \nabla (\Delta \Phi_C)]f_s(R) - \Delta \Phi_C(n_C \cdot \nabla f_s) \right\} dA - \]
\[ \int_{S_w} \Delta \Phi_w[n_w \cdot \nabla f_s(R)] dA \]  

(3.22)

where \( \Delta \Phi_B \) is the difference between the flow potential of the body and the camberline, \( \Delta \Phi_C \) is the difference between the flow potential above and below the camberline, and \( \Delta \Phi_w = \Phi_w - \Phi^*_w \).

By defining the source and doublet strengths \((\sigma, \mu)\) as:

\[ \sigma = n \cdot \nabla (\Delta \Phi) \]  

(3.23)

\[ \mu = \Delta \Phi \]  

(3.24)

one can obtain:

\[ \Phi = \Phi_\infty + \int_{S_B \cup S_C} [\sigma f_s(R) + \mu f_D(R)] dA + \int_{S_w} \mu f_D(R) dA \]  

(3.25)

where \( f_D(R) \) is the potential of a unit strength doublet:

\[ f_D(R) = -n \cdot \nabla f_s(R) = -\frac{1}{2\pi} \frac{n \cdot R}{R^2} \]  

(3.26)

and \( f_s(R) \) is the potential of a unit strength source:

\[ f_s(R) = \frac{1}{2\pi} \ln R \]  

(3.27)

One may note that the fluid velocity at the locations of the source and doublets tends to infinity, and these points are in fact the singularities. Equation (3.25) can be recast by employing equation (3.16) as follows:

\[ \varphi = \int_{S_B} \sigma f_s(R) dA + \int_{S_C \cup S_w} \mu f_D(R) dA \]  

(3.28)
Hence, in order to obtain the velocity potential, the distribution of the singularities (source and doublets) should be determined. It can be observed that the source singularity is used to model the effects of the thickness of the airfoil and the doublet singularity represents the camber of the airfoil. If the thickness of the airfoil is considered negligible, the source distribution is not required. Therefore, equation (3.28) results in:

\[
\varphi = \int_{S_{\text{c}}} \mu f_D(R) \, dA + \int_{S_{\text{w}}} \mu_w f_D(R) \, dA \tag{3.29}
\]

where \(\mu_w\) is the wake doublet strength and can be considered constant. The doublet distribution can also be equivalently replaced by vortex singularity as shown in Figure 3.3. The potential of line \(AB\) can be expressed as (Mateescu 2008b):

\[
\varphi_{AB} = \int_{AB} \mu f_D(R) \, dA = \frac{1}{2\pi} \left[ \mu(s_A)\theta_A(s_A) - \mu(s_B)\theta_B(s_B) + \int_{s_A}^{s_B} \frac{d\mu}{ds} \theta(s) \, ds \right] \tag{3.30}
\]

where \(\theta_A\) and \(\theta_B\) denote the inclination angles at points \(A\) and \(B\), respectively. If \(\Gamma_A\) and \(\Gamma_B\) denote the strengths of the lumped vortices at points \(A\) and \(B\), one can write:

\[
\varphi_{AB} = \frac{1}{2\pi} \left[ \Gamma_A \theta_A(s_A) - \Gamma_B \theta_B(s_B) + \int_{s_A}^{s_B} \gamma(s) \theta(s) \, ds \right] \tag{3.31}
\]

where:

\[
\gamma(s) = \frac{d\mu}{ds} \quad ; \quad \Gamma_A = \mu(s_A) \quad ; \quad \Gamma_B = -\mu(s_B) \tag{3.32}
\]

Note that \(\Gamma_A, \Gamma_B,\) and \(\gamma(s)\) are positive in the trigonometric sense.
In this thesis, the vortex singularities are used in the panel method. In the following sections, first the vortex solution of the potential flow problem is discussed and then the panel method for steady and unsteady two-dimensional flow is described.

### 3.3 Vortex Solution of the Potential Flows

As explained in Chapter 2, Laplace’s equation for two-dimensional flow can be given by:

\[
\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0
\]  

(3.33)
where \( \varphi \) is the perturbation velocity potential. One of the basic solutions of the Laplace’s equation for two-dimensional flow is the point vortex singularity as illustrated in Figure 3.4. Knowing that the vortex singularity has only tangential velocity component, the velocity potential and the velocity components for a point vortex at the origin can be obtained as:

\[
V_r = 0; \quad V_\theta = -\frac{\Gamma}{2\pi r}; \quad \Phi = -\frac{\Gamma}{2\pi} \theta \tag{3.34}
\]

where \( V_r \) is the radial velocity, \( V_\theta \) the tangential velocity, \( \Gamma \) the circulation (positive clockwise) and \( \theta \) is the angular coordinate. In Cartesian coordinates, for a vortex located at \( (x_j, z_j) \) with vortex strength \( \Gamma_j \), the velocity potential \( \varphi \) and the velocity components \( u_{ij}, w_{ij} \) at point \( (x_i, z_i) \) can be stated as follows:

\[
\varphi = -\frac{\Gamma_j}{2\pi} \tan^{-1} \frac{z_i - z_j}{x_i - x_j} \tag{3.35}
\]

\[
u_{ij} = \frac{\Gamma_j}{2\pi} \frac{z_i - z_j}{(x_i - x_j)^2 + (z_i - z_j)^2} \tag{3.36}
\]

\[
w_{ij} = -\frac{\Gamma_j}{2\pi} \frac{x - x_j}{(x_i - x_j)^2 + (z_i - z_j)^2}
\]
3.4 Two-dimensional Panel Method for Steady Flows

As explained briefly in Chapter 2, the unsteady aerodynamic loads on the oscillating plate are calculated for subsonic incompressible flow using a panel method, in which the velocity field is solved by considering the singularities distributed on the surface panels. Based on the choice of singularity, the level of approximation of the singularity distribution, surface geometry, and type of boundary conditions, numerous methods can be constructed.

In this thesis, the Discrete Vortex Method is used to model the 2-D potential flows which is a useful approach and can be expanded to study the three-dimensional steady and unsteady flows. This approach is used to solve the thin lifting airfoil problem based on the lumped vortex element. In this approach, the boundary conditions can be defined on the airfoil camber surface without a need for the small perturbation approximation, which is one of the advantages of this method (Katz and Plotkin 2001).

The problem is to calculate the generalized aerodynamic load on the plate oscillating in two-dimensional subsonic incompressible flow. It is assumed that the viscosity effects are negligible; hence the Laplace equation applies which has been introduced in equation (3.33). As shown in the previous section, the vortex singularity is one of the basic and useful solutions of the Laplace equation,

![Figure 3.5 Discrete vortex representation of a thin airfoil model (Katz and Plotkin 2001)](image_url)
where the velocity field using this singularity can be obtained using equation (3.36). This equation can be used in the Discrete Vortex Method to define the velocity field for each panel.

In the Discrete Vortex Method, the surface of the solid body is discretized into $N$ panels as shown in Figure 3.5 where each panel is represented by only one vortex. Accordingly, the boundary condition can be specified at only one point, too, which is called the collocation point. As we know from the lifting problem, since the lift of the flat plate acts at the center of pressure which lies at the one-quarter of the chord, the lumped vortex is placed at the quarter-chord of each panel. Obviously, when the lifting flat plate airfoil is represented by only one concentrated vortex, $\Gamma$, the vertical velocity induced by this vortex varies along the plate and hence the boundary condition requiring zero normal velocity at the plate surface can be specified at only one point called collocation point. It can be shown that the collocation point is located at the three-quarter of the chord (Mateescu 2008a).

Each panel in the Discrete Vortex Method includes a point vortex with an unknown strength of $\Gamma_j (j = 1, ..., N)$. For each panel, if the vortex is located at $(x_j, z_j)$, the induced velocity at the point $(x_i, z_i)$ due to this vortex can be obtained using equation (3.36), which is shown in the matrix form as follows:

$$\begin{bmatrix}
u_{ij} \\
w_{ij} \
\end{bmatrix} = \frac{\Gamma_j}{2\pi r_j^2} \begin{bmatrix}
0 & 1 \\
-1 & 0 \
\end{bmatrix} \begin{bmatrix} x_i - x_j \\
z_i - z_j \
\end{bmatrix}$$

$$(3.37)$$

where $r_j^2 = (x_i - x_j)^2 + (z_i - z_j)^2$. Here $\nu_{ij}$ and $w_{ij}$ are the induced velocity components at the point $(x_i, z_i)$ due to the vortex $\Gamma_j$ in the $x$ and $z$ directions, respectively. Since the values of $\Gamma_j$ are unknown at this point, unit velocity components $\nu_{ij}^*, w_{ij}^*$ can be defined by considering $\Gamma_j = 1$ in equation (3.37) as shown below:
\[
\begin{bmatrix}
u_{ij}^* \\
w_{ij}^*
\end{bmatrix} = \frac{1}{2\pi r^2} \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
x_i - x_j \\
z_i - z_j
\end{bmatrix}
\]  
(3.38)

Hence:
\[
u_{ij} = \Gamma_j u_{ij}^* ; \quad w_{ij} = \Gamma_j w_{ij}^*
\]  
(3.39)

The boundary condition for steady flow is the zero normal flow on the airfoil surface which can be shown as:
\[
V \cdot n = 0
\]  
(3.40)

where \(V = \nabla \Phi\) is the fluid velocity vector in the body-fixed coordinate system and \(n = n_x \hat{i} + n_z \hat{k}\) is the unit vector normal to the surface. Here \(\hat{i}\) and \(\hat{k}\) are the unit vectors in a 2-D Cartesian coordinates system and are along \(x\) and \(z\) directions, respectively.

It should be noted that for steady flow, only one coordinate system is required which is the body-fixed frame attached to the airfoil as illustrated in Figure 3.5. It is worth mentioning that here the 2-D airfoil is described in the \(x, z\) axes instead of the common \(x, y\) axes, in order to be consistent with the 3-D system representation used in Chapter 4, where the spanwise axis \(y\) will be added to the system to describe the 3-D wing.

The fluid velocity vector \(V\) consists of two components of free-stream velocity \(V_\infty\) and the perturbation velocity \(q\) as shown below:
\[
V = V_\infty + q
\]  
(3.41)

The perturbation velocity \(q\) is the induced velocity due to the vortex singularities of all the panels. For unsteady flow, the perturbation velocity includes the induced velocity due to both the body and the wake vortices \((q = q_b + q_w)\); however, since the starting vortex in the steady flow moves to the far-field, its strength does not affect the velocity distribution on the airfoil and can be neglected. Hence, in
steady flow, the perturbation velocity is only due to the body panels: \( q = q_b \) and equation (3.41) can be written as:

\[
V = V_\infty + q_b
\]  
(3.42)

where:

\[
V_\infty = U_\infty \hat{i} + W_\infty \hat{k}
\]  
(3.43)

\[
q_b = u \hat{i} + w \hat{k}
\]  
(3.44)

Consequently, the boundary condition in equation (3.40) can be expressed as:

\[
V_\infty \cdot n + q_b \cdot n = 0
\]  
(3.45)

The first part of this equation \( V_\infty \cdot n \) is the normal velocity induced by the free-stream which is known and can be transferred to the right-hand side of the equation. The second part \( q_b \cdot n \) is the normal perturbation velocity induced by the body vortices which contains the unknown terms of vortex strengths.

The perturbation velocity at panel \( i \) due to all the body vortices can be denoted by \( (q_b)_i \) which includes the contribution of each vortex \( j \) on the perturbation velocity of panel \( i \) as shown below:

\[
(q_b)_i = \sum_{j=1}^{N} (u_{ij} \hat{i} + w_{ij} \hat{k})
\]  
(3.46)

where \( u_{ij}, w_{ij} \) were defined in equation (3.37). By substituting equation (3.39), one can obtains:

\[
(q_b)_i = \sum_{j=1}^{N} \Gamma_j (u_{ij}^* \hat{i} + w_{ij}^* \hat{k})
\]  
(3.47)

Therefore, the normal perturbation velocity on the panel \( i \) can be expressed as:
Here, the values of $\Gamma_j$ are the only unknown terms and the rest of the terms which are known can be represented by a coefficient called the influence coefficient $C_{ij}$ which represents the normal perturbation velocity due to a unit strength vortex ($\Gamma_j = 1$) on the panel $i$. The influence coefficient can be defined as the perturbation velocity component normal to the surface of panel $i$, due to a unit strength vortex on the panel $j$ as follows:

$$C_{ij} = (u_{ij}^* \hat{t} + w_{ij}^* \hat{k}) \cdot (n_x \hat{t} + n_z \hat{k})_i$$  \hspace{1cm} (3.49)

where $u_{ij}^*, w_{ij}^*$ is obtained using equation (3.38). Hence, equation (3.48) can be stated as:

$$(q_b \cdot n)_i = C_{i1} \Gamma_1 + C_{i2} \Gamma_2 + C_{i3} \Gamma_3 + \cdots + C_{iN} \Gamma_N \hspace{1cm} (3.50)$$

As mentioned before, at this point, the strengths of $\Gamma_j$ are unknown; however, the influence coefficients $C_{ij}$ are known. The boundary condition requires that at each collocation point the normal velocity component goes to zero. Specifying this condition for collocation point $i$, as stated in equation (3.45), yields:

$$(q_b \cdot n)_i = -(V_\infty \cdot n)_i \hspace{1cm} (3.51)$$

or by replacing equation (3.50):

$$C_{i1} \Gamma_1 + C_{i2} \Gamma_2 + C_{i3} \Gamma_3 + \cdots + C_{iN} \Gamma_N = B_i \hspace{1cm} (3.52)$$

where:

$$B_i = -(V_\infty \cdot n)_i = -(U_\infty \hat{t} + W_\infty \hat{k}) \cdot (n_x \hat{t} + n_z \hat{k})_i \hspace{1cm} (3.53)$$
which is known. Here index \( i \) assures that the normal vector can be different for each panel. By writing equation (3.52) for all the \( N \) collocation points, a set of algebraic equations is obtained as shown below:

\[
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1N} \\
C_{21} & C_{22} & \cdots & C_{2N} \\
C_{31} & C_{32} & \cdots & C_{3N} \\
\vdots & \vdots & \ddots & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NN}
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\vdots \\
\Gamma_N
\end{bmatrix}
=
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\vdots \\
B_N
\end{bmatrix}.
\]

(3.54)

This algebraic equation is solved to acquire the vortex strengths of the surface panels. To obtain the aerodynamic pressure and loads on the plate, the Kutta-Joukowski theorem can be employed for each panel \( j \):

\[
\Delta L_j = \rho V_{\infty} \Gamma_j
\]

(3.55)

\[
\Delta p_j = \rho V_{\infty} \Gamma_j / \Delta l ; \quad \Delta l = c/N
\]

(3.56)

where \( \rho \) is the air density, \( c \) is the plate chord, and \( V_{\infty} \) is the magnitude of the free-stream velocity. The non-dimensional coefficient of pressure difference for each panel is obtained as shown below:

\[
\Delta C_{p,j} = \frac{\Delta p_j}{\frac{1}{2} \rho V_{\infty}^2} = \frac{2\Gamma_j}{\rho V_{\infty} \Delta l}
\]

(3.57)

The total lift and moment per unit span is obtained as the summation of \( \Delta L_j \) from equation (3.55):

\[
L = \sum_{j=1}^{N} \Delta L_j = \rho V_{\infty} \sum_{j=1}^{N} \Gamma_j
\]

(3.58)

\[
M_0 = \sum_{j=1}^{N} \Delta L_j x_j \cos \alpha = \rho V_{\infty} \cos \alpha \sum_{j=1}^{N} \Gamma_j x_j
\]

(3.59)
The moment is about the leading edge. The non-dimensional lift and moment coefficients can also be found as follows:

\[ C_L = \frac{L}{\frac{1}{2} \rho V_\infty^2 c} = \frac{2}{V_\infty c} \sum_{j=1}^{N} \Gamma_j \]  

(3.60)

\[ C_{m_0} = \frac{M}{\frac{1}{2} \rho V_\infty^2 c^2} = \frac{2 \cos \alpha}{V_\infty c^2} \sum_{j=1}^{N} \Gamma_j x_j \]  

(3.61)

### 3.5 Two-dimensional Panel Method for Unsteady Flows

As explained in the previous section, in order to study the steady flow using the panel method, the surface of the airfoil is divided into finite number of panels with vortex singularities. The solution technique used for the steady flows can be extended to treat the unsteady flows, by applying the modifications regarding the effects of time-dependant terms.

In steady flow, the starting vortex moves to the far-field and its strength does not affect the pressure distribution on the airfoil, but in unsteady flow, at each instant, a new vortex is created at the trailing edge, which creates a downwash on the airfoil panels. Therefore, the surface of the wake should be also discretized into panels with vortex singularities. Figure 3.6 shows the body and wake panel vortices for the two-dimensional unsteady panel method.

Figure 3.6 Panel method representation of the unsteady potential flow around a 2-D airfoil
To study the unsteady flow, the solution method is based on the time-stepping method. At each time step, the unknown vortex strengths are calculated by solving the set of equation obtained by applying the known boundary condition on the airfoil. This set of equation is the modified form of equation (3.54) which is explained later. This procedure will consequently result in defining the entire flow field.

3.5.1 Boundary Conditions

As indicated initially in the previous chapter, in order to describe the motion of the body in the unsteady flow, two inertial and body-fixed coordinate systems can be introduced. Figure 3.7 illustrates these two coordinate systems for the 2-D airfoil flying in the shown flight path.

The unsteady fluid flow around the two-dimensional airfoil is assumed to be inviscid, irrotational, and incompressible over the entire flow field, therefore the Laplace equation holds [equation (3.33)]. As shown in equation (2.11), the boundary condition in the body-fixed frame can be stated as:

\[ V \cdot n = V_b \cdot n \]  

(3.62)

Figure 3.7 Body-fixed and inertial coordinates in the unsteady motion of a 2-D airfoil
CONTROL OF AEROELASTIC OSCILLATIONS OF WING STRUCTURES USING BONDED PIEZOELECTRIC STRIPS

where \( \mathbf{V} = \nabla \Phi \) is the fluid velocity and \( \mathbf{n} = n_x \hat{i} + n_z \hat{k} \) is the unit vector normal to the moving surface in 2-D flow. Here \( \mathbf{V}_b \) is the velocity of the airfoil in the body-fixed frame defined as:

\[
\mathbf{V}_b = \omega \times \mathbf{r} + \mathbf{v}_{rel}
\]  

(3.63)

which can be shown in simpler form in terms of its components as:

\[
\mathbf{V}_b = U_b \hat{i} + W_b \hat{k}
\]  

(3.64)

where \( U_b, W_b \) are the time-dependant components of surface velocity along \( x \) and \( z \) axes, respectively. Henceforth, this more concise form will be used in the formulation of the 2-D unsteady aerodynamic model.

As mentioned earlier, in unsteady flows, at each time step, a new wake vortex is created at the trailing edge, which then moves downstream with the local fluid velocity, and the influence of this wake vortex on the velocity distribution of the body panels should be considered. Hence, in unsteady flow, the perturbation velocity vector \( \mathbf{q} \) in equation (3.41) consists of the perturbation velocity induced by both the body and the wake panels: \( \mathbf{q} = \mathbf{q}_b + \mathbf{q}_w \), and the fluid velocity vector can be shown as below:

\[
\mathbf{V} = \mathbf{V}_\infty + \mathbf{q}_b + \mathbf{q}_w
\]  

(3.65)

where:

\[
\mathbf{V}_\infty = U_\infty \hat{i} + W_\infty \hat{k}
\]  

(3.66)

\[
\mathbf{q}_b = u \hat{i} + w \hat{k}
\]  

(3.67)

\[
\mathbf{q}_w = u_w \hat{i} + w_w \hat{k}
\]  

(3.68)

Therefore, the boundary condition of equation (3.62) can result in:

\[
\mathbf{q}_b \cdot \mathbf{n} + \mathbf{q}_w \cdot \mathbf{n} = (\omega \times \mathbf{r} + \mathbf{v}_{rel}) \cdot \mathbf{n} - \mathbf{V}_\infty \cdot \mathbf{n}
\]  

(3.69)

In this equation, the first term \( \mathbf{q}_b \cdot \mathbf{n} \) is the induced normal velocity component due to the body panels. The contribution of the normal perturbation
velocity on the panel \( i \) can be represented by a linear combination of influence coefficients and singularity strengths on the panel which is similar to equation (3.50) in steady flow. For instance, the induced normal velocity component at \( i \)th collocation point, due to all \( N \) body vortices \( \Gamma_j \) can be written in the following form:

\[
(q_b \cdot n)_i = \sum_{j=1}^{N} C_{ij} \Gamma_j = C_{i1} \Gamma_1 + C_{i2} \Gamma_2 + \ldots + C_{iN} \Gamma_N
\]  

(3.70)

where \( C_{ij} \) is the influence coefficient and is defined in equation (3.49). One may note that equation (3.70) should be updated at each time step based on the modifications in the flow or geometry specification.

At each time step, a new wake vortex is created at the trailing edge with an unknown vortex strength \( \Gamma_{wt} \). For all the other wake vortices, we assume that the values of vortex strengths are known from previous time steps. The induced normal velocity component at \( i \)th collocation point, due to all wake panels can be written as:

\[
(q_w \cdot n)_i = (u_{w0} \hat{i} + w_{w0} \hat{k})_i \cdot (n_x \hat{i} + n_z \hat{k})_i + C_{iw_{wt}} \Gamma_{wt}
\]  

(3.71)

where \( C_{iw_{wt}} \) is the influence coefficient of the latest wake and \( u_{w0} \) and \( w_{w0} \) are the velocity components induced by all the previous wake vortices. The strengths of the wake vortices except the latest one in equation (3.71) are known at each time step, which can go to the right hand side of equation (3.69) along with other known terms related to the flight path of the airfoil. Eventually, the contribution of the equation (3.69) for panel \( i \) can be expressed as below:

\[
(q_b \cdot n)_i + (q_w \cdot n)_i = [(\omega \times r + v_{rel}) \cdot n]_i - [V_\infty \cdot n]_i
\]  

(3.72)

and by employing equations (3.64), (3.66), (3.70) and (3.71) and transferring all the known terms to the right hand side of the equation, one can obtain:
\[
\sum_{j=1}^{N} C_{ij} \Gamma_j + C_{i\infty} \Gamma_{\infty} = B_i
\]

or in the more concise form as following:

\[
\sum_{j=1}^{N} C_{ij} + C_{i\infty} = B_i
\]  \hspace{1cm} (3.74)

where:

\[
B_i = \left( (U_b \hat{\Gamma} + W_b \tilde{\Gamma}) - (U_\infty \hat{\gamma} + W_\infty \tilde{\gamma}) - (u_{w0} \hat{\gamma} + w_{w0} \tilde{\gamma}) \right) \cdot \mathbf{n}_i
\]  \hspace{1cm} (3.75)

Finally, equation (3.74) for all the body panels \((i = 1..N)\) can be expressed in the matrix form to gain the following set of equations:

\[
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1N} & C_{1\infty} \\
C_{21} & C_{22} & \cdots & C_{2N} & C_{2\infty} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NN} & C_{N\infty}
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_N \\
\Gamma_{\infty}
\end{bmatrix}
= 
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_N
\end{bmatrix}
\]  \hspace{1cm} (3.76)

This set of equations has \(N\) equation but \(N + 1\) unknown, and in order to be solved explicitly, an extra equation is needed. This additional equation can be obtained by employing the Thomson’s circulation theorem around the solid body. This theorem states that the circulation \(\Gamma_c\) around a closed material contour is constant. It denotes that at each instant the total circulation around the airfoil and its wake is constant which is illustrated in Figure 3.8. This theorem is described in mathematical form as shown below:

\[
\frac{d\Gamma_c}{dt} = \frac{d\Gamma_{\infty}}{dt} + \frac{d\Gamma_{\infty}}{dt} = 0
\]  \hspace{1cm} (3.77)
where $\Gamma$ and $\Gamma_w$ are the circulation of the airfoil and of the wake, respectively. So, for each time step, the following equation holds:

$$\Gamma(t) - \Gamma(t - \Delta t) + \Gamma_{w_t} = 0$$  \hspace{1cm} (3.78)

or:

$$\Gamma(t) + \Gamma_{w_t} = \Gamma(t - \Delta t)$$  \hspace{1cm} (3.79)

The instantaneous airfoil circulation $\Gamma(t)$ is the sum of the airfoil’s vortices:

$$\Gamma(t) = \sum_{j=1}^{N} \Gamma_j$$  \hspace{1cm} (3.80)

By adding equation (3.79) to (3.76), it can result in the following form:

$$
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1N} & C_{1W} \\
C_{21} & C_{22} & \cdots & C_{2N} & C_{2W} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{N1} & C_{N2} & \cdots & C_{NN} & C_{NW}
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_N
\end{bmatrix}
= 
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_N
\end{bmatrix}

= 
\begin{bmatrix}
\Gamma(t - \Delta t)
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
= 
\begin{bmatrix}
\Gamma_w(t)
\end{bmatrix}
$$  \hspace{1cm} (3.81)
3.5.2 Computation of the Pressure Distribution and Loads

At each time step, the vortex strengths of the surface panels $\Gamma_j$ ($j = 1, \ldots, N$) and the latest wake $\Gamma_{\text{w}}$ can be acquired by solving equation (3.81). Once the flow field is determined and the vortex strengths are obtained, Bernoulli equation can be used near the panel surface to calculate the pressure distribution. The Bernoulli equation for unsteady flow can be shown as below:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} V^2 + \frac{p}{\rho_\infty} = \frac{1}{2} \frac{V_\infty^2}{\rho_\infty} + \frac{p_\infty}{\rho_\infty}$$  \hspace{1cm} (3.82)

where $p_\infty$ and $\rho_\infty$ are the far-field reference pressure and density and $p$ is the local fluid pressure. Here $V_\infty$ and $V$ are the magnitude of the free-stream velocity and the fluid velocity vectors, respectively. As indicated in equation (3.65), the fluid velocity $V$ can be written as:

$$V = V_\infty + q$$  \hspace{1cm} (3.83)

where $q = q_b + q_w$ is the total perturbation velocity. Hence:

$$V^2 = V_\infty^2 + q^2 + 2(V_\infty \cdot q)$$  \hspace{1cm} (3.84)

Substituting this equation in equation (3.82) yields:

$$p - p_\infty = -\rho_\infty \left( \frac{\partial \varphi}{\partial t} + V_\infty \cdot q + \frac{1}{2} q^2 \right)$$  \hspace{1cm} (3.85)

Since based on the small perturbation assumption, the perturbation velocity $q$ has a small value, thus its squared value $q^2$ can be ignored in this equation ($q^2 \ll 1$):

$$p - p_\infty = -\rho_\infty \left( \frac{\partial \varphi}{\partial t} + V_\infty \cdot q \right)$$  \hspace{1cm} (3.86)

This equation can be used to obtain the pressure difference between the upper and lower surfaces of the airfoil:
\[ \Delta p = p_u - p_l = -\rho_\infty \left[ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_l + (V_\infty \cdot q)_u - (V_\infty \cdot q)_l \right] \] (3.87)

where \( \Delta p \) is the pressure difference and the subscript \( u \) and \( l \) denotes the upper and lower surface, respectively. This equation should be written for each panel, for instance, the pressure difference on the \( j \)th panel is shown below:

\[ \Delta p_j = -\rho_\infty \left\{ \left[ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_l \right]_j + [(V_\infty \cdot q)_u - (V_\infty \cdot q)_l]_j \right\} \] (3.88)

In order to determine the value of \( V_\infty \cdot q \), it is found more convenient to write the vectors in terms of their components in the normal-tangential \((n, \tau)\) coordinate system. If \( n, \tau \) are the normal and tangential unit vectors, by defining:

\[ q = \nabla \varphi = \frac{\partial \varphi}{\partial n} n + \frac{\partial \varphi}{\partial \tau} \tau \] (3.89)

\[ V_\infty = (V_\infty \cdot n) n + (V_\infty \cdot \tau) \tau \] (3.90)

one can obtain:

\[ V_\infty \cdot q = (V_\infty \cdot n) \frac{\partial \varphi}{\partial n} + (V_\infty \cdot \tau) \frac{\partial \varphi}{\partial \tau} \] (3.91)

where \( \nabla \varphi \) is the perturbation velocity potential. Since the velocity terms normal to the surface are identical on the upper and lower surfaces, they can be removed from equation (3.91) and only the tangential terms remains in the equation (3.88) as indicated below:

\[ (V_\infty \cdot q)_u - (V_\infty \cdot q)_l = (V_\infty \cdot \tau) \left[ \left( \frac{\partial \varphi}{\partial \tau} \right)_u - \left( \frac{\partial \varphi}{\partial \tau} \right)_l \right] = (V_\infty \cdot \tau) \left[ \frac{\partial}{\partial \tau} \Delta \varphi \right] \] (3.92)

which for panel \( j \) is given by:

\[ [(V_\infty \cdot q)_u - (V_\infty \cdot q)_l]_j = (V_\infty \cdot \tau_j) \left[ \frac{\partial}{\partial \tau_j} \Delta \varphi \right] \] (3.93)
Figure 3.9 Definition of the panel surface vectors in the Discrete Vortex Method (Katz and Plotkin 2001)

where \( \boldsymbol{\tau}_j \) is the tangential surface vector of panel \( j \) and \( \Delta \varphi \) is the difference of the velocity potential between the upper and lower surfaces. Figure 3.9 indicates the definition of the normal and tangential vectors of panels in the Discrete Vortex Method. The relation between the surface vectors \( \boldsymbol{n}_j, \boldsymbol{\tau}_j \) and the incidence angle of the \( j \)th panel \( \beta_j \) can be specified as follows:

\[
\begin{align*}
\boldsymbol{n}_j &= \sin \beta_j \hat{i} + \cos \beta_j \hat{k} \\
\boldsymbol{\tau}_j &= \cos \beta_j \hat{i} - \sin \beta_j \hat{k}
\end{align*}
\]  
(3.94)

To obtain \( \Delta \varphi \) in equation (3.93), the definition of the circulation can be used as below:

\[
\Gamma = \oint \mathbf{V} \cdot d\mathbf{s} = \oint \nabla \Phi \cdot d\mathbf{s} = \oint \frac{d\Phi}{ds} ds
\]  
(3.95)

where the integration is performed around a closed contour enclosing the airfoil, it leads to:

\[
\Gamma = \Phi_u - \Phi_l = \varphi_u - \varphi_l = \Delta \varphi
\]  
(3.96)

Consequently, the tangential derivative of the velocity potential can be obtained as shown below:

\[
\frac{\partial}{\partial \tau_j} \Delta \varphi = \frac{\Gamma_j}{\Delta l_j}
\]  
(3.97)
where $\Delta l_j$ is the length of the $j$th panel. Thus, equation (3.93) is restated as:

$$
[(V_{\infty} \cdot q)_u - (V_{\infty} \cdot q)_l]_j = [U_\infty \hat{i} + W_\infty \hat{k}]_j \cdot \tau_j \frac{\Gamma_j}{\Delta l_j}
$$  \hspace{1cm} (3.98)

This equation determines the second part of equation (3.88). In order to calculate the first term of equation (3.88), the time derivative of the velocity potential $\frac{\partial \varphi}{\partial t}$ should be obtained using equation (3.96):

$$
\left[ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_l \right]_j = \frac{\partial}{\partial t} \Delta \varphi_j = \frac{\partial}{\partial t} \sum_{k=1}^{j} \Gamma_k
$$  \hspace{1cm} (3.99)

It should be noted, in this case, that the local potential difference $\Delta \varphi_j$ is the sum of the vortex strengths from the leading edge to the $j$th panel. Finally, the aerodynamic pressure difference on each panel is obtained as follows:

$$
\Delta p_j = -\rho_\infty \left[ \frac{\partial}{\partial t} \sum_{k=1}^{j} \Gamma_k + \left[ U_\infty \hat{i} + W_\infty \hat{k} \right]_j \cdot \tau_j \frac{\Gamma_j}{\Delta l_j} \right]
$$  \hspace{1cm} (3.100)

Summation of the pressure difference along the chord line can generate the total lift and moment on the airfoil:

$$
L = \sum_{i=1}^{N} \Delta p_j \Delta l_j \cos \beta_j
$$  \hspace{1cm} (3.101)

$$
M_0 = \sum_{i=1}^{N} \Delta p_j \Delta l_j x_j \cos \beta_j
$$  \hspace{1cm} (3.102)

It should be noted that since the wake vortex carries no load, it moves with the local fluid velocity. Hence, at each time step, the induced velocity on the wake should be computed and the new force-free location of the wake should be obtained. So, the wake vortex motion can be obtained using the following simple equations:

$$
\Delta x_i = u_i \Delta t \hspace{1cm} \Delta z_i = w_i \Delta t
$$  \hspace{1cm} (3.103)
where $\Delta x_i$ and $\Delta z_i$ are the translation of the $i$th wake vortex and $u_i$ and $w_i$ are the velocity components induced at the $i$th wake vortex due to the panel vortices and other wake vortices.

3.6 Model Validation

In order to verify the two-dimensional aerodynamic model presented in the previous sections, a program in FORTRAN is generated. The code solves the associated equations to obtain the steady and unsteady pressure distributions and the generalized aerodynamic forces. To validate the aerodynamic model, the results of the present method are compared with similar results of the published literature. The comparison is firstly performed for the steady flow and then for the unsteady flow. The results show that the present method is in good agreement with the published results.

3.6.1 Steady Aerodynamic Results

To validate the two-dimensional aerodynamic model for steady flows, a 2-D flat plate in a steady flow is considered at a 5 (deg) angle of attack relative to the air flow. The results in the form of the variation of the pressure difference coefficient [defined in equation (3.57)] along the chordwise direction is shown in Figure 3.10. The results are compared with the thin airfoil theory (Mateescu 2008a) and show very good agreement.
3.6.2 Unsteady Aerodynamic Results

In order to validate the unsteady effects of the aerodynamic model, the results of a 2-D unsteady panel method is compared with similar results of the published literature. In this case, the rigid pitching oscillations is considered and the results of the pressure distribution and generalized loading are compared with Mateescu and Neculita (2006), Mateescu (2011) as well as with the Theodorsen method.

A thin airfoil is considered and assumed to have a translational velocity with constant speed while executing pitching oscillations. The translational velocity of the airfoil in inertial frame is assumed being along $x$ axis in the negative direction $V_0 = -U_\infty t$ where $U_\infty = 100$ (m/s). The pitching oscillations are assumed to be caused by changes of the flow angle of attack $\alpha(t)$ due to the gusts in the general cosine form as:

$$\alpha(t) = \alpha_0 + \alpha_A \cos 2\pi f t$$  \hspace{1cm} (3.104)
where for the present case, values of $\alpha_0 = 0$ and $\alpha_A = 5 \, (deg)$ is considered. The frequency $f$ can be related to the non-dimensional reduced frequency as follows:

$$ k = \frac{2\pi fc}{2U_\infty} = \frac{\pi fc}{U_\infty} $$

(3.105)

The period of oscillation can be obtained from frequency and time step can be presumed as shown below:

$$ T = \frac{1}{f} \Rightarrow \Delta t = \frac{T}{80} $$

(3.106)

The transformation between the inertial and body coordinate systems presented in equations (2.1) and (2.3) can be written for the 2-D motion as:

$$ \begin{bmatrix} x \\ z \end{bmatrix} = T_y(t) \begin{bmatrix} X - X_0 \\ Z - Z_0 \end{bmatrix} = \begin{bmatrix} \cos \alpha(t) & -\sin \alpha(t) \\ \sin \alpha(t) & \cos \alpha(t) \end{bmatrix} \begin{bmatrix} X - X_0 \\ Z - Z_0 \end{bmatrix} $$

(3.107)

For two-dimensional pitching oscillation when the plate has one degree of rotation with angle of attack $\alpha(t)$ about $y$ axis, the angular velocity of the body frame $\omega$ can be defined as: $\omega = \dot{\alpha} \hat{j}$. Hence, having the position vector as $r = x\hat{i} + z\hat{k}$, one can obtain:

$$ \omega \times r = \dot{\alpha} z\hat{i} - \dot{\alpha} x\hat{k} $$

(3.108)

To compare the results with the present literature, the rigid oscillation of the airfoil is considered. Hence, the relative airfoil velocity $v_{rel} = 0$ and the surface velocity can be found as:

$$ V_b = \begin{bmatrix} U_b \\ W_b \end{bmatrix} = \begin{bmatrix} \cos \alpha(t) & -\sin \alpha(t) \\ \sin \alpha(t) & \cos \alpha(t) \end{bmatrix} \begin{bmatrix} U_0 \\ W_0 \end{bmatrix} - \begin{bmatrix} \dot{\alpha} z \\ -\dot{\alpha} x \end{bmatrix} $$

(3.109)

It is also assumed that the distance between the position of the first shedding vortex and the airfoil trailing edge is stated as:

$$ \Delta x_w = 0.3U_\infty \Delta t $$

(3.110)

which is about one third of the distance that airfoil travels in one time step.
The results of the present method for 2-D unsteady flow have been compared with results of Mateescu and Neculita (2006) and Mateescu (2011) for reduced frequency $k = 0.05$ and incompressible flow ($M = 0$). Figure 3.11 presents the variation of the real part of the reduced pressure difference coefficient $\frac{\Delta C_p}{\alpha_A}$ along the chordwise direction. Since the oscillations of the present method are in the cosine form, the obtained result is compared with the real part of $\frac{\Delta C_p}{\alpha_A}$ in the published paper which is in the form of $\alpha(t) = \alpha_A \exp(i\omega t)$. It can be observed that the results have a good accuracy with the published results.

Figure 3.12 shows the variation of the unsteady lift and moment coefficients during oscillatory cycle for the same case of the pitching oscillations and are compared with similar results from Mateescu (2011) and the Theodorsen method. The presented results show a good agreement with both methods which indicates that the unsteady model works very well.

![Figure 3.11 Chordwise distribution of the real part of the reduced pressure difference coefficient on a thin plate executing 2-D unsteady oscillations](image_url)
Figure 3.12 Variation of unsteady lift and moment coefficients during one cycle of the pitching oscillation of a 2-D thin plate

Figure 3.13 Unsteady variation of the lift and moment coefficients with time
The unsteady lift and moment coefficients have oscillatory variation with time which is shown in Figure 3.13. It should be noted that the lift coefficient curve shows a transient response in the first cycle of oscillations which is due to the wake build-up. After one cycle, the magnitude of the oscillations reaches its steady-state value.
CHAPTER 4

AERODYNAMIC MODEL FOR STEADY AND UNSTEADY 3-D FLOWS

4.1 Introduction

In this chapter, the details of the aerodynamic modeling for three-dimensional unsteady flows are discussed. Firstly, the vortex line solution of the 3-D potential flows and the vortex ring elements are introduced and then the details of the aerodynamic model for 3-D flows for the steady case are presented. Then, the required modifications are applied to include the effects of time-depending terms and the unsteady aerodynamic model is obtained. Finally, the aerodynamic models developed for both steady and unsteady cases are validated by comparing the results with the existing literature.
As shown in Chapter 3, the general solution of the Laplace’s equation for two-dimensional flow can be obtained by source and doublet distributions where doublet singularities can be equivalently replaced by vortex elements. Similar to 2-D flows, here the vortex line elements for 3-D flows are considered. Figure 4.1 shows that a quadrilateral doublet element can be considered equivalent to a vortex ring.

In this thesis, for the 3-D panel method, the vortex ring elements are used. In the following sections, first the vortex ring solution of the 3-D potential flow problem is explained and the velocity field is obtained. Then the 3-D panel method for steady and unsteady flows is discussed.

**4.2 Vortex Line Solution of the 3-D Potential Flows**

For three-dimensional flow, Laplace’s equation can be stated as shown below:

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0
\]

(4.1)

where \( \Phi \) is the fluid potential velocity. Similar to 2-D flows, \( \Phi \) is the summation of the free-stream velocity potential \( \Phi_\infty \) and the perturbation velocity potential \( \varphi \), as follows:
\[ \Phi = \Phi_\infty + \varphi \]  

Laplace’s equation for 3-D flows can be written in terms of the perturbation velocity potential \( \varphi \) as follows:

\[ \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \]  

which can be solved to obtain the velocity potential field and eventually the velocity field. One of the useful solutions to the 3-D Laplace’s equation is that of a vortex line where the induced velocity can be obtained using the Biot-Savart law. We can consider a small straight vortex segment \( dl \) with a constant circulation \( \Gamma \) as shown in Figure 4.2 and assume that \( r \) is the vector connecting this vortex segment and an arbitrary point \( K \). Then, the differential velocity \( dV \) at point \( K \) induced by the vortex element can be expressed as follows:

\[ dV = \frac{1}{4\pi} \frac{r \times \Gamma dl}{r^3} \]  

where the sign \( \times \) denotes the vector cross product. This equation can appear in the scalar form as:

\[ dV = |dV| = \frac{\Gamma}{4\pi} \frac{r \sin \theta \, dl}{r^3} = \frac{\Gamma}{4\pi r^2} \, dl \sin \theta \]  

where \( dV \) is the magnitude of the vector \( dV \). From Figure 4.2, the following relations can be found:

![Figure 4.2 Induced velocity due to a straight vortex element](image-url)
AERODYNAMIC MODEL FOR STEADY AND UNSTEADY 3-D FLOWS

\[ h_0 = r \sin \theta \quad \Rightarrow \quad \frac{1}{r} = \frac{\sin \theta}{h_0} \quad (4.6) \]

\[ -l = h_0 \cotan \theta \quad \Rightarrow \quad dl = \frac{h_0 \, d\theta}{\sin^2 \theta} \quad (4.7) \]

which can be replaced in equation (4.5) to get:

\[ dV = \frac{\Gamma}{4\pi} \frac{\sin^2 \theta \cdot h_0 \, d\theta}{h_0^2} \frac{\sin \theta}{\sin^2 \theta} = \frac{\Gamma}{4\pi h_0} \sin \theta \, d\theta \quad (4.8) \]

The vortex line which is located between points 1 and 2 (shown in Figure 4.2) can be referred as the vortex line element 12. The velocity induced by this straight vortex segment can be obtained by integrating equation (4.8) over the entire element, which means between points 1 and 2, as stated below:

\[ V_{12} = \int_1^2 dV = \int_{\theta_1}^{\theta_2} \frac{\Gamma}{4\pi h_0} \sin \theta \, d\theta \quad (4.9) \]

and consequently as follows:

\[ V_{12} = \frac{\Gamma}{4\pi h_0} (\cos \theta_1 - \cos \theta_2) \quad (4.10) \]

where the normal distance \( h_0 \) and angles \( \theta_1 \) and \( \theta_2 \) are defined in Figure 4.3.

Figure 4.3 Nomenclature of a 3-D vortex line element
For a general three-dimensional case, equation (4.10) can be modified to a more convenient form using the position vectors of the edges of the vortex element as illustrated in Figure 4.3. If two edges of the vortex element are positioned by \( r_1 \) and \( r_2 \) and \( r_0 = r_1 - r_2 \) is the vector connecting the edges, other terms in equation (4.10) can be defined as:

\[
h_0 = \frac{|r_1 \times r_2|}{r_0} \quad ; \quad \cos \theta_1 = \frac{r_0 \cdot r_1}{r_0 r_1} \quad ; \quad \cos \theta_2 = \frac{r_0 \cdot r_2}{r_0 r_2}
\]

(4.11)

where \( r_0 = |r_0| \) is the magnitude of the \( r_0 \) vector. The direction of the induced velocity, represented by directional unit vector \( \hat{V}_{12} \), is normal to the plane of \( r_1 \) and \( r_2 \) and can be shown as:

\[
\hat{V}_{12} = \frac{r_1 \times r_2}{|r_1 \times r_2|}
\]

(4.12)

Hence, the induced velocity vector on point \( K \) due to the vortex element 12 is obtained as:

\[
V_{12} = V_{12} \hat{V}_{12} = \frac{\Gamma}{4\pi} \frac{r_1 \times r_2}{|r_1 \times r_2|^2} \left( \frac{r_0 \cdot r_1}{r_1} - \frac{r_0 \cdot r_2}{r_2} \right)
\]

(4.13)

Equation (4.13) is the analytical formulation to calculate the velocity field for the three-dimensional flows and can be used to develop a numerical description of the 3-D aerodynamic loading. In order to this, a 3-D Cartesian coordinate system is used and the parameters are expressed in the Cartesian coordinates. If the vortex element is located between two points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\), for an arbitrary point \(K(x_K, y_K, z_K)\), the distances \( r_1 \) and \( r_2 \) can expressed as:

\[
r_1 = \sqrt{(x_K - x_1)^2 + (y_K - y_1)^2 + (z_K - z_1)^2}
\]

\[
r_2 = \sqrt{(x_K - x_2)^2 + (y_K - y_2)^2 + (z_K - z_2)^2}
\]

(4.14)

Knowing that \( r_0 = r_1 - r_2 \), the other terms in equation (4.13) can be stated as:

\[
r_0 \cdot r_1 = (x_2 - x_1)(x_K - x_1) + (y_2 - y_1)(y_K - y_1) + (z_2 - z_1)(z_K - z_1)
\]

(4.15)
\[ \mathbf{r}_0 \cdot \mathbf{r}_2 = (x_2 - x_1)(x_K - x_2) + (y_2 - y_1)(y_K - y_2) + (z_2 - z_1)(z_K - z_2) \]  \hspace{1cm} (4.16)

\[ |\mathbf{r}_1 \times \mathbf{r}_2|^2 = (\mathbf{r}_1 \times \mathbf{r}_2)_x^2 + (\mathbf{r}_1 \times \mathbf{r}_2)_y^2 + (\mathbf{r}_1 \times \mathbf{r}_2)_z^2 \]  \hspace{1cm} (4.17)

where:

\[ (\mathbf{r}_1 \times \mathbf{r}_2)_x = (y_K - y_1) \cdot (z_K - z_2) - (z_K - z_1) \cdot (y_K - y_2) \]

\[ (\mathbf{r}_1 \times \mathbf{r}_2)_y = -(x_K - x_1) \cdot (z_K - z_2) + (z_K - z_1) \cdot (x_K - x_2) \]  \hspace{1cm} (4.18)

\[ (\mathbf{r}_1 \times \mathbf{r}_2)_z = (x_K - x_1) \cdot (y_K - y_2) - (y_K - y_1) \cdot (x_K - x_2) \]

Thus, the components of the resulting velocity \( \mathbf{v}_{12} = u_{12}\mathbf{i} + v_{12}\mathbf{j} + w_{12}\mathbf{k} \) can be presented as:

\[ u_{12} = C (\mathbf{r}_1 \times \mathbf{r}_2)_x \]

\[ v_{12} = C (\mathbf{r}_1 \times \mathbf{r}_2)_y \]  \hspace{1cm} (4.19)

\[ w_{12} = C (\mathbf{r}_1 \times \mathbf{r}_2)_z \]

where:

\[ C = \frac{\Gamma}{4\pi |\mathbf{r}_1 \times \mathbf{r}_2|^2} \left( \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_1} - \frac{\mathbf{r}_0 \cdot \mathbf{r}_2}{r_2} \right) \]  \hspace{1cm} (4.20)

For the numerical calculations, the induced velocity obtained using equations (4.19) and (4.20) can be stored in a subroutine called \textit{VLine}. If we assume that the vortex line 12 is located between two end points of \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\), the subroutine \textit{VLine} can be called to calculate the induced velocity on a collocation point \(K(x_K, y_K, z_K)\) due to this vortex line. In this case, the input of the subroutine is the coordinates of two end points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) and the collocation point \(K(x_K, y_K, z_K)\). Hence, the output of the subroutine is the components of the induced velocity \( \mathbf{v}_{12} = u_{12}\mathbf{i} + v_{12}\mathbf{j} + w_{12}\mathbf{k} \). The command line used in the numerical code to call this subroutine can be written as follows:

\[ (u_{12}, v_{12}, w_{12}) = \text{VLine}(x_K, y_K, z_K, x_1, y_1, z_1, x_2, y_2, z_2, \Gamma) \]
In the 3-D panel method similar to the 2-D case, the values of vortex strengths are unknown and are obtained by solving a linear set of equations. Therefore, the velocity components $u_{12}^*, v_{12}^*, w_{12}^*$ can be defined by considering $\Gamma = 1$ in equations (4.19) and (4.20) which results in the following equations:

$$u_{12}^* = C^* (\mathbf{r}_1 \times \mathbf{r}_2)_x$$
$$v_{12}^* = C^* (\mathbf{r}_1 \times \mathbf{r}_2)_y$$
$$w_{12}^* = C^* (\mathbf{r}_1 \times \mathbf{r}_2)_z$$

where $\mathbf{V}_{12} = u_{12}^* \hat{i} + v_{12}^* \hat{j} + w_{12}^* \hat{k}$ is the velocity induced by a vortex line with the unit vortex strength. In addition:

$$C^* = \frac{1}{4\pi |\mathbf{r}_1 \times \mathbf{r}_2|^2} \left( \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{r_1} - \frac{\mathbf{r}_0 \cdot \mathbf{r}_2}{r_2} \right)$$

As a result:

$$\mathbf{V}_{12} = \Gamma \mathbf{V}_{12}^*$$

Hence, the required command line can be expressed as shown below:

$$(u_{12}^*, v_{12}^*, w_{12}^*) = V\text{Line}(x_K, y_K, z_K, x_1, y_1, z_1, x_2, y_2, z_2, \Gamma = 1)$$

### 4.3 Three-dimensional Panel Method for Steady Flows

In general, in order to develop a 3-D panel model in steady flow, the lifting surface is divided into a finite number of surface panels with singularity distributions and a set of equations is solved to determine the velocity field. The procedure of the steady panel method presented here for 3-D flow is similar, in principle, to the 2-D case described in the previous chapter, but here, the vortex ring elements are chosen as singularities to solve the thin lifting surface problem.
Figure 4.4 A vortex ring consisting of four vortex line elements (Katz and Plotkin 2001)

The advantage of using the vortex ring elements is in the simple programming effort that it requires. Also, the exact boundary condition is satisfied on the actual wing surface which can have camber and various planform shapes (Katz and Plotkin 2001).

As illustrated in Figure 4.4, a vortex ring consists of four constant-strength vortex line elements. Therefore, the velocity induced by a vortex ring is the summation of the velocities induced by each vortex line. Equations (4.19) and (4.20) can be used to calculate the induced velocity at point $K$ due to the vortex line 12 (between points 1 and 2). Similar equations can be applied to calculate the induced velocity due to the vortex lines 23, 34, and 41 by entering the coordinates of the appropriate end points (points 2, 3, 4, and 1). If the vortex ring consisting of these four vortex lines is referred as vortex ring $L$ with the vortex strength of $\Gamma_L$, the total induced velocity on point $K$ due to the vortex ring $L$ ($V_{KL}$) can be expressed as presented below:

$$V_{KL} = V_{12} + V_{23} + V_{34} + V_{41}$$

(4.24)

where subscripts 12 to 41 refers to the induced velocity of the vortex lines 12 to 41, respectively. This equation can be written in terms of the velocity components as follows:
Equation (4.25) can be used in another subroutine called $VRing$, resulting in the following command line:

$$(u_{KL}, v_{KL}, w_{KL}) = VRing(x_K, y_K, z_K, i, j, \Gamma_L)$$

where indices $i, j$ determines the location of the vortex ring on the wing surface (explained later). Equation (4.24) can be recast by replacing equation (4.23) as follows:

$$V_{KL} = \Gamma_L V^*_{KL} = \Gamma_L (V_{12}^* + V_{23}^* + V_{34}^* + V_{41}^*)$$

where $V_{KL} = u_{KL}^* \hat{i} + v_{KL}^* \hat{j} + w_{KL}^* \hat{k}$ is the induced velocity on point $K$ due to the vortex ring $L$ with the unit vortex strength and can be found using the following equation:

$$V_{KL}^* = V_{12}^* + V_{23}^* + V_{34}^* + V_{41}^*$$

where the unit induced velocities $V_{12}^*, V_{23}^*, V_{34}^*$, and $V_{41}^*$ can be obtained from equations (4.21) and (4.22) and other similar equations. Consequently, the pertinent command line to obtain the unit induced velocity components can be called as:

$$(u_{KL}^*, v_{KL}^*, w_{KL}^*) = VRing(x_K, y_K, z_K, i, j, \Gamma_L = 1)$$

It was discussed in Chapter 3 that the point vortex in the 2-D panel method is placed at the center of pressure of each panel which lies at the panel quarter-chord and the collocation point is placed at the panel third-quarter chord. Similarly here, the leading vortex-line segment is placed on the quarter-chord line of the panel and the collocation point is at the center of the three-quarter chord line. Figure 4.5 shows the normal vector and vortex ring placement on the wing panels.
Figure 4.5 Vortex ring placement on the wing panels (Katz and Plotkin 2001)

In Figure 4.5, $\Delta x_{ij}$ and $\Delta y_{ij}$ are the lengths of the panel $ij$ along the $x$ and $y$ axes, respectively. Figure 4.6 presents the vortex ring elements and the panel distribution on a trapezoidal wing surface moving in a steady 3-D flow. If the wing is divided into $M$ panels along the chord and $N$ panels along the semi-span,
the total number of panels is \( G = M \times N \). We can select \( i \) and \( j \) to be the counters for the rows (along the chord direction) and the columns (along the semi-span direction), respectively, where the values \( i \) and \( j \) varies from 1 to \( M \) and from 1 to \( N \), respectively. Then, the properties related to each panel can be stored in a two-dimensional array, where \((i,j)\) is the index of this array. Figure 4.7 demonstrates the vortex rings numbering on the surface of the wing.

All the singularity elements contribute to the induced velocity of all the collocation points. So, the effect of all the vortex rings should be included in the induced velocity on a certain collocation point. Hence, a procedure is needed to scan both the collocation points and the vortex rings. Since two counters are needed to scan both the collocation points and the vortex rings, it is often found useful to combine two indices \( i, j \) (the counters for the rows and columns) into one index \( L \) which is the counter for the vortex rings. So, \( L \) scans from 1 to \( M \times N \) where \( M \times N \) is the total number of the vortex rings (or panels). The counter \( L \) can be related to the row and column counters \( i, j \) as shown below:

\[
L = (i - 1)N + j
\]  

(4.28)

Similarly, another counter \( K \) can be introduced to scan the collocation points and can be obtained using an equation similar to equation (4.28). The order of scanning the vortex rings is shown in Figure 4.8 which exhibits the procedure of changing the \( L \) counter from 1 to \( M \times N \) for each collocation point (for instance for \( K = 1 \)).

It can be observed that using this procedure, first the contribution of the vortex rings situated on the first row is considered in the induced velocity on a certain collocation point and then the contribution of the other vortex rings on other rows is taken into account. The procedure of scanning the collocation points (with the \( K \) counter) is also similar to that of the vortex rings shown in Figure 4.8.

From the numerical point of view, these two procedures can be represented as two loops, the outer loop with the \( K \) counter and the inner one with the \( L \) counter.
Figure 4.7 Vortex ring numbering of the wing surface

More precisely, for the first collocation point when \( K = 1 \), the inner loop starts to scan the vortex rings \( (L = 1 \rightarrow M \times N) \) and obtains the induced velocity on the first collocation points due to all the scanned vortex rings.

As discussed in the previous section, the vortex line and consequently the vortex ring element is a solution of the Laplace’s equation. Thus, the boundary condition that must be satisfied is the zero normal flow across the wing surface as expressed below:

\[
V \cdot n = (V_\infty + q) \cdot n = 0 \tag{4.29}
\]

where \( V \) is the fluid velocity vector which consists of the free-stream velocity \( V_\infty \) and the perturbation velocity \( q \), and \( n = nx\hat{i} + ny\hat{j} + nz\hat{k} \) is the vector normal to the panel surface. The free-stream velocity can be expressed in the general form as:

\[
V_\infty = U_\infty\hat{i} + V_\infty\hat{j} + W_\infty\hat{k} \tag{4.30}
\]

In this thesis, it is assumed that the flow velocity is only along the \( x \) direction, hence: \( V_\infty = W_\infty = 0 \) and equation (4.30) can be restated as:
Figure 4.8 Scanning order for the vortex ring counter (Katz and Plotkin 2001)

\[ \mathbf{V}_\infty = U_\infty \hat{\mathbf{l}} \] (4.31)

In steady flow, the velocity vectors are considered in the body-fixed frame. Here, the perturbation velocity vector \( \mathbf{q} \) is the induced velocity due to the vortex ring elements of all the panels. Similar to the 2-D case, the starting vortex in 3-D steady flow moves to the far field and its effect on the velocity distribution on the wing is negligible. Hence, \( \mathbf{q} \) is due to the vortex rings of the body panels only.

Equation (4.29) can be restated as follows:

\[ \mathbf{V}_\infty \cdot \mathbf{n} + \mathbf{q} \cdot \mathbf{n} = 0 \] (4.32)

The first part of this equation \( \mathbf{V}_\infty \cdot \mathbf{n} \) is the normal velocity induced by the free-stream which is known if the flight path is specified. Accordingly, this term can go to the right side of the equation (4.32). The second part \( \mathbf{q} \cdot \mathbf{n} \) is the normal perturbation velocity induced by the body panels and is unknown because the strengths of vortex rings are unknown at this point.

For the 3-D flow, the body perturbation velocity on panel \( K \) \( (\mathbf{q})_K \) is the velocity induced by all the body panels on panel \( K \) and can be written similar to equation (3.14) as follows:
where $V_{KL}$ is the velocity induced by the vortex ring $L$ on the collocation point $K$ and can be calculated using equation (4.24). Since the strengths of the vortex lines $\Gamma_L$ are unknown at this step, it is preferred to write this equation in terms of the unit induced velocities introduced by equation (4.26). Hence, one can write the equation (4.33) as shown below:

$$
(q)_K = \sum_{L=1}^{M \times N} \Gamma_L V^*_{KL} = \sum_{L=1}^{M \times N} \Gamma_L \left( u_{KL}^* \hat{i} + v_{KL}^* \hat{j} + w_{KL}^* \hat{k} \right) \quad (4.34)
$$

Accordingly, the normal body perturbation velocity on the panel $K$ can be stated as:

$$
(q \cdot n)_K = \sum_{L=1}^{M \times N} \Gamma_L V^*_{KL} \cdot n_K \quad (4.35)
$$

where $n_K$ is the surface vector normal to the panel $K$. Equation (4.35) can be written as a linear combination of the unknown vortex strengths $\Gamma_L$ by introducing the influence coefficients $C_{KL}$ as explained in Chapter 3, which results in:

$$
(q \cdot n)_K = \sum_{L=1}^{G} C_{KL} \Gamma_L = C_{K1} \Gamma_1 + C_{K2} \Gamma_2 + \cdots + C_{KG} \Gamma_G \quad (4.36)
$$

where $G = M \times N$ is the total number of wing panels and $C_{KL}$ is the influence coefficient which can be defined as the perturbation velocity component normal to the surface of panel $K$, due to a unit strength vortex ring element on the panel $L$ as follows:

$$
C_{KL} = V^*_{KL} \cdot n_K = \left( u_{KL}^* \hat{i} + v_{KL}^* \hat{j} + w_{KL}^* \hat{k} \right) \cdot \left( n_x \hat{i} + n_y \hat{j} + n_z \hat{k} \right)_K \quad (4.37)
$$

where $V^*_{KL}$ can be calculated from equation (4.27). As a result, the boundary condition equation (4.32) is written for panel $K$ as follows:
CONTROL OF AEROELASTIC OSCILLATIONS OF WING STRUCTURES USING BONDED PIEZOELECTRIC STRIPS

\[ (q \cdot n)_K = -(V_\infty \cdot n)_K \quad (4.38) \]

or by substituting from equation (4.36) as:

\[ \sum_{L=1}^{G} C_{KL} \Gamma_L = -(V_\infty \cdot n)_K \quad (4.39) \]

where the right hand side of the equation is related to the flight path conditions, so it is known for each panel. Finally, one can write:

\[ \sum_{L=1}^{G} C_{KL} \Gamma_L = C_{K1} \Gamma_1 + C_{K2} \Gamma_2 + C_{K3} \Gamma_3 + \cdots + C_{KG} \Gamma_G = B_K \quad (4.40) \]

where

\[ B_K = -(V_\infty \cdot n)_K = -(U_\infty i) \cdot (n_x \hat{i} + n_y \hat{j} + n_z \hat{k})_K \quad (4.41) \]

The normal vector \( n = n_x \hat{i} + n_y \hat{j} + n_z \hat{k} \) is defined in the collocation point which falls in the center of the vortex ring and can be calculated using the shape of the wing surface as shown in Chapter 2. Another general and useful method for evaluation of the normal vector, especially when the surface shape is not specified, is shown in Figure 4.9. If two vectors which connect the two opposite corners of panel \( K \) are called \( A_K \) and \( B_K \), the normal vector \( n_K \) can be obtained using the vector product of these two vectors as shown below:

\[ n_K = \frac{A_K \times B_K}{|A_K \times B_K|} \quad (4.42) \]

where:

\[ A_K = (x_{i+1,j+1} - x_{i,j})\hat{i} + (y_{i+1,j+1} - y_{i,j})\hat{j} + (z_{i+1,j+1} - z_{i,j})\hat{k} \quad (4.43) \]

\[ B_K = (x_{i,j+1} - x_{i+1,j})\hat{i} + (y_{i,j+1} - y_{i+1,j})\hat{j} + (z_{i,j+1} - z_{i+1,j})\hat{k} \quad (4.44) \]
As explained before, the starting wake vortex in the steady flow, moves to the far-field and has no effect on the pressure distribution of the surface panels. However, for the last row of the vortex rings, one of the spanwise vortex lines lies at the trailing edge, which makes the vortex strength at the trailing edge to be non-zero. This violates the Kutta condition. To cancel the effect of this vortex line, we consider another spanwise vortex line in its vicinity with the same vortex strength (shown in Figure 4.10). This vortex line belongs to a vortex ring called free wake which can extend to infinity.

To consider the influence of the wake panel, the velocity induced by the wake vortex ring can be added to the induced velocity of the trailing-edge vortex ring. If there are $M$ rows of panels, then $i = M$ represents the trailing-edge row and $i = M + 1$ indicates the row of the wake panels. These values of $i$ can be used as the input of the $VRing$ subroutine to obtain the unit induced velocities of the trailing-edge and the wake panels as presented in the following command lines:

$$ (u_{KL}^{*}, v_{KL}^{*}, w_{KL}^{*})_{TE} = VRing(x_K, y_K, z_K, M, j, \Gamma_L = 1) $$

$$ (u_{KL}^{*}, v_{KL}^{*}, w_{KL}^{*})_w = VRing(x_K, y_K, z_K, M + 1, j, \Gamma_L = 1) $$

where the subscripts $TE$ and $w$ indicates the trailing edge and the wake panels, respectively. Therefore, the velocity components due to the effect of wake should be added to those of the trailing edge panels as shown below:
Figure 4.10 Introducing the vortex wake panel to satisfy the Kutta condition

\[
(u^*_K L \hat{i} + v^*_K L \hat{j} + w^*_K L \hat{k})_{i=M} = (u^*_K L \hat{i} + v^*_K L \hat{j} + w^*_K L \hat{k})_{TE} + (u^*_K L \hat{i} + v^*_K L \hat{j} + w^*_K L \hat{k})_w
\]

These velocity components \((u^*_K L \hat{i} + v^*_K L \hat{j} + w^*_K L \hat{k})_{i=M}\) can be used in equation (4.37) to calculate the influence coefficient for the trailing-edge row.

Finally, writing equation (4.40) for all the collocation points leads to a set of \(G\) equations and \(G\) unknowns which can be shown in the matrix form as:

\[
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1G} \\
C_{21} & C_{22} & \cdots & C_{2G} \\
\vdots & \vdots & \ddots & \vdots \\
C_{G1} & C_{G2} & \cdots & C_{GG}
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
\vdots \\
\Gamma_G
\end{bmatrix} =
\begin{bmatrix}
B_1 \\
B_2 \\
\vdots \\
B_G
\end{bmatrix}
\]

Once this set of equations is solved, the vortex strengths of the vortex rings can be determined to define the aerodynamic loads on the wing surface using the Kutta-Joukowski theorem. At this point, it is advisable to return to the two-index
form of the panel counters $i, j$ (the row and column counter). If the vortex ring located at row $i$ and column $j$ is denoted by $ij$, the lift of this vortex ring $\Delta L_{ij}$ can be calculated as:

$$\Delta L_{ij} = \rho U_\infty (\Gamma_{ij} - \Gamma_{i-1,j}) \Delta y_{ij}$$

where $\rho$ is the air density, $U_\infty$ is the magnitude of the free-stream velocity vector, and $\Delta y_{ij}$ is the length of panel $ij$ along the semi-span direction ($y$ axis). In this equation, $\Gamma_{ij}$ is the vortex strength of the vortex ring $ij$ and $\Gamma_{i-1,j}$ is the vortex strength of the vortex ring located at the previous row, the $(i-1)$th row and the same column, $j$th column.

For the panels located at the leading edge, $i = 1$, the interaction of the vortex rings of the previous row does not exist. Therefore, the lift generated due to the vortex ring located on the first row $i = 1$ and column $j$, $\Delta L_{1j}$, can be obtained as:

$$\Delta L_{1j} = \rho U_\infty \Gamma_{1j} \Delta y_{1j}$$

Finally, the pressure difference between the upper and lower surface of each panel $ij$ can be stated as:

$$\Delta p_{ij} = \frac{\Delta L_{ij}}{\Delta S_{ij}}$$

where $\Delta S_{ij}$ is the area of the panel $ij$.

### 4.4 Three-dimensional Panel Method for Unsteady Flows

As discussed in the unsteady part of Chapter 3, the unsteady 3-D panel method can be obtained by modifying the steady panel method and including the effects of the time-dependent terms. To model the unsteady 3-D flow, the surface of the wing and the wake is discretized into a finite number of panel elements containing the vortex ring singularities similar to what explained in the previous chapter.
Since in unsteady flow, at each instant new vortices are created at the trailing edge, a downwash due to these vortices is created on the wing panels which should be taken into account. Therefore, the surface of the wake should also be divided into panels with vortex ring elements. Figure 4.11 shows the surface and the wake vortex rings on the three-dimensional unsteady wing model.

4.4.1 Boundary Conditions

The solution of the 3-D unsteady problem is based on the time-stepping method and the initial condition is the 3-D steady-state motion. More precisely, at the beginning of the motion, \( t_1 = 0 \), there are no wake panels and only the wing bound vortex rings exist. If the wing is represented by \( K \) unknown vortex rings, then by specifying the zero normal flow boundary condition on \( K \) collocation points, a solution at \( t_1 = 0 \) is possible. During the second time step or \( t_2 = \Delta t \),
the wing is moved along its known flight path and each trailing edge vortex panel sheds a wake panel with the a vortex strength equal to its circulation in the previous time step. During this time step, there is only one row of wake vortices, but with known strengths from the previous time step. Therefore, the wing bound vortices can be calculated for this time step by specifying the boundary conditions on the collocation points.

To describe the 3-D unsteady motion, two inertial and body-fixed coordinate systems are considered, similar to the 2-D case. The body-fixed coordinate system is attached to the wing body and coincides with the inertial coordinate at the first instance, \( t = 0 \), but moves away from the inertial frame for \( t > 0 \). Figure 4.12 illustrates these two coordinate systems for a 3-D wing model moving in an unsteady flow.

As explained before, the boundary condition in the body-fixed frame can be specified as the zero normal flow on the wing surface, as shown below:

\[
V \cdot n = V_b \cdot n \tag{4.50}
\]

Figure 4.12 Inertial and body-fixed coordinate systems in an unsteady motion of a 3-D wing
where \( V = \nabla \Phi \) is the fluid velocity and \( n = n_x \hat{i} + n_y \hat{j} + n_z \hat{k} \) is the unit vector normal to the wing surface in the 3-D flow. As mentioned in Chapter 2, \( V_b \) is the wing velocity in the body-fixed frame defined as:

\[
V_b = \omega \times r + v_{rel} \quad (4.51)
\]

The simpler form of \( V_b \) in terms of its components can be written as:

\[
V_b = U_b \hat{i} + V_b \hat{j} + W_b \hat{k} \quad (4.52)
\]

where \( U_b, V_b, W_b \) are the time-dependent components of surface velocity along \( x, y \) and \( z \) axes, respectively.

As mentioned in the 2-D unsteady flow, the fluid velocity vector \( V \) consists of the free-stream velocity \( V_\infty \) and the perturbation velocity \( q \) where \( q \) includes the perturbation velocity induced by both the body and the wake panels: \( q = q_b + q_w \), hence:

\[
V = V_\infty + q_b + q_w \quad (4.53)
\]

where:

\[
V_\infty = U_\infty \hat{i} \quad (4.54)
\]

\[
q_b = u \hat{i} + v \hat{j} + w \hat{k} \quad (4.55)
\]

\[
q_w = u_w \hat{i} + v_w \hat{j} + w_w \hat{k} \quad (4.56)
\]

Therefore, the boundary condition of equation (4.50) can result in:

\[
q_b \cdot n + q_w \cdot n = (\omega \times r + v_{rel}) \cdot n - V_\infty \cdot n \quad (4.57)
\]

In this equation, the first term \( q_b \cdot n \) is the induced normal velocity component due to the body panels. The contribution of this term on the panel \( K \) can be written as a linear combination of the unknown vortex strengths \( \Gamma_L \) and the influence coefficients \( C_{KL} \) as expressed in equation (4.36) for steady flow:
\[
(q_b \cdot n)_K = \sum_{L=1}^{G} C_{KL} \Gamma_L = C_{K1} \Gamma_1 + C_{K2} \Gamma_2 + \cdots + C_{KG} \Gamma_G
\]  \tag{4.58}

where \( G = M \times N \) is the number of all the wing panels and \( K \) and \( L \) are the counters for scanning the collocation points and the vortex rings, respectively. Here the values of \( C_{KL} \) can be obtained from equation (4.37) by calling two subroutines \( VLine \) and \( VRing \), as explained the previous section. Equation (4.58) should be updated at each time step based on the necessary modifications in the flow or the wing geometry.

The second term in equation (4.57), shown as \( q_w \cdot n \), is the contribution of the wake panels in the induced normal velocity. At each time step, a new wake vortex is created at the trailing edge with an unknown vortex strength \( \Gamma_{wt} \) whose effect should be considered in addition to the effect of the old wake vortices. For all the old wake vortices created in the previous time steps, we assume that the values of vortex strengths are known from the problem solution in the previous time steps. The induced normal velocity component at \( K \)th collocation point, due to all wake panels can be written as:

\[
(q_w \cdot n)_K = (q_{w_0} \cdot n)_K + C_{Kw_t} \Gamma_{wt}
\]  \tag{4.59}

where \((q_{w_0} \cdot n)_K\) is the normal induced velocity due to the wake vortices created in the previous time steps and its value is known from the previous time steps. Here \( C_{Kw_t} \) is the influence coefficient of the latest wake where its strength \( \Gamma_{wt} \) is unknown at this point. All the known terms in equation (4.59) can go to the right hand side of the boundary condition equation. If the contribution of equation (4.57) for panel \( K \) is shown as below:

\[
(q_b \cdot n)_K + (q_w \cdot n)_K = [(\omega \times r + v_{rel}) \cdot n]_K - [V_\infty \cdot n]_K
\]  \tag{4.60}

by substituting equations (4.58) and (4.59), one may obtain:
\[
\sum_{L=1}^{G} \left( C_{KL} \Gamma_L + C_{Kw_t} \Gamma_{w_t} \right) = \left( [\omega \times r + \nu_{rel}] \cdot n \right)_K - \left( V_\infty \cdot n \right)_K - \left( q_{w_0} \cdot n \right)_K 
\] (4.61)

The right hand side of this equation can be expressed in terms of the Cartesian components by employing equations (4.54) and (4.55) as follows:

\[
\sum_{L=1}^{G} C_{KL} \Gamma_L + C_{Kw_t} \Gamma_{w_t} = B_K 
\] (4.62)

where:

\[
B_K = \left( \left( U_b i + V_b j + W_b k \right)_K - \left( U_\infty i \right)_K \right.
\]
\[
- \left( u_{w_0} i + v_{w_0} j + w_{w_0} k \right)_K \] \cdot \left( n_K \right) 
\] (4.63)

where \( u_{w_0}, v_{w_0}, w_{w_0} \) are the components of the \( q_{w_0} \cdot n \), the normal induced velocity due to the wake vortices created in the previous time steps.

Finally, equation (4.62) can be written for all the collocation points (\( K = 1, \cdots, G \)) which results in a set of \( G \) equations and \( G \) unknowns, shown in the matrix form as below:

\[
\begin{bmatrix}
C_{11} & C_{12} & \cdots & C_{1G} \\
C_{21} & C_{22} & \cdots & C_{2G} \\
C_{31} & C_{32} & \cdots & C_{3G} \\
\vdots & \vdots & \ddots & \vdots \\
C_{G1} & C_{G2} & \cdots & C_{GG} \\
\end{bmatrix}
\begin{bmatrix}
\Gamma_1 \\
\Gamma_2 \\
\Gamma_3 \\
\vdots \\
\Gamma_G \\
\end{bmatrix}
= \begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\vdots \\
B_G \\
\end{bmatrix} 
\] (4.64)

where the values of \( B_K \) are defined in equation (4.63).

It is worth mentioning that although for the 2-D unsteady aerodynamic model, one additional equation was considered to satisfy the Thomson condition, for the 3-D unsteady model, the vortex ring inherently fulfils the Thomson condition. So, there is no need to add an additional equation to equation (4.64). In addition, since the induced velocity due to the wake panels are considered at each time step, there
is no need to modify the trailing edge induced velocities, as it was done for the steady flow in equation (4.45).

### 4.4.2 Computation of the Pressure Distribution and Loads

Once equation (4.64) is solved, the vortex strengths of the body panels $\Gamma_L (L = 1, ..., G)$ and the latest wake vortex $\Gamma_{W_r}$ can be obtained at each time step. Subsequently, by employing the Bernoulli equation near the wing surface, the pressure distribution can be calculated. Similar to the 2-D unsteady Bernoulli equation, the Bernoulli equation for the 3-D unsteady flow can be stated as follows:

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} V^2 + \frac{p}{\rho_\infty} = \frac{1}{2} U_\infty^2 + \frac{p_\infty}{\rho_\infty}$$  \hspace{1cm} (4.65)

where $\varphi$ is the perturbation velocity potential, $p_\infty$ and $\rho_\infty$ are the far-field reference pressure and density, $p$ is the local fluid pressure, and $U_\infty$ and $V$ are the magnitude of the free-stream velocity and the fluid velocity vectors, respectively.

Following a similar procedure to that of the 2-D unsteady flow, equation (3.87) can be used for the 3-D unsteady flow to determine the pressure difference between the upper and lower wing surfaces as shown below:

$$\Delta p = p_u - p_l = -\rho_\infty \left[ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_l + (V_\infty \cdot q)_u - (V_\infty \cdot q)_l \right]$$  \hspace{1cm} (4.66)

where $V_\infty = U_\infty \hat{i}$ is the free-stream velocity and $q = \nabla \varphi = \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k}$ is the total perturbation velocity where $\varphi$ is the perturbation velocity potential. Accordingly, the pressure difference on the panel $ij$ can be stated as:

$$\Delta p_{ij} = -\rho_\infty \left\{ \left[ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_l \right]_{ij} + [(V_\infty \cdot q)_u - (V_\infty \cdot q)_l]_{ij} \right\}$$  \hspace{1cm} (4.67)
As mentioned in Chapter 3, it is often found more convenient to write the velocity vectors $V_\infty$, $q$ in terms of their components in the normal-tangential coordinate system. For the 3-D case, the normal unit vector $n$ is normal to the plane of the wing surface and the tangential unit vector $\tau$ lies in the plane of the wing surface and has two components which can be considered as $\tau_i$ and $\tau_j$ along $i$ and $j$ directions, respectively. Thus:

$$q = \nabla \varphi = \frac{\partial \varphi}{\partial n} n + \frac{\partial \varphi}{\partial \tau_i} \tau_i + \frac{\partial \varphi}{\partial \tau_j} \tau_j$$

$$V_\infty = (V_\infty \cdot n) n + (V_\infty \cdot \tau_i) \tau_i + (V_\infty \cdot \tau_j) \tau_j$$

and consequently:

$$V_\infty \cdot q = (V_\infty \cdot n) \frac{\partial \varphi}{\partial n} + (V_\infty \cdot \tau_i) \frac{\partial \varphi}{\partial \tau_i} + (V_\infty \cdot \tau_j) \frac{\partial \varphi}{\partial \tau_j}$$

Since $\frac{\partial \varphi}{\partial n}$ is the normal component of the perturbation velocity potential which is identical on the upper and lower surfaces, it can be removed and the second term in equation (4.67) can be presented as:

$$(V_\infty \cdot q)_u - (V_\infty \cdot q)_i$$

$$= (V_\infty \cdot \tau_i) \left[ \left( \frac{\partial \varphi}{\partial \tau_i} \right)_u - \left( \frac{\partial \varphi}{\partial \tau_i} \right)_i \right]$$

$$+ (V_\infty \cdot \tau_j) \left[ \left( \frac{\partial \varphi}{\partial \tau_j} \right)_u - \left( \frac{\partial \varphi}{\partial \tau_j} \right)_i \right]$$

or:

$$(V_\infty \cdot q)_u - (V_\infty \cdot q)_i$$

$$= (V_\infty \cdot \tau_i) \left[ \frac{\partial}{\partial \tau_i} \Delta \varphi \right] + (V_\infty \cdot \tau_j) \left[ \frac{\partial}{\partial \tau_j} \Delta \varphi \right]$$

Hence, the pressure difference for panel $ij$ can be expressed as:
\[ \Delta p_{ij} = -\rho_\infty \left\{ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_l \right\} 
+ \left[ \left( \mathbf{V}_\infty \cdot \mathbf{t}_i \right) \left( \frac{\partial}{\partial t} \Delta \varphi \right) + \left( \mathbf{V}_\infty \cdot \mathbf{t}_j \right) \left( \frac{\partial}{\partial t} \Delta \varphi \right) \right]_{ij} \]  

(4.73)

where \( \mathbf{t}_i \) and \( \mathbf{t}_j \) are the tangential surface vectors along the \( i \) and \( j \) directions, respectively. Here \( \Delta \varphi \) is the difference of the velocity potential between the upper and lower surfaces which can be obtained by applying the definition of the circulation described in equation (3.96):

\[ \Gamma = \varphi_u - \varphi_i = \Delta \varphi \]  

(4.74)

As a result, the tangential derivative of the velocity potential along the \( x \) direction can be approximated by:

\[ \frac{\partial}{\partial t_i} \Delta \varphi = \frac{\partial \Gamma}{\partial t_i} \approx \frac{\Delta \Gamma}{\Delta t_i} \]  

(4.75)

If the circulation of the panel \( ij \) which is located at the \( i \)th row and the \( j \)th column is denoted by \( \Gamma_{ij} \), and the length of this panel along the chord direction is referred as \( \Delta c_{ij} \), then equation (4.75) can be stated as follows:

\[ \frac{\partial}{\partial t_i} \Delta \varphi \approx \frac{\Delta \Gamma}{\Delta t_i} = \frac{\Gamma_{ij} - \Gamma_{i-1,j}}{\Delta c_{ij}} \]  

(4.76)

It is worth noting that for the leading edge panel, \( \Delta \Gamma \) is equal to \( \Gamma_{ij} \) and for all the elements behind it is equal to \( \Gamma_{ij} - \Gamma_{i-1,j} \). Similarly if the width of panel \( ij \) along the span direction is \( \Delta b_{ij} \), one can find:

\[ \frac{\partial}{\partial t_j} \Delta \varphi \approx \frac{\Delta \Gamma}{\Delta t_j} = \frac{\Gamma_{ij} - \Gamma_{i,j-1}}{\Delta b_{ij}} \]  

(4.77)

In order to calculate the time derivative of the velocity potential \( \frac{\partial \varphi}{\partial t} \), equation (4.74) can be used again, hence:
\[
\left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_t = \frac{\partial}{\partial t} \Delta \varphi = \frac{\partial \Gamma}{\partial t}
\] (4.78)

which can be written for the panel \( ij \) as

\[
\left[ \left( \frac{\partial \varphi}{\partial t} \right)_u - \left( \frac{\partial \varphi}{\partial t} \right)_{t_{ij}} \right] = \frac{\partial \Gamma_{ij}}{\partial t} = \frac{(\Gamma_{ij})_t - (\Gamma_{ij})_{t-1}}{\Delta t}
\] (4.79)

where \( \Delta t \) is the time step and \( (\Gamma_{ij})_t \) and \( (\Gamma_{ij})_{t-1} \) are the circulation of the panel \( ij \) at the present and the previous time steps, respectively.

Finally, by substituting equations (4.76), (4.77), and (4.79) in equation (4.73), the pressure difference for the panel \( ij \) can be obtained as follows:

\[
\Delta p_{ij} = -\rho_\infty \left\{ \left( \frac{(\Gamma_{ij})_t - (\Gamma_{ij})_{t-1}}{\Delta t} \right) + \left( V_\infty \cdot \tau_i \right) \left( \frac{\Gamma_{ij} - \Gamma_{i-1,j}}{\Delta c_{ij}} \right) \right. \\
\left. + \left( V_\infty \cdot \tau_j \right) \left( \frac{\Gamma_{ij} - \Gamma_{i,j-1}}{\Delta b_{ij}} \right) \right\}
\] (4.80)

Consequently, the lift force on the panel \( ij \) can be obtained as below:

\[
\Delta L_{ij} = \Delta p_{ij} S_{ij} \cos \beta_{ij} = \Delta p_{ij} \Delta c_{ij} \Delta b_{ij} \cos \beta_{ij}
\] (4.81)

where \( S_{ij} = \Delta c_{ij} \Delta b_{ij} \) is the area of the panel \( ij \) where \( \Delta c_{ij} \) and \( \Delta b_{ij} \) are the length and width of the panel, respectively. Here \( \beta_{ij} \) is the local angle of attack of the panel \( ij \).

In order to obtain the spanwise distribution of the lift force, the lift on the panels located on the \( j \)th row can be added as follows:

\[
\Delta L_j = \sum_{i=1}^{M} \Delta p_{ij} \Delta c_{ij} \Delta b_{ij} \cos \beta_{ij}
\] (4.82)

where \( M \) is the number of panels along chord. Thus, the total lift force on the wing surface can be obtained as shown below:
AERODYNAMIC MODEL FOR STEADY AND UNSTEADY 3-D FLOWS

\[ L = \sum_{j=1}^{N} \sum_{i=1}^{M} \Delta p_{ij} \Delta c_{ij} \Delta b_{ij} \cos \beta_{ij} \]  

(4.83)

where \( N \) is the number of panels along span. The total lift coefficient is expressed as presented below:

\[ C_L = \frac{L}{\frac{1}{2} \rho U^2 \delta S} = \frac{L}{\rho U^2 (bc)} \]  

(4.84)

where \( S = 2bc \) is the total wing surface, \( b \) is the wing half-span and \( c \) is the wing chord.

Here also, the wake vortex moves with the local fluid velocity. So, at each time step, the wake vortex displacement for the \( m \)th wake vortex can be obtained as follows:

\[ \Delta x_m = u_m \Delta t ; \quad \Delta y_m = v_m \Delta t; \quad \Delta z_m = w_m \Delta t \]  

(4.85)

where \( \Delta x_m, \Delta y_m, \Delta z_m \) are the translation of the \( m \)th wake vortex along \( x, y, z \) directions, respectively. Here \( u_m, v_m, w_m \) are the velocity components induced at the \( m \)th wake vortex due to the panel vortices and all the wake vortices which can be defined as shown below:

\[ (u_m, v_m, w_m) = \sum_{L=1}^{G} (u_{mL}, v_{mL}, w_{mL}) + \sum_{r=1}^{N_W} (u_{mWR_r}, v_{mWR_r}, w_{mWR_r}) \]  

(4.86)

or using the command line for \textit{VRing} subroutine as below:

\[ (u_m, v_m, w_m) \]

\[ = \sum_{L=1}^{G} \text{VRing}(x_m, y_m, z_m, i, j, \Gamma_L) \]

\[ + \sum_{r=1}^{N_W} \text{VRing}(x_m, y_m, z_m, i_w, j_w, \Gamma_{WR_r}) \]

where \( G \) is the total number of the wing panels and \( N_W \) is the total number of wake panels.
4.5 Model Validation

By writing a program in FORTRAN, the 3-D aerodynamic model presented in the previous sections can be validated by comparing the results of the present method with similar results in the published literature. In order to do this, the FORTRAN code solves the set of equations to calculate the steady and unsteady pressure distributions and consequently the generalized aerodynamic loadings. The following sections present the comparisons of the results related to firstly the steady 3-D flow, and then the unsteady 3-D flow.

It is worth mentioning that although some of the comparisons such as those related to very high-aspect-ratio wings or very low amplitude oscillations are not the main focus of this thesis, they are performed merely due to the lack of the related present literature. However, it is shown in the comparisons presented that the method developed in this thesis has good agreement with the published results.

4.5.1 Steady Aerodynamic Results

In order to verify the 3-D steady aerodynamic model, a trapezoidal wing is considered at an angle of attack $\alpha_0 = 5$ (deg) relative to the steady air flow. The results in the form of the spanwise variation of the non-dimensional panel circulation $\bar{\Gamma}(y)$ are presented where $\bar{\Gamma}(y)$ can be defined as follows:

$$\bar{\Gamma}(y) = \frac{\Gamma_j}{2b\alpha_0 U_\infty} \quad (4.87)$$

where $b$ is the wing half-span and $U_\infty$ is the constant flow-stream velocity. Here, $\Gamma_j$ is the spanwise distribution of the circulation which can be defined as the summation of the vortex strengths in each column $j$:

$$\Gamma_j = \sum_{i=1}^{M} \Gamma_{ij} \quad (4.88)$$
The results of the present method are compared with the results of Mateescu et al. (2003) for two values of aspect ratios ($AR = 9, 12$) and two values of the taper ratios ($\lambda = 0, 0.6$). Figure 4.13 shows the spanwise distribution of $\Gamma(y)$ for two values of aspect ratios $AR = 9, 12$ for a rectangular wing (taper ratio $\lambda = 0$).

Similar results are presented in Figure 4.14 for a trapezoidal wing with a taper ratio of $\lambda = 0.6$. Both graphs show a good agreement with the results of the published literature.

In addition, to have a better understanding of the vortex distribution on the wing, the chordwise distribution of wing circulation, $\Gamma_{ij}$ are shown in Figure 4.15 for a wing with an aspect ratio of $AR = 9$. The number of panels along the chord and span are $M = 20$ and $N = 5$, respectively. The value of $\Gamma_{ij}$ can be defined as:

$$\Gamma_{ij} = \frac{\Gamma_{ij}}{2b\alpha_0 U_\infty} \quad (4.89)$$

where $\Gamma_{ij}$ is the circulation of the panel $ij$. The results of $\Gamma_{ij}$ are presented for two columns of panels, close to the wing root ($j = 1$) and close to the wing tip ($j = 5$).
Figure 4.13 Spanwise distribution of $\bar{\Gamma}(y)$ for a rectangular wing ($\lambda = 0$) for 2 aspect ratios

Figure 4.14 Spanwise distribution of $\bar{\Gamma}(y)$ for a tapered wing ($\lambda = 0.6$) for 2 aspect ratios
Figure 4.15 Chordwise distribution of the $\Gamma_i^j$ on a rectangular wing with $AR = 9$.

### 4.5.2 Unsteady Aerodynamic Results

In this section, firstly the unsteady 3-D aerodynamic model developed in Section 4.4 is validated with the published literature, and secondly, the results for various cases which are more focus of the present thesis are presented.

As mentioned earlier, due to the lack of the related existing published results for the unsteady motion of the 3-D wings, the results of the present model are compared with similar results of the steady 3-D and unsteady 2-D results. For this reason, the unsteady results for the limit case of very small amplitude oscillations are compared with the steady results of Mateescu et al. (2003). In addition, the unsteady results of the 3-D wing are compared with the unsteady results of the 2-D airfoils published by Mateescu (2011) for the case of high-aspect-ratio wings.

Initially, the aerodynamic loadings of a 3-D wing situated in the unsteady flow are compared with those presented by Mateescu et al. (2003) for the steady flow.
The wing is executing the rigid pitch oscillations in the cosine form where the amplitude of oscillation is considered to be very small. The pitching oscillations are assumed to be caused by changes of the flow angle of attack \( \alpha(t) \) due to the gusts in the general cosine form as:

\[
\alpha(t) = \alpha_0 + \alpha_A \cos 2\pi f t
\]  

(4.90)

where \( \alpha_0 = 5 \) (deg) and \( \alpha_A = 0.01 \) (deg) where the small value of 0.01 degree has been chosen to better simulate the steady case. Here \( f \) is the frequency of the oscillations which can be related to the non-dimensional reduced frequency as shown below:

\[
k = \frac{\pi f c}{U_\infty}
\]  

(4.91)

The period of oscillation is the inverse of the frequency and the time step can be assumed as follows:

\[
T = \frac{1}{f} \quad \Rightarrow \quad \Delta t = \frac{T}{80}
\]  

(4.92)

The time variation of lift coefficient for various aspect ratios from 3 to 12 are compared with Mateescu et al. (2003) and are presented in Figure 4.16. It is worth mentioning that there is a transient part at the first time steps which is removed to facilitate the comparison. The results obtained demonstrate a close agreement between the two models for various wing aspect ratios.
To examine the validity of the model when the unsteady effects are of more importance, a comparison is performed between the three-dimensional unsteady results of the present model and two-dimensional unsteady results of Mateescu (2011) as well as the Theodorsen method. In order to be able to compare the aerodynamic loading of a 3-D model with a 2-D one, first the wing is considered to have a high aspect ratio. Secondly, since the effect of the wing tip vortex shedding can be considered negligible close to the wing root, the results related to the first column of panels close to the wing root is compared with the 2-D results.

The comparison is performed for the lift coefficient values of a wing with aspect ratios of 6 and 12, which execute the rigid pitching oscillating in the cosine form as described in equation (4.90). The variation of the local lift coefficient in one cycle of oscillations is compared with the 2-D lift coefficient of Mateescu (2011) and Theodorsen method and are shown in Figures 4.17 and 4.18 for $AR = 12$ and $AR = 6$, respectively.
The local lift coefficient for the first column of panels close to the wing root $C_{L_1}$ can be defined as follows:

\[ C_{L_1} = \frac{\Delta L_1}{\left(\frac{1}{2} \rho U_\infty^2\right) S_1} \]  

(4.93)

where $\Delta L_1$ is the lift on the first column and $S_1$ is the area of the first column. If the number of panels along the span direction ($y$ axis) is $N$ and it is assumed that the panels have the equal width, the value of $S_1$ can be obtained as: $S_1 = cb/N$ where $b, c$ are the half span and chord of the wing. Hence:

\[ C_{L_1} = \frac{\Delta L_1}{\left(\frac{1}{2} \rho U_\infty^2\right) (bc/N)} \]  

(4.94)

Here, the free-stream velocity $U_\infty = 50$ (m/s) is considered being along $x$ axis. The pitching oscillations are considered for $\alpha_0 = 0, \alpha_A = 5$ (deg) and the frequency is assumed to be $f = 5$ (Hz). The number of panels along the chord and span directions are $M = 10, N = 30$, respectively, and the results are compared for the case of incompressible flow ($M = 0$) where the reduced frequency is $k = 0.05$, where $k$ can be defined as shown in equation (4.91).

It can be observed from Figures 4.17 and 4.18 that the results of the present method have close agreement with the published literature, especially for the higher value of aspect ratio ($AR = 12$).

The time history of the local lift coefficients for the cases shown in Figures 4.17 and 4.18, are presented versus time in Figures 4.19 and 4.20. It should be noted that the transient response in the first cycle of oscillations, which is due to the wake build-up, vanishes after one cycle.

Additionally, the results of the unsteady flexural oscillations are compared to similar results of Mateescu (2011). The wing surface is considered having the flexural parabolic deformations in the following form:
Figure 4.17 Variation of unsteady lift coefficient with angle of attack \((k = 0.05, AR = 12)\)

Figure 4.18 Variation of unsteady lift coefficient with angle of attack \((k = 0.05, AR = 6)\)
Figure 4.19 Variation of unsteady lift coefficient with time \((k = 0.05, \ AR = 12)\)

Figure 4.20 Variation of unsteady lift coefficient with time \((k = 0.05, \ AR = 6)\)
\[ z = \eta(x, y, t) = e_0 \left( \frac{x}{c} \right)^2 \exp(i2\pi ft) \]  

(4.95)

where \( \eta \) is the coordinate of surface, \( e_0 \) is the amplitude of the oscillations, and \( i = \sqrt{-1} \).

The chordwise distribution of the pressure difference coefficient \( \Delta \hat{C}_p/e_0 \) is compared for the case of \( k = 0.05 \) with Mateescu (2011) and is shown in Figures 4.21 and 4.22 for real and imaginary parts of pressure coefficients, respectively. The reduced pressure difference coefficient is defined as:

\[ \Delta C_p = \Delta \hat{C}_p \exp(i2\pi ft) \]  

(4.96)

which is a time independent term. It can be observed that the results presented in Figures 4.21 and 4.22 show good agreement with the published literature.

\[ \]

Figure 4.21 Chordwise distribution of the real part of the pressure difference coefficient on a wing executing parabolic flexural oscillations
Figure 4.22 Chordwise distribution of the imaginary part of the pressure difference coefficient on a wing executing parabolic flexural oscillations
5.1 Introduction

In this chapter, the dynamics model and the finite element formulation for the wing structure with bonded piezoelectric strips are presented. In the first section, the analytical expressions to calculate the voltage in the sensor and actuator strips for the special case of a beam are derived using the variational principle. These expressions give an understanding of the pertinent parameters in computing the sensor/actuator voltages, and are compared later to the results of the finite element formulation in Chapter 7.
In the subsequent sections, the finite element formulation for the piezoelectric sensors/actuators is presented by defining the interpolating functions for each element of the structure. The finite element model of the wing structure is then described. Finally, the implementation of this formulation in the finite element software ANSYS is explained and the choice of elements is discussed.

Although most of the material in this chapter are available in advanced literature, they have been compiled here to provide the readers with certain background and prepare them for the next chapter.

## 5.2 Piezoelectric Sensors and Actuators:

In Chapter 2, an introduction to the piezoelectric materials and their properties was presented and a comparison between the properties of the PZT and PVDF was carried out. It was also mentioned that, in this thesis, the piezoelectric materials are used as both actuators and sensors placed on the top and bottom surface of the wing, respectively. The piezoelectric strips are assumed to be made of PZT and are bonded onto the wing surface. The contact between the piezoelectric strips and the surface of the wing structure is assumed to be ideal.

In this thesis, the output voltage of the piezoelectric sensors is used in the feedback control law to determine the voltage to be applied to the piezoelectric actuators. Therefore, in the following sections the required formulations to calculate the voltage of the piezoelectric sensors and actuators are described to better understand the behavior of the active structures and the involved parameters. Analytical expressions are obtained for a simplified case of a beam undergoing pure bending.

### 5.2.1 Calculation of Sensor Voltage

To calculate the output voltage of a piezoelectric sensor, a piezoelectric strip with zero electric field and non-zero load is considered. As explained in Chapter
2, the constitutive equation of the piezoelectric materials in a one-dimensional medium can be expressed in the Strain-Charge form as follows (Preumont 2002):

\[
\begin{align*}
T &= c_E S - eE \\
D &= eS + \varepsilon_S E
\end{align*}
\] (5.1)

where \( T \) and \( S \) are the mechanical stress and strain, respectively, and \( E \) is the electric field. Here, \( D \) is the electric displacement or the charge per unit area, \( c_E = 1/s_E \) is the elastic constant under constant electric field, and \( e = d/s_E \) is the constant relating the electric displacement to the strain, in the absence of the electric field. When the electric field \( E \) is equal to zero, the second equation in equation (5.1) yields:

\[
D = eS
\] (5.2)

The geometrical arrangement of the piezoelectric strip during the manufacturing process determines the direction of the polarization and causes the useful direction of expansion to be normal to that of the electric field. Hence, if the polarization direction is along the \( x_3 \) (or \( z \)) axis (refer to Figure 2.5), the useful expansion is along the \( x_1 \) (or \( x \)) axis and the activation capability is governed by the piezoelectric constant \( d_{31} \). Equation (5.2) can be expressed using the conventional mechanical engineering notations as follows:

\[
D = E_p d_{31} \varepsilon_{11}
\] (5.3)

where \( E_p \) is the Young’s modulus of the piezoelectric material, \( d_{31} \) is the piezoelectric constant, and \( \varepsilon_{11} \) is the axial strain. Based on the Euler-Bernoulli assumption, the axial deformation and the curvature are related by:

\[
\varepsilon_{11} = -z \omega''
\] (5.4)

Substitution of the strain \( \varepsilon_{11} \) into equation (5.3) can lead to the following equation for charge per unit area:

\[
D = -E_p d_{31} t_b \omega''
\] (5.5)
where \( t_b \) is the thickness of the wing structure and \( \omega \) is the transverse deformation. Here, the double prime symbol denotes second derivative with respect to \( x \). Equation (5.5) can be integrated over the piezoelectric element area to obtain the electric charge as:

\[
Q = \int_{x_0}^{x_0+w_p} Dl_p \, dx = -E_p t_b l_p \int_{x_0}^{x_0+w_p} d_{31} \omega'' \, dx \tag{5.6}
\]

where \( w_p \) and \( l_p \) are the sensor length along \( x \) and \( y \) axes, as illustrated in Figure 5.1. Assuming that the polarization profile is uniform, \( d_{31} \) would be constant and one can obtain:

\[
Q = -E_p t_b l_p d_{31} \int_{x_0}^{x_0+w_p} \omega'' \, dx \tag{5.7}
\]

The electrodes can be connected to either a current amplifier (Figure 5.2) or a charge amplifier (Figure 5.3). The output voltage of a current amplifier can be expressed as shown below (Preumont 2002):

\[
V_s(t) = -R_f i_s(t) = -R_f \dot{Q} \tag{5.8}
\]
where $V_s$ is the sensor output voltage, $R_f$ is the current amplifier constant and $i_s$ is the current, defined by $i_s(t) = \dot{Q}$, where dot denotes differentiation with respect to time. Equation (5.7) can be employed in equation (5.8) to obtain:

$$V_s(t) = -R_f \dot{Q} = R_f E_p t_b l_p d_{31} \int_{x_0}^{x_0+w_p} \dot{\omega}'' \, dx$$

(5.9)

Consequently, by integrating over the length of the piezoelectric sensor one can write:

$$V_s(t) = K_c \left[ \dot{\omega}'(x_0 + w_p) - \dot{\omega}'(x_0) \right]$$

(5.10)

where $K_c = R_f E_p t_b l_p d_{31}$ is a constant.
If the piezoelectric sensor is connected to a charge amplifier (Figure 5.3), the output sensor voltage is similarly calculated as:

$$V_s(t) = -\frac{Q}{C_f} = \frac{E_p t_b l_p d_{31}}{C_f} \int_{x_0}^{x_0+w_p} \omega'' \, dx$$

(5.11)

where $C_f$ is the charge amplifier gain. Integrating over the length of the piezoelectric sensor can result in the following expression:

$$V_s(t) = \frac{E_p t_b l_p d_{31}}{C_f} \left[ \omega'(x_0 + w_p) - \omega'(x_0) \right]$$

(5.12)

or simply:

$$V_s(t) = K_f \left[ \omega'(x_0 + w_p) - \omega'(x_0) \right]$$

(5.13)

where $\omega'$ is the slope at any point of the piezoelectric strip and $K_f = \frac{E_p t_b l_p d_{31}}{C_f}$ is a constant which is related to the properties of the structure and the sensor. If $\theta = \omega'$ is the slope at the end point of the sensor, the difference of slopes at the ends of the sensor is denoted by $\Delta \theta = \theta_{x_0+w_p} - \theta_{x_0}$. Hence, one can observe that the output charge and consequently the output voltage of the piezoelectric sensor is proportionally related to $\Delta \theta$.

In this thesis, the charge amplifier is utilized to predict the analytical values of the sensor voltages and compare the results with the finite element formulation.

### 5.2.2 Calculation of Actuator Voltage

As explained in Chapter 2, there are two arrangements for piezoelectric strips: collocated and non-collocated. In the collocated arrangement, two identical piezoelectric strips are placed at the same location on the upper and lower surfaces of the wing. The piezoelectric sensor and actuator are 180 degrees out of phase with the same applied voltage which causes one strip to expand while the
other one contracts. This out-of-phase behavior is shown in Figure 5.4 for a beam with a pair of collocated piezoelectric strips. Since it is assumed that the strips are glued perfectly to the surface, by applying equal voltage of 180 degrees phase difference, the same tension and compression occurs on the upper and lower surfaces of the structure which is equivalent to applying a concentrated moment.

By applying a voltage $V_a$ to an unconstrained piezoelectric strip in the polarization direction, the electric field created is $E = V_a/t_p$ and the strain in the $x$ direction can be obtained as (Piefort 2001):

$$\varepsilon_p = \frac{d_{31}V_a}{t_p}$$  \hspace{1cm} (5.14)

where $t_p$ is the thickness of the piezoelectric strip. Assuming that the strain distribution is linear across the thickness of the beam, one can write (Moheimani and Fleming 2006):

$$\varepsilon(x) = C_0z + \varepsilon_0$$  \hspace{1cm} (5.15)

where $C_0$ is the slope, $z$ is the distance from the neutral axis, and $\varepsilon_0$ is an intercept. If the piezoelectric strips are glued symmetrically about the neutral axis (as shown in Figure 5.4), the strain distribution is symmetric and $\varepsilon_0$ is equal to zero.
Using Hooke’s law, the bending stress distribution $\sigma_b$ within the beam is written as:

$$\sigma_b(z) = E_b C_0 z \quad (5.16)$$

where $E_b$ is the Young’s modulus of the beam. When the piezoelectric strip is bonded to the beam, its movements are constrained by the stiffness of the beam. Hence, the stress distribution within the upper and lower piezoelectric strips can be expressed in terms of the total strain in each strip as:

$$\sigma_p^u(z) = E_p (C_0 z - \varepsilon_p) \quad (5.17)$$

$$\sigma_p^l(z) = E_p (C_0 z + \varepsilon_p) \quad (5.18)$$

where $\sigma_p$ is the stress of the piezoelectric strip, $E_p$ is the Young’s modulus of the piezoelectric material and superscripts $u$ and $l$ denotes the upper and lower strips, respectively.

The unknown parameter $C_0$ can be found by applying the moment equilibrium about the center of the beam as follows:

$$\int_{-t_b}^{-t_b-t_p} \sigma_p^l(z) z \, dz + \int_{-t_b}^{t_b} \sigma_b(z) z \, dz + \int_{t_b}^{t_b+t_p} \sigma_p^u(z) z \, dz = 0 \quad (5.19)$$

By substituting equations (5.16) to (5.18), one can find:

$$C_0 = \frac{\varepsilon_p E_p t_p (2t_b + t_p)}{2[E_p (3t_b^2 t_p + 3t_b t_p^2 + t_p^3) + E_b t_p^3]} \quad (5.20)$$

The induced moment distribution (moment per unit length) $M$ in the beam can be determined by (Moheimani and Fleming 2006):

$$M = E_b I C_0 \quad (5.21)$$
where $I$ is the moment of inertia of the cross section of the beam about the neutral axis of the beam. As a result, by replacing the value of $\varepsilon_p$ from equation (5.14), equation (5.21) can be recast as:

$$M = E_b I C_1 \mathcal{V}_a$$

(5.22)

where:

$$C_1 = \frac{C_0}{\mathcal{V}_a}$$

(5.23)

or:

$$C_1 = \frac{3E_p d_{31}(2t_b + t_p)}{2[E_p(3t_b^2 t_p + 3t_b t_p^2 + t_p^3) + E_b t_b^3]}$$

(5.24)

It is worth noting that in equation (5.21), all the terms except the actuator applied voltage $\mathcal{V}_a$ are related to the geometrical and material properties of the structure.

### 5.3 Finite Element Approach

Although a closed-form solution can be obtained for simple cases similar to the one considered in section 2.6, the dynamic equations for complicated structures, particularly in the presence of smart materials, cannot be solved analytically for most of the cases. Hence, a numerical approach such as the finite element method must be used.

The basic steps in the finite element methods can be described as follows. First the continuum is divided into a finite number of elements with simple geometrical shapes. The elements are connected together through a finite number of nodes. For each element, an interpolation function can be selected to describe the unknown displacement field. By replacing the applied loads with equivalent elemental forces, the differential equations can be solved for each element to calculate the unknown nodal displacements.
In the following sections, the finite element formulation for the piezoelectric strips first, and then for the wing structure, are presented.

5.4 Finite Element Model of the Piezoelectric Sensors and Actuators

As mentioned earlier, in the finite element approach, the structure is divided into a finite number of elements. To obtain the finite element formulation, a variational method is applied for each element. The appropriate variational method here is the Hamilton’s principle that yields the equations of motion of the piezoelectric continuum, while taking into account the constitutive equations of the piezoelectric material which was discussed in Chapter 2. The Hamilton’s principle is the generalized form of the virtual work principle in which, the Lagrangian and the virtual work are properly adapted to include the electrical effects as well as the mechanical ones [Allik and Hughes (1970), Piefort (2001)].

5.4.1 Equations of Motion of the Piezoelectric Continuum

The equations of motion of a piezoelectric continuum can be derived using the Hamilton’s principle (Tiersten 1967):

\[
\int_{t_1}^{t_2} (\delta L + \delta W) dt = 0
\]  

(5.25)

where \( L \) is the Lagrangian, \( \delta W \) is the virtual work done by the external mechanical and electrical forces. Here, \( \delta \) is the variational operator and \( t_1 \) and \( t_2 \) are the limits of time. In the variational method, all the variations should vanish at \( t = t_1 \) and \( t = t_2 \). Moreover, for the special case of conservative systems:

\[
\delta \int_{t_1}^{t_2} L dt = 0
\]  

(5.26)
The Lagrangian $\mathcal{L}$ can be defined as the kinetic energy $K$ minus the electrical enthalpy $H$:

$$\mathcal{L} = \int_V (K - H) dV \quad (5.27)$$

The kinetic energy $K$ can be written as:

$$K = \frac{1}{2} \rho \{\dot{\mathbf{u}}\}^T \{\dot{\mathbf{u}}\} \quad (5.28)$$

where $\{\dot{\mathbf{u}}\}$ is the velocity field and $\rho$ is the mass density.

If $U$ is the strain energy density for the piezoelectric continuum, the electrical enthalpy $H$ can be expressed as (Tiersten 1967):

$$H = U - E_i D_i \quad (5.29)$$

where the Cartesian components of the electric field intensity and the electric displacement are denoted by $E_i$ and $D_i$, respectively. Here, indices $i = 1,2,3$ indicate the components of the Cartesian coordinate frame where $x_1, x_2, x_3$ are equivalent to $x, y, z$ axes, respectively. Repeated indices imply summation.

For the linear piezoelectric continuum, the conversion of energy can result in the first law of thermodynamics for piezoelectric medium (Tiersten 1969):

$$dU = T_{ij} dS_{ij} + E_i dD_i \quad (5.30)$$

where $T_{ij}$ and $S_{ij}$ are respectively the stress and the strain matrices. By combining equations (5.29) and (5.30), one can write:

$$dH = T_{ij} dS_{ij} - D_i dE_i \quad (5.31)$$

As a result, the following expressions for $T_{ij}$ and $D_i$ can be obtained:

$$T_{ij} = \frac{\partial H}{\partial S_{ij}} \quad (5.32)$$

$$D_i = -\frac{\partial H}{\partial E_i} \quad (5.33)$$
As mentioned in Chapter 2, the constitutive equations of the piezoelectric materials can be shown in the Stress-Charge form as follows:

\[
\{T\} = [c_E][S] - [e]^T[E] \\
\{D\} = [e][S] + [\varepsilon_S][E]
\] (5.34)

Motivated by (5.32) to (5.34), we assume (Tiersten 1967):

\[
H = \frac{1}{2} c_{ijkl} S_{ij} S_{kl} - e_{klj} E_k S_{ij} - \frac{1}{2} \varepsilon_{ij}^S E_i E_j
\] (5.35)

where \( c_{ijkl} \), \( e_{klj} \), and \( \varepsilon_{ij}^S \) are the elastic, piezoelectric, and dielectric constants, respectively. Equation (5.35) can be written in the matrix form as shown below by utilizing equation (5.34):

\[
H = \frac{1}{2} [(S)^T{T} - (E)^T{D}]
\] (5.36)

Consequently, the Lagrangian can be stated as:

\[
L = \int_V (K - H) dV = \int_V \left[ \frac{1}{2} \rho (\dot{u})^T{T}\{ \dot{u} \} - \frac{1}{2} (S)^T{T}\{T\} + \frac{1}{2} (E)^T{D}\right] dV
\] (5.37)

The virtual work done by the external mechanical load and the electrical charge, for an arbitrary variation of the displacement field \( \delta u \) and of the electrical potential \( \delta \phi_E \) can be expressed as (Allik and Hughes 1970):

\[
\delta W = [\delta u]^T\{F\} - \delta \phi_E \sigma
\] (5.38)

where \( \delta W \) denotes the virtual work density, \( \{F\} \) is the external mechanical force, \( \sigma \) is the electrical charge density and \( \delta \) is variational operator. There is a useful analogy between the electrical and mechanical variables which is presented in Table 5.1. Note that the electrical tensors in Table 5.1 are one tensorial degree lower than the corresponding mechanical ones.

The external mechanical loads which are applied on the structure can include the body forces \( F_b \), surface forces \( F_s \), and point forces \( F_p \). The total electric
charge also includes the body charges $\sigma_b$, surface charges $\sigma_s$ and point charges $q_E$. By substituting all the mechanical and electrical loads in equation (5.38), the virtual work $\delta W$ can be given by:

$$\delta W = \int_V \left[ \{\delta u\}^T \{F_b\} - \delta \phi_E \sigma_b \right] dV + \int_{S_1} \{\delta u\}^T \{F_s\} dS$$

$$- \int_{S_2} \delta \phi_E \sigma_s dS + \{\delta u\}^T \{F_p\} - \delta \phi_E q_E \tag{5.39}$$

The variation of the kinetic energy $\rho \{\delta \dot{u}\}^T \{\dot{u}\}$ can be integrated by parts over the time interval $t_1$ and $t_2$ as shown below:

$$\int_{t_1}^{t_2} \rho \{\delta \dot{u}\}^T \{\dot{u}\} \, dt = \int_{t_1}^{t_2} \left[ \rho \{\delta \dot{u}\}^T \{\dot{u}\} \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho \{\delta \dot{u}\}^T \{\dot{u}\} \, dt \tag{5.40}$$

In this equation, the first term in the right-hand side is equal to zero because $\{\delta u\}$ vanishes at $t = t_1$ and $t = t_2$.

By substituting the Lagrangian and virtual work expressions into equation (5.25) while taking into account the constitutive equations of (5.34), one can get:

$$\int_V \left( -\rho \{\delta u\}^T \{\dot{u}\} - \{\delta S\}^T [c_E] \{S\} + \{\delta S\}^T \{e\} \{E\} \right.$$

$$+ \{\delta E\}^T \{e\} \{S\} + \{\delta E\}^T [\varepsilon_S] \{E\} + \{\delta u\}^T \{F_b\} - \delta \phi_E \sigma_b \right) \, dV \tag{5.41}$$
\begin{equation}
+ \int_{S_1} \{\delta \mathbf{u}\}^T \{\mathbf{F}_s\} \, dS - \int_{S_2} \delta \phi_E \sigma_s \, dS + \{\delta \mathbf{u}\}^T \{\mathbf{F}_p\} - \delta \phi_E q_E = 0
\end{equation}

5.4.2 Finite Element Formulation

Introducing \([\mathbf{N}_u]\) and \([\mathbf{N}_\phi]\) as shape functions to relate the displacement field \(\{\mathbf{u}\}\) and electric potential \(\phi_E\) to their corresponding nodal values \(\{u_i\}\) and \(\{\phi_i\}\), one obtains:

\begin{equation}
\{\mathbf{u}\} = [\mathbf{N}_u] \{u_i\} \quad (5.42)
\end{equation}

\begin{equation}
\phi_E = [\mathbf{N}_\phi]^T \{\phi_i\} \quad (5.43)
\end{equation}

By defining \([\mathbf{D}]\) as the derivation operator, the following relation for the strain field holds:

\begin{equation}
\{\mathbf{S}\} = [\mathbf{D}] \{\mathbf{u}\} \quad (5.44)
\end{equation}

where (Cook 2001):

\begin{equation}
[\mathbf{D}] = \begin{bmatrix}
\partial_x & 0 & 0 \\
0 & \partial_y & 0 \\
0 & 0 & \partial_z \\
0 & \partial_z & \partial_y \\
\partial_z & 0 & \partial_x \\
\partial_y & \partial_x & 0
\end{bmatrix}
\end{equation}

Substitution of equation (5.42) into equation (5.44) results in:

\begin{equation}
\{\mathbf{S}\} = [\mathbf{D}] [\mathbf{N}_u] \{u_i\} = [\mathbf{B}_u] \{u_i\} \quad (5.46)
\end{equation}

where \([\mathbf{B}_u]\) is the derivative of the shape functions \([\mathbf{N}_u]\). Similarly, the electrical field \(\{\mathbf{E}\}\) can be stated by defining the gradient operator \(\nabla\) as follows:

\begin{equation}
\{\mathbf{E}\} = -\nabla [\mathbf{N}_\phi] \{\phi_i\} = -[\mathbf{B}_\phi] \{\phi_i\} \quad (5.47)
\end{equation}

Here, the derivative of the shape functions \([\mathbf{N}_\phi]\) is denoted by \([\mathbf{B}_\phi]\) similar to \([\mathbf{B}_u]\).
Equation (5.41) can now be recast by replacing equations (5.42), (5.43), (5.46), and (5.47) as shown below:

\[- \{ \delta \mathbf{u}_i \}^T \int_V \rho [\mathbf{N}_u]^T [\mathbf{N}_u] dV \{ \ddot{\mathbf{u}}_i \} - \{ \delta \mathbf{u}_i \}^T \int_V [\mathbf{B}_u]^T [\mathbf{c}_E] [\mathbf{B}_u] dV \{ \mathbf{u}_i \} \]

\[- \{ \delta \mathbf{u}_i \}^T \int_V [\mathbf{B}_u]^T [\mathbf{e}] [\mathbf{B}_\phi] dV \{ \phi_i \} - \{ \delta \phi_i \}^T \int_V [\mathbf{B}_\phi]^T [\mathbf{e}]^T [\mathbf{B}_u] dV \{ \mathbf{u}_i \} \]

\[+ \{ \delta \phi_i \}^T \int_V [\mathbf{B}_\phi]^T [\mathbf{e}_S] [\mathbf{B}_\phi] dV \{ \phi_i \} + \{ \delta \mathbf{u}_i \}^T \int_V [\mathbf{N}_u]^T \{ \mathbf{F}_b \} dV \]

\[- \{ \delta \phi_i \}^T \int_V [\mathbf{N}_\phi]^T \sigma_b dV + \{ \delta \mathbf{u}_i \}^T \int_{S_1} [\mathbf{N}_u]^T \{ \mathbf{F}_s \} dS \]

\[- \{ \delta \phi_i \}^T \int_{S_2} [\mathbf{N}_\phi]^T \sigma_s dS + \{ \delta \mathbf{u}_i \}^T [\mathbf{N}_u]^T \{ \mathbf{F}_p \} - \{ \delta \phi_i \}^T [\mathbf{N}_\phi]^T q_E = 0 \quad (5.48) \]

This equation should be valid for any arbitrary variation of the displacement field \{ \delta \mathbf{u}_i \} and electrical potential \{ \delta \phi_i \}. Hence, equation (5.48) for an element can be written as:

\[
[M] \{ \ddot{\mathbf{u}}_i \} + [K_{uu}] \{ \mathbf{u}_i \} + [K_{u\phi}] \{ \phi_i \} = \{ f_i \} \\
[K_{\phi u}] \{ \mathbf{u}_i \} + [K_{\phi \phi}] \{ \phi_i \} = \{ g_i \} \quad (5.49)
\]

where \([M]\) is the mass matrix, \([K_{uu}]\) is the stiffness matrix, \([K_{u\phi}]\) is the piezoelectric coupling matrix, and \([K_{\phi \phi}]\) is the capacitance matrix and are defined as follows:

\[
[M] = \int_V \rho [\mathbf{N}_u]^T [\mathbf{N}_u] dV \quad (5.50)
\]

\[
[K_{uu}] = \int_V [\mathbf{B}_u]^T [\mathbf{c}_E] [\mathbf{B}_u] dV \quad (5.51)
\]

\[
[K_{u\phi}] = \int_V [\mathbf{B}_u]^T [\mathbf{e}] [\mathbf{B}_\phi] dV \quad (5.52)
\]

\[
[K_{\phi \phi}] = -\int_V [\mathbf{B}_\phi]^T [\mathbf{e}_S] [\mathbf{B}_\phi] dV \quad (5.53)
\]
It should be noted that: \( [K_{\phi u}] = [K_{uu}]^T \). In addition, \( \{f_i\} \) is the external mechanical force, and \( \{g_i\} \) is the applied electric charge as defined below:

\[
\{f_i\} = \int_V [N_u]^T \{F_b\} dV + \int_{S_1} [N_u]^T \{F_s\} dS + [N_u]^T \{F_p\}
\]

\[
\{g_i\} = -\int_V [N_\phi]^T \sigma_b dV - \int_{S_2} [N_\phi]^T \sigma_z dS - [N_\phi]^T q_E
\]

Equivalently, the governing equation (5.49) can be written in the compact form as shown below:

\[
\begin{bmatrix}
[M] & [0] \\
[0] & [0]
\end{bmatrix}
\begin{bmatrix}
\{\ddot{u}\}\{\dot{u}\}\{u\}
\end{bmatrix} +
\begin{bmatrix}
[K_{uu}] & [K_{u\phi}] \\
[K_{\phi u}] & [K_{\phi\phi}]
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{\phi\}
\end{bmatrix} =
\begin{bmatrix}
\{f\} \\
\{g\}
\end{bmatrix}
\]

Equation (5.56) applies to each element in the model. The element matrices can be assembled for the entire system to yield:

\[
\begin{bmatrix}
[M^*] & [0] \\
[0] & [0]
\end{bmatrix}
\begin{bmatrix}
\{\dddot{U}\} \\
\{\dot{U}\}
\end{bmatrix} +
\begin{bmatrix}
[K_{uu}^*] & [K_{u\phi}^*] \\
[K_{\phi u}^*] & [K_{\phi\phi}^*]
\end{bmatrix}
\begin{bmatrix}
\{U\} \\
\{\phi\}
\end{bmatrix} =
\begin{bmatrix}
\{F^*\} \\
\{G^*\}
\end{bmatrix}
\]

where \( \{U^*\} \) and \( \{\phi^*\} \) comprise the global degrees of freedom of the model. Here, \( [M^*] \), \( [K_{uu}^*] \), \( [K_{u\phi}^*] \), and \( [K_{\phi\phi}^*] \) are the assembled matrices of mass, stiffness, piezoelectric coupling, and capacitance matrices, respectively.

It is sometimes more convenient to work with the non-compact form of equation (5.57) as expressed below:

\[
[M^*]\{\dddot{U}\} + [K_{uu}^*]\{U\} + [K_{u\phi}^*]\{\phi\} = \{F^*\}
\]

\[
[K_{\phi u}^*]\{U\} + [K_{\phi\phi}^*]\{\phi\} = \{G^*\}
\]

Equations (5.58) and (5.59) clearly show the coupling between the mechanical and electrical variables in the model. For particular cases of the electrical boundary conditions, such as short-circuited or open electrode, this coupled governing equation set can be solved explicitly.
For voltage driven electrodes where the electric potential $\phi^*$ is specified, equation (5.58) results in:

$$[M^*]\{\ddot{U}^*\} + [K_{uu}^*]\{U^*\} = \{F^*\} - [K_{u\phi}^*]\{\phi^*\}$$

(5.60)

where $[K_{u\phi}^*]\{\phi^*\}$ represents the equivalent piezoelectric loads. The mechanical displacement can be solved form this equation and consequently substituted in equation (5.59) to calculate the electric charges. It can be observed that for short-circuited electrodes where $\{\phi^*\} = 0$, equation (5.60) is identical to the case where no piezoelectric electromechanical coupling exists. Then, one can obtain the eigenvalue problem:

$$([K_{uu}^*] - \omega^2[M^*])\{U^*\} = 0$$

(5.61)

which shows that the natural frequencies and mode shapes for this short-circuited system is the same as those for non-piezoelectric system.

For open electrodes, the electric charge is zero ($\{G^*\} = 0$), hence, $\{\phi^*\}$ can be obtained from equation (5.59) as shown below:

$$\{\phi^*\} = -[K_{\phi\phi}^*]^{-1}[K_{\phi u}^*]\{U^*\}$$

(5.62)

Substituting equation (5.62) in equation (5.58), one can obtain:

$$[M^*]\{\ddot{U}^*\} + \left([K_{uu}^*] - [K_{u\phi}^*][K_{\phi\phi}^*]^{-1}[K_{\phi u}^*]\right)\{U^*\} = \{F^*\}$$

(5.63)

Since the matrix $[K_{\phi\phi}^*]$ is negative definite, the overall stiffness of the system increases if the electrodes are open. The natural frequencies of the system in this case are also greater than those of the short-circuited electrodes.

Equation (5.56) is used in the finite element software ANSYS to solve the piezoelectric coupled problems. For completeness of the equations of motion, damping matrices can be added as shown below:
where \([ C]\) is the structural damping matrix and \([ C_{\phi}]\) is the dielectric damping matrix (ANSYS 2009).

## 5.5 Finite Element Model of the Wing Structure

As explained in Chapter 2, in this thesis, the wing structure is considered as a cantilever plate. The finite element formulation of the cantilever plate can be obtained from the principle of the virtual work similar to that for the piezoelectric strips and is not repeated here. Hence, the elemental finite element formulation of the wing structure for nodal displacements \(\{u_i\}\) under the external mechanical loads \(\{f_i\}\) can be represented by:

\[
[M][\ddot{u}_i] + [C][\dot{u}_i] + [K][u_i] = \{f_i\}
\]

(5.65)

where subscript \(w\) denotes the wing structure and \(\{u_i\}\) is the nodal displacement of the wing structure. Here, \([M_w]\), \([C_w]\), and \([K_w]\) are the elemental mass, damping, and stiffness matrices of the wing structure, respectively. In addition, \(\{f_w\}\) represents the applied mechanical load on each element.

As was the case for the piezoelectric material, equation (5.65) for each element in the wing model can be assembled for all the elements to obtain the global governing equation:

\[
[M_w^*][\ddot{U}^*] + [C_w^*][\dot{U}^*] + [K_w^*][U^*] = \{F_w^*\}
\]

(5.66)

where \(U^*\) is the global degrees of freedom of the structure which its components in Cartesian coordinates are \(U_X^*, U_Y^*, U_Z^*\). Here, the total applied force is denoted by \(\{F_w^*\}\) which in this study is the aerodynamic load. In addition, \([M_w^*]\), \([C_w^*]\), and
are the global mass, damping, and stiffness matrices of the wing structure, respectively.

The damping matrix \( [C_w^*] \) can be defined by the common Rayleigh damping as follows:

\[
[C_w^*] = \alpha_D [M_w^*] + \beta_D [K_w^*]
\] (5.67)

where \( \alpha_D \) and \( \beta_D \) are the constant multipliers for damping which are related to the modal damping ratio \( (\zeta_m) \) and natural frequency \( (\omega_m) \) of the wing system at \( m \)th mode as follows:

\[
2\zeta_m \omega_m = \alpha_D + \beta_D \omega_m^2
\] (5.68)

More precisely, if the natural frequencies and the damping ratios of two modes \( m_1 \) and \( m_2 \) are denoted by \( \omega_{m_1}, \omega_{m_2}, \zeta_{m_1} \) and \( \zeta_{m_2} \) respectively, the values of the constant multipliers can be found as following:

\[
\beta_D = \frac{2(\zeta_{m_1} \omega_{m_1} - \zeta_{m_2} \omega_{m_2})}{\omega_{m_1}^2 - \omega_{m_2}^2}
\] (5.69)

\[
\alpha_D = 2\zeta_{m_1} \omega_{m_1} - \beta_D \omega_{m_1}^2
\] (5.70)

### 5.6 Implementation of Finite Element Formulation Using ANSYS

In order to model the aeroelastic oscillations, the full transient analysis of the finite element software ANSYS is used. The wing structure is modeled as a cantilever plate with brick elements and piezoelectric strips as small patches bonded on the wing surface. Figure 5.5 shows the finite element mesh of a cantilever wing with piezoelectric strips bonded on the wing surface.

In this thesis, the three-dimensional element SOLID45 is chosen to model the cantilever plate in ANSYS. For the piezoelectric strips, the 8-node element
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SOLID5 is chosen which has eight nodes up to six degrees of freedom at each node. When used in structural and piezoelectric analyses, SOLID5 has large deflection and stress stiffening capabilities.

The SOLID5 element has a 3-D magnetic, thermal, electric, piezoelectric and structural field capability with limited coupling between the fields. In the present study, the structural and electric effects of piezoelectric elements have been considered in the analysis and the magnetic and thermal effects have been neglected.

In ANSYS, the Stress-Charge form of the piezoelectric constitutive equations is used as presented in equation (5.34). Hence, the pertinent parameters for the piezoelectric materials are the stiffness matrix $[c_E]$, the piezoelectric stress matrix $[e]$, and the dielectric matrix $[\varepsilon_S]$. The value of these matrices should be entered as the input of the piezoelectric material properties.

The array values of the stiffness matrix $[c_E]$ can be inputted as data tables in ANSYS, by using the TB and TBDATA commands. Since, the PZT material is an anisotropic material the TB command should be used as: TB, ANEL. The unit of the stiffness matrix is $N/m^2$.

Figure 5.5 Finite element mesh of the wing structure with bonded piezoelectric strips
The piezoelectric stress matrix \([e]\) relates the electric field to stress and has units of \(C/m^2\). This matrix is a \(6 \times 3\) array for a 3-D model and the data can be entered into ANSYS as a material data table type of TB, PIEZ. The dielectric matrix \([\varepsilon_3]\) defines the electrical permittivity in typical units of \(F/m\) or \(nF/m\). For a 3-D model, this matrix is a \(3 \times 3\) diagonal array where the data are entered into ANSYS as an orthotropic material property with PERX, PERY, and PERZ commands. The material properties of a sample PZT material which is used in this thesis can be found in Appendix A.

It is worth mentioning that in order to simulate the unsteady aeroelastic oscillations of the wing, the structural model presented in this chapter should be coupled with the aerodynamic models presented in Chapters 3 and 4 for two- and three-dimensional cases, respectively. This coupling is discussed in the next chapter.
CHAPTER 6

ACTIVE AEROELASTIC OSCILLATIONS CONTROL

6.1 Introduction

This chapter discusses the active control model to suppress the vibration of a wing structure with bonded piezoelectric sensors and actuators. The finite element model, described in the previous chapter, is used here. In the following sections, first the coupling between the finite element model of the structure and the aerodynamic model is described. For the control approach, the output voltage of the sensor strips is used as feedback to obtain the applied voltage of the actuator strips. A systematic method is also presented to obtain the feedback gains by using the system matrices.
In Section 6.4, in order to investigate the control procedure on a simpler case, some test results for the case of the elastic oscillations, in the absence of the aerodynamic loads, are presented. An appropriate PID control is designed and the effect of the feedback gains in the control performance is discussed.

6.2 Coupling the Structural and Aerodynamic Models

As explained in Chapter 5, to simulate the aeroelastic oscillation, the structural and aerodynamic models should be coupled to transfer the data at each time step. Figure 6.1 demonstrates a schematic flow chart of the coupling between the structural and the aerodynamic models.

It can be observed that at each time step, the nodal coordinate data are sent to the aerodynamic model to calculate the influence coefficients and the flight path which are used in a set of algebraic equation. By solving the aerodynamic set of equations, the unsteady aerodynamic pressure and forces are calculated and transferred to the structural model to be applied on the surface of the wing. As a result, the finite element model of the wing with bonded piezoelectric strips is solved to compute the structural deformations. These deformations are considered in the finite element model to update the nodal coordinates for the next time step. This loop continues until the time step reaches the final time step specified in the program.

In this thesis, a FORTRAN program has been written to calculate the aerodynamic loads due to the unsteady three-dimensional flow. The finite element modeling of the wing structure consisting of the bonded piezoelectric strips has been performed in a program using the ANSYS package. In order to transfer the data at each time step, two programs are connected using User Programmable Features in the ANSYS package. More precisely, at each time step, the FORTRAN code is called as a function in the ANSYS program where the updated nodal coordinate are the input and the aerodynamic pressures/loads are the output of this function (See Figure 6.1).
Figure 6.1 Schematic flow chart for the coupling of structural and aerodynamic models

6.3 Control Model

6.3.1 Feedback Control Model

As indicated in Chapter 2, a PID control method is used in this thesis to suppress the aeroelastic oscillations. Both sensing and actuating parameters are the voltages in the piezoelectric strips which are placed in a collocated
arrangement, i.e. actuators on the top and sensors on the bottom surface of the wing (Figure 6.2). Considering that $K_p$, $K_I$, and $K_D$ are the proportional, integral and derivative gains, respectively, the control law for the PID control is described as follows:

$$V_a(t) = K_p V_d(t) + K_I \int_0^t V_d(\tau) d\tau + K_D \dot{V}_d(t)$$

(6.1)

where $V_a$ is the actuator voltage and $V_d$ denotes the difference voltage between the sensor voltage $V_s$ and a setpoint or reference voltage $V_{so}$ as shown below:

$$V_d(t) = V_s(t) - V_{so}$$

(6.2)

Here, $V_{so}$ is the constant reference voltage which is used to regulate the time-depndant sensor voltage. In the present study, it is assumed that at $t = 0$, the wing structure is flying in a steady flow subject to steady aerodynamic loads. This steady motion is considered as the reference point and the goal of the control system is to regulate the unsteady oscillations around this reference point. The sensor voltages corresponding to the initial steady motion is employed as the constant setpoint voltage $V_{so}$.
It is of importance to note that for the large-scale structures with theoretically infinite number of the vibration modes, due to the computational limitations in modeling and the restricted number of the district sensors and actuators, it is more convenient to use an output feedback control where the output signal of the sensors modified by the control gains are feedback to the control actuators.

The control law in equation (6.1) is a continuous-time dynamical system which needs to be discretized to a discrete-time system in order to be used in a time-stepping code. The actuator command in equation (6.1) is the sum of the proportional, integral and derivative terms respectively which can be expressed as:

\[ V_d(t) = P(t) + I(t) + D(t) \]  \hspace{1cm} (6.3)

By representing the time step as \( t_k \), the proportional term can be replaced with the sampled version as follows:

\[ P(t_k) = K_P [V_s(t_k) - V_{s_0}] \]  \hspace{1cm} (6.4)

The integral term can be approximated as:

\[ I(t_k) = I(t_{k-1}) + K_I \Delta t [V_s(t_k) - V_{s_0}] \]  \hspace{1cm} (6.5)

or:

\[ I(t_k) = K_I \Delta t \sum_{i=1}^{k} [V_s(t_i) - V_{s_0}] \]  \hspace{1cm} (6.6)

where \( \Delta t = t_k - t_{k-1} \) is the time increment. The time derivative part in the derivative term can be approximated by backward difference formula as:

\[ \dot{V}_d(t_k) = \frac{[V_d(t_k) - V_d(t_{k-1})]}{\Delta t} \]  \hspace{1cm} (6.7)

By replacing the value of the voltage difference from equation (6.2) and considering that the reference voltage is constant, the derivative term can be expressed as:
\[ D(t_k) = K_D \frac{V_s(t_k) - V_s(t_{k-1})}{\Delta t} \] (6.8)

Summation of the proportional, integral and derivative terms from equations (6.4), (6.5), and (6.8) results in the discretized form of the PID control law as follows:

\[ V_a(t_k) = K_P [V_s(t_k) - V_{s0}] + K_I \Delta t \sum_{i=1}^{k} [V_s(t_i) - V_{s0}] \]

\[ + \frac{K_D}{\Delta t} [V_s(t_k) - V_s(t_{k-1})] \] (6.9)

At each time step, the applied actuator voltage is obtained using the sensor voltage and the setpoint voltage at this time step and the previous ones.

As discussed in Chapter 2, the proper PID gains can be obtained by applying the empirical methods or trial and error approaches. In this work, the feedback gains are initially selected by trial and error, but have been validated by a systematic approach which uses the system matrices to calculate the proper feedback gains. This approach is presented in the following section.

### 6.3.2 Gain Calculation Approach

As discussed in Chapter 5, the finite element formulation of a piezoelectric continuum can be expressed as:

\[ [M^*][\ddot{U}^*] + [K_{uu}^*][U^*] + [K_{u\phi}^*][\phi^*] = \{ F^* \} \] (6.10)

\[ [K_{\phi u}^*][U^*] + [K_{\phi \phi}^*][\phi^*] = \{ G^* \} \] (6.11)

The electric potential \( \{ \phi^* \} \) can be obtained from equation (6.11) as:

\[ \{ \phi^* \} = [K_{\phi \phi}^*]^{-1} \{ G^* \} - [K_{\phi u}^*][U^*] \] (6.12)

where for sensors, for which \( \{ G^* \} \) is equal to zero, it can result in:

\[ \{ \phi_s^* \} = -[K_{\phi \phi}^*]^{-1} [K_{\phi u}^*][U_s^*] \] (6.13)
The above equation can be re-written as:

\[
\{\phi_s^*\} = [C_s]\{U_s^*\} \tag{6.14}
\]

where:

\[
[C_s] = -[K_{\phi\phi}]^{-1}[K_{\phi u}]
\]

is a coefficient matrix which relates the piezoelectric sensor voltage \(\{\phi_s^*\}\) to the sensor displacement vector \(\{U_s^*\}\). Since the piezoelectric sensors and actuators are collocated, it can be assumed that the transversal deflection at their nodes is the same for sensors and actuators at a certain location. Therefore, the displacement vector of sensors \(\{U_s^*\}\) is equal to that of actuators \(\{U_a^*\}\) which can be denoted by \(\{U_{pz}^*\}\) for convenience. Thus:

\[
\{\phi_s^*\} = [C_s]\{U_{pz}^*\} \tag{6.16}
\]

As discussed in the previous chapter, in this thesis the output voltage of the piezoelectric sensors is used in the feedback control law to calculate the voltage to be applied by the piezoelectric actuators. Hence, the piezoelectric actuator voltage \(\{\phi_a^*\}\) can be represented as:

\[
\{\phi_a^*\} = [G_p]\{\phi_s^*\} + [G_d]\{\phi_s^*\}
\]

where \([G_p]\) is the proportional gain and \([G_d]\) is the derivative gain. Here \(\{\phi_s^*\}\) denotes the time derivative of the sensor voltage and can be obtained by differentiating equation (6.16) with respect to time. Considering that the coefficient matrix \([C_s]\) is independent of time, one can write:

\[
\{\phi_s^*\} = [C_s]\{\dot{U}_{pz}^*\} \tag{6.18}
\]

By substituting equations (6.16) and (6.18), equation (6.17) can be recast as follows:

\[
\{\phi_a^*\} = [\tilde{G}_p]\{U_{pz}^*\} + [\tilde{G}_d]\{\dot{U}_{pz}^*\} \tag{6.19}
\]
where:

\[
\begin{align*}
\mathbf{[G_p]} = \mathbf{[G_p][C_s]} &= -\mathbf{[G_p][K_{\phi\phi}^{-1}][K_{\phi u}]} \\
\mathbf{[G_d]} = \mathbf{[G_d][C_s]} &= -\mathbf{[G_d][K_{\phi\phi}^{-1}][K_{\phi u}]} 
\end{align*}
\] (6.20)

Equation (6.19) shows that the applied voltage of the piezoelectric actuators is related to the displacement and velocity vectors. In order to obtain the actuator voltage to be applied to control the aeroelastic oscillations, one needs to find the feedback gains \([G_p]\) and \([G_d]\) (or equivalently \([\tilde{G}_p]\) and \([\tilde{G}_d]\)).

In voltage driven actuators, when the voltage is directly applied to the actuator, equation (6.10) can be rewritten by substituting the actuator voltage from equation (6.19) as follows:

\[
\begin{align*}
\mathbf{[M^*]}\{\ddot{U}_{pz}\} + \mathbf{[K_{\phi\phi}]}\{\tilde{G}_d\}\{\dot{U}_{pz}\} + \left(\mathbf{[K_{uu}]} + \mathbf{[K_{u\phi}][C_s]}\right)\{U_{pz}\} &= \{F_{pz}\} 
\end{align*}
\] (6.21)

Similar equation for sensor can be expressed by substituting equation (6.16) into equation (6.10) as given below:

\[
\begin{align*}
\mathbf{[M^*]}\{\dot{U}_{pz}\} + \left(\mathbf{[K_{uu}]} + \mathbf{[K_{u\phi}][C_s]}\right)\{U_{pz}\} &= \{F_{pz}\} 
\end{align*}
\] (6.22)

Since the size and material properties of the sensor and actuator strips are identical, the mass, stiffness and other matrices are identical. As a result, equations (6.21) and (6.22) can be added as follows:

\[
\begin{align*}
2\mathbf{[M^*]}\{\ddot{U}_{pz}\} + \mathbf{[K_{u\phi}]}\{\tilde{G}_d\}\{\dot{U}_{pz}\} + \\
\left(2\mathbf{[K_{uu}]} + \mathbf{[K_{u\phi}][C_s]}\right)\{U_{pz}\} &= 2\{F_{pz}\} 
\end{align*}
\] (6.23)

where subscript \(pz\) represents the piezoelectric strips.

As explained in Chapter 5, the finite element formulation for the wing structure can be given by:

\[
\begin{align*}
\mathbf{[M_w]}\{\ddot{U}_w\} + \mathbf{[C_w]}\{\dot{U}_w\} + \mathbf{[K_w]}\{U_w\} &= \{F_w\} 
\end{align*}
\] (6.24)
By combining the last two equations, (6.23) and (6.24), the finite element formulation for the whole structure, including the wing and the piezoelectric strips can be described as shown below:

\[
[\mathcal{M}]\{\ddot{U}\} + [\mathcal{C}]\{\dot{U}\} + [\mathcal{K}]\{U\} = \{F\} \tag{6.25}
\]

where \(\{U\}\) is the global displacement vector for the combined wing-piezo structure:

\[
\{U\} = \begin{bmatrix} \{U_w\} \\ \{U_{pz}\} \end{bmatrix} \tag{6.26}
\]

and \([\mathcal{M}], [\mathcal{C}], [\mathcal{K}]\) are respectively the global mass, damping and stiffness matrices corresponding to the combined wing-piezo structure, which are defined as follows:

\[
[\mathcal{M}] = \begin{bmatrix} [M_w] & [0] \\ [0] & 2[M_{pz}] \end{bmatrix} \tag{6.27}
\]

\[
[\mathcal{C}] = \begin{bmatrix} [C_w] & [0] \\ [0] & [K_{w\phi}][\mathcal{G}_d] \end{bmatrix} \tag{6.28}
\]

\[
[\mathcal{K}] = \begin{bmatrix} [K_w] & [0] \\ [0] & 2[K_{w\phi}][\mathcal{G}_p] + [K_{s}] \end{bmatrix} \tag{6.29}
\]

Here, the global load \(\{F\}\) can be defined as shown below:

\[
\{F\} = \begin{bmatrix} \{F_w\} \\ 2\{F_{pz}\} \end{bmatrix} \tag{6.30}
\]

Since the nodes of the piezoelectric strips are attached to the wing nodes at their common surface, it can be assumed that the external load, which in our case is the aerodynamic load, is applied only to the wing surface. So, no external mechanical force is applied to the piezoelectric strips and \(\{F_{pz}\} = 0\).
6.3.3 Model Reduction

The number of degrees of freedom in the equation of motion described in equation (6.25) is usually large for the finite element models; hence, it is not very suitable for the control synthesis. Therefore, in order to improve the numerical efficiency, modal analysis should be employed where the system is transformed into a reduced system with fewer number of degrees of freedom.

In this analysis, the physical coordinates \( \{ \mathbf{U} \} \) can be changed to modal coordinates using the following equation:

\[
\{ \mathbf{U} \} = [\Psi]\{ \mathbf{Z} \} = \sum_{i=1}^{m} \Psi_i \mathbf{Z}_i
\]  

(6.31)

where \( \mathbf{Z}_i \) represents the modal amplitude of mode \( i \). The mode shape \( \Psi_i \) and the corresponding natural frequencies \( \omega_i \) are the solutions of the eigenvalue problem of equation (6.25) by neglecting the damping terms as:

\[
([\mathbf{\bar{K}}] - \omega_i^2[\mathbf{\bar{M}}])\{ \mathbf{Z}_i \} = \{0\}
\]  

(6.32)

The modal matrix \([\Psi]\) whose columns are the eigenvectors normalized with respect to the mass matrix can be given by:

\[
[\Psi] = \{ \Psi_1 \ \Psi_2 \ \cdots \ \Psi_m \}
\]  

(6.33)

where \( m \) is the number of modes. If \( n_w \) and \( n_{pz} \) are the number of the wing nodes and the number of the nodes of the piezoelectric strips, respectively, the number of degrees of freedom for equation (6.25) can be shown by \( n_{w+pz} \). Hence, the number of modes is significantly smaller than the number of generalized degrees of freedom: \( m \ll n_{w+pz} \).

Here, it is more advantageous to use the uncombined format of the degree of freedom, and separate the degrees of freedom of the wing and the piezoelectric strips. Similarly, the modal transformation matrix can also be decomposed as follows:
\[
\{\bar{U}\} = \begin{pmatrix} \{U^*_w\} \\ \{U^*_pz\} \end{pmatrix} = \begin{bmatrix} [\Psi_w] \\ [\Psi_{pz}] \end{bmatrix} \{Z\} \tag{6.34}
\]

or equivalently:

\[
\{U^*_w\}_{n_w \times 1} = [\Psi_w]_{n_w \times m} \{Z\}_{m \times 1} \tag{6.35}
\]

\[
\{U^*_pz\}_{n_{pz} \times 1} = [\Psi_{pz}]_{n_{pz} \times m} \{Z\}_{m \times 1} \tag{6.36}
\]

where the subscript denotes the order of the vector or matrix. Substituting equation (6.35) into equation (6.24) results in:

\[
[M^*_w][\Psi_w]\{\ddot{Z}\} + [C^*_w][\Psi_w]\{\dot{Z}\} + [K^*_w][\Psi_w]\{Z\} = \{F^*_w\} \tag{6.37}
\]

Pre-multiplying equation (6.37) by \([\Psi_w]^T\) and using the orthogonality relationships (6.39) and (6.40) can result in:

\[
[\bar{M}_w]\{\ddot{Z}\} + [\bar{C}_w]\{\dot{Z}\} + [\bar{K}_w]\{Z\} = [\Psi_w]^T\{F^*_w\} \tag{6.38}
\]

Based on the orthogonality of the natural modes and assuming that the modes are distinct, one can write:

\[
[\bar{M}_w] = [\Psi_w]^T[M^*_w][\Psi_w] \tag{6.39}
\]

\[
[\bar{K}_w] = [\Psi_w]^T[K^*_w][\Psi_w] \tag{6.40}
\]

where \([\bar{M}_w]\) and \([\bar{K}_w]\) are diagonal matrices. Here, \([\bar{C}_w] = [\Psi_w]^T[C^*_w][\Psi_w]\) is also assumed to be an orthogonal matrix. It is worth noticing that the matrices \([\bar{M}_w], [\bar{C}_w], \) and \([\bar{K}_w]\) are all of \(m \times m\) order.

Similar procedure can be applied to the equations of motion of the piezoelectric strips indicated in equation (6.23) by replacing equation (6.36) as shown below:

\[
[\bar{M}_{pz}][\ddot{Z}] + [\bar{C}_{pz}][\dot{Z}] + [\bar{K}_{pz}][Z] = 0 \tag{6.41}
\]

where:
\[
\begin{align*}
[\tilde{M}_{pz}] &= 2[\Psi_{pz}]^T[M_{pz}]^*\Psi_{pz} \\
[\tilde{C}_{pz}] &= [\Psi_{pz}]^T[K_{u\phi}^*][\tilde{G}_d][\Psi_{pz}] \\
[\tilde{K}_{pz}] &= [\Psi_{pz}]^T(2[K_{uu}^*] + [K_{u\phi}^*][\tilde{G}_p] + [K_{u\phi}^*][C_s])\Psi_{pz}
\end{align*}
\] (6.42)
(6.43)
(6.44)

where the matrices \([\tilde{M}_{pz}], [\tilde{C}_{pz}],\) and \([\tilde{K}_{pz}]\) are all of \(m \times m\) order as well.

Equations (6.38) and (6.41) can be combined to obtain the equation of motion of the controlled wing-piezo structure in the modal form which is shown below:

\[
[M_{ctrl}][\ddot{\Psi}] + [C_{ctrl}][\dot{\Psi}] + [K_{ctrl}][\Psi] = [\Psi_{w}]^T[F_w]
\] (6.45)

where:

\[
[M_{ctrl}] = [\tilde{M}_w] + [\tilde{M}_{pz}] = [\Psi_w]^T[M_w]^*[\Psi_w] + 2[\Psi_{pz}]^T[M_{pz}]^*[\Psi_{pz}]
\] (6.46)

\[
[C_{ctrl}] = [\tilde{C}_w] + [\tilde{C}_{pz}] = [\Psi_w]^T[C_w]^*[\Psi_w] + [\Psi_{pz}]^T[K_{u\phi}^*][\tilde{G}_d][\Psi_{pz}]
\] (6.47)

\[
[K_{ctrl}] = [\tilde{K}_w] + [\tilde{K}_{pz}] = [\Psi_w]^T[K_w]^*[\Psi_w] + \\
[\Psi_{pz}]^T(2[K_{uu}^*] + [K_{u\phi}^*][\tilde{G}_p] + [K_{u\phi}^*][C_s])\Psi_{pz}
\] (6.48)

By comparing this equation to the uncontrolled structure, the gains of the feedback control can be obtained.

\[
[\tilde{G}_d] = [K_{u\phi}^*]^{-1}([\Psi_{pz}]^T)^{-1}([C_{ctrl}] - [\Psi_w]^T[C_w][\Psi_w])[\Psi_{pz}]^{-1}
\] (6.49)

\[
[\tilde{G}_p] = [K_{u\phi}^*]^{-1}([\Psi_{pz}]^T)^{-1}([K_{ctrl}] - [\Psi_w]^T[K_w]^*[\Psi_w]
- 2[\Psi_{pz}]^T[K_{uu}^*][\Psi_{pz}]
- [\Psi_{pz}]^T[K_{u\phi}^*][C_s][\Psi_{pz}])[\Psi_{pz}]^{-1}
\] (6.50)
Finally, the values of the proportional and derivative gains $[G_p]$ and $[G_d]$ can be obtained from equation (6.20).

It should be noted that the controlled matrices $[M_{ctrl}]$, $[C_{ctrl}]$, and $[K_{ctrl}]$ are not obtained from equations (6.46) to (6.48) in practice; they are obtained by solving the equations of the wing-piezo structure. The uncontrolled matrices are also found as the solution of the wing structure equations, by deleting the piezoelectric terms. Comparing these two solutions can lead to calculating the proportional and derivative gains presented in equations (6.49) and (6.50).

It is worth mentioning that this systematic approach calculates the proportional and derivative gains of the PID control, but not the integral gain. However, since the integral gain is more effective only in removing the steady state error, but not much in the amplitude reduction, it can be assumed that the values of the P and D gains obtained from this approach are very close to those in the presence of the integral gain.

In the present thesis, the effect of various values of the PID gains in suppression of oscillations is compared. Moreover, an analysis is performed to examine the effects of actuator placement on the wing surface which are presented in Chapter 7.

Prior to the results of the control of the aeroelastic oscillations, first some test results are performed to test the control procedure and study the effect of the feedback gains. Hence, a simpler case of forced vibrations cause by harmonic loads in the absence of the aerodynamic flows is considered and controlled, which the results are shown in the next section. In this test study, the harmonic loads are applied on the surface of the wing instead of the aerodynamic loads and a PID controller is applied to suppress the elastic oscillations. Also, the effect of the feedback gains in the performance of the controller is discussed.
6.4 Test Results on the Control of the Elastic Oscillations

For the wing structure, a cantilever Aluminum plate is used which is clamped in the wing root. The piezoelectric strips are made of lead zirconate titanate (PZT) and are used as either sensors or actuators on the top and bottom surfaces of the wing. Sensors and actuators are assumed to be collocated, i.e. positioned at the same locations and bonded perfectly to the wing surface.

Figure 6.3 Configuration of the wing structure with piezoelectric sensors and actuators

Figure 6.4 Finite element mesh of the wing structure and sensor/actuator numbering scheme
Table 6.1 Dimensions and location of piezoelectric strips and wing structure

<table>
<thead>
<tr>
<th>Wing structure</th>
<th>Piezoelectric strips</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (span), $l_b$</td>
<td>1.0 (m)</td>
</tr>
<tr>
<td>Width (chord), $w_b$</td>
<td>0.3 (m)</td>
</tr>
<tr>
<td>Thickness, $t_b$</td>
<td>2.0 (mm)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Location of piezoelectric strips</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>0.03 (m)</td>
</tr>
<tr>
<td>$y_0$</td>
<td>0.11 (m)</td>
</tr>
</tbody>
</table>

Figure 6.3 shows the configuration of the wing structure and the piezoelectric strips. The system parameters such as the dimensions of the wing structure and the locations of the piezoelectric strips are given in Table 6.1. In the simulations presented for the elastic case, fifteen pairs of piezoelectric sensors/actuators are chosen which are shown in Figure 6.4 with their numbering scheme.

To model the oscillations, the transient analysis of ANSYS is used and the time history of the results such as the transverse displacement at the wing tip and the voltages of the piezoelectric sensor/actuator are presented.

### 6.4.1 Control of Oscillations Arising From Harmonic Loads

To study the control of oscillations arising from harmonic loads, a time dependant load is applied sinusoidally as a distributed pressure on the wing structure in the following form:

$$p(t) = p_0 \sin 2\pi f_e t$$  \hspace{1cm} (6.51)

where the constant pressure $p_0$ is the uniform distributed pressure and $f_e$ is the
frequency of exciting force of the forced oscillations. The values of $p_0 = 10 \, (N/m^2)$ and $f_e = 3.35 \, (Hz)$ are considered in the present simulations.

The time history of the transverse (vertical) displacement of the wing is obtained for three points located at the half-chord at various locations along the span, near the wing root, at the middle of the span and near the wing tip. The coordinates of these points are shown below:

Point near root: $x = w_b/2, \ y = l_b/5, \ z = 0$

Point at middle: $x = w_b/2, \ y = l_b/2, \ z = 0$ \hfill (6.52)

Point near tip: $x = w_b/2, \ y = l_b, \ z = 0$

where $l_b$ is the wing span. The transverse displacement of the uncontrolled oscillations is presented in Figure 6.5. The amplitude of the transient oscillations decreases slightly due to the presence of the structural damping. The values of the Rayleigh damping parameters are considered to be $\alpha_D = 0.1, \beta_D = 0.0001$.

![Figure 6.5 Uncontrolled oscillations of the wing TE due to harmonic loads](image-url)
The output voltages of the piezoelectric sensors are shown for the middle row of sensors (sensors #2, #5, #8, #11 and #14) are shown in Figure 6.6. The piezoelectric sensors close to the wing root have lower voltages compared to the ones close to the wing tip. As discussed in Section 5.2, this is mainly due to the fact that the output voltage of the piezoelectric sensor is proportional to $\Delta \theta$, the difference of slopes at the two ends of the sensor, which is greater near to the clamped end of a cantilever structure, or in this case, the root of the wing.

In order to suppress the oscillations, a PID controller is applied at $t = 0.1 \text{ s}$. The transverse displacement of the uncontrolled oscillations at the wing tip is compared with two values of the proportional gain, $K_p = 10$ and 15, and is presented in Figure 6.7. The values of the derivative and integral gains are considered respectively as $K_d = 0.005$ and $K_i = 5$. Figure 6.8 shows the voltages applied to the actuators in the middle row for two values of proportional gains mentioned above.

![Figure 6.6 Comparison of the output voltage of sensors at different locations on the wing for uncontrolled oscillations subject to harmonic loads](image-url)
Figure 6.7 Transverse displacement for the case of PID controller subject to harmonic loads (a) $K_p = 10$ (b) $K_p = 15$ ($K_i = 5$, $K_d = 0.005$)
Figure 6.8 Actuator voltages for the case of PID controller for different actuator locations on the wing (a) $K_p = 10$ (b) $K_p = 15$ ($K_I = 5, K_D = 0.005$)
It is evident that the PID controller can effectively reduce the amplitude of the oscillations in small amount of time. For instance, for the case of controller with $K_p = 15$, the magnitude of the oscillations are reduced to 50% after 6 seconds. It can also be noted that increasing the proportional gain increases the damping effect of the control system and reduces the settling time.

It can be observed that higher values of actuator voltage are needed at the beginning of applying the control which is mainly due to higher magnitudes of the oscillations. However, after a couple of cycles, as the oscillations are reduced, the actuator voltage decreases. By comparing the voltages of the different actuators, it can be found that the actuators close to wing root require higher values of voltage which is mainly due to the concentration of shear strain and stress in that area.

In Figure 6.9, the results of the controlled oscillations using a PD controller are presented and compared with the similar results of using PID controller. Figure 6.9 (a) shows the transverse displacement of the oscillations using a PD controller when $K_p = 15, K_d = 0.005$ and there is no integral gain ($K_i = 0$). It can be noted that the amplitude of the oscillations is reduced, but there is a small steady-state error, which can be eliminated by adding an integral feedback to the PD controller which are shown in Figure 6.9 (b). The actuator voltages of the PD and PID controller are presented in Figure 6.10. It should be noted that the values of actuator voltage oscillate around a non-zero steady-state value which are removed in Figure 6.10 (b) by adding an integral gain ($K_i = 5$) to the control system.

In Figure 6.11 the transverse displacement for two different values of derivative gain $K_d = 0.001$ and 0.005 is compared and the actuator voltages are presented in Figure 6.12. It can be observed that increasing $K_d$ improves the controller performance to some extent, but it is not very significant. It is worth mentioning that the values of the derivative gain are relatively very small which is mainly due to higher values of voltage rate.
Figure 6.9 Transverse displacement subject to harmonic loads for the case of (a) PD controller \((K_I = 0)\) (b) PID controller \((K_I = 5)\) where \(K_P = 15, K_D = 0.005\)
Figure 6.10 Actuator voltages for different actuator locations on the wing for the case of (a) PD controller ($K_f = 0$) (b) PID controller ($K_f = 5$) where $K_p = 15, K_d = 0.005$
Figure 6.11 Transverse displacement for the case of PID controller subject to harmonic loads (a) $K_D = 0.001$ (b) $K_D = 0.005$ ($K_P = 15, K_I = 5$)
Figure 6.12 Actuator voltages for the case of PID controller for different actuator locations on the wing (a) $K_D = 0.001$ (b) $K_D = 0.005$ ($K_p = 15, K_I = 5$)
CHAPTER 7

RESULTS ON THE CONTROL OF THE AEROELASTIC OSCILLATIONS

7.1 Introduction

This chapter presents the results of the simulations for control of aeroelastic oscillations using bonded piezoelectric sensors/actuators. In the first section, aeroelastic oscillations of a wing structure subject to two-dimensional unsteady subsonic aerodynamic loads are studied. An appropriate PID control is designed and applied to suppress the oscillations. For the 2-D unsteady aerodynamic loads, the aerodynamic model presented in Chapter 3 is used.

In Section 7.3, the more complicated case of the control of aeroelastic oscillations of finite span wings subject to the three-dimensional unsteady
aerodynamic loads is investigated. The aerodynamic model discussed in Chapter 4 for the 3-D unsteady flows is employed. For the structural model, the model presented in Chapter 5 is used. The aeroelastic oscillations are discussed for two types of vertical gust loads and an appropriate PID controller is applied to suppress the oscillations. In addition, an analysis is performed to examine the effects of actuator placement on the wing surface in suppression of oscillations.

### 7.2 Control of Aeroelastic Oscillations of Wings Based on the 2-D Unsteady Aerodynamic Model

#### 7.2.1 Geometry Definition

In this study, a cantilever Aluminum plate is used for the wing structure. In order to use the 2-D model of unsteady aerodynamic loading, the leading edge of the wing is considered to be constrained as a cantilever and a relatively high aspect ratio is considered for the wing.

![Figure 7.1 Configuration of the wing structure with piezoelectric sensors and actuators](image)
Table 7.1 Dimensions and location of piezoelectric strips and wing structure

<table>
<thead>
<tr>
<th>Wing structure</th>
<th>Piezoelectric strips</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (span), $l_b$</td>
<td>1.0 (m)</td>
</tr>
<tr>
<td>Width (chord), $w_b$</td>
<td>0.24 (m)</td>
</tr>
<tr>
<td>Thickness, $t_b$</td>
<td>2.0 (mm)</td>
</tr>
</tbody>
</table>

Location of piezoelectric strips

| $x_0$ | 0.024 (m) | $\Delta x_0$ | 0.032 (m) |
| $y_0$ | 0.080 (m) | $\Delta y_0$ | 0.160 (m) |

Figure 7.2 Finite element mesh of the wing structure and sensor/actuator numbering scheme

The piezoelectric strips are made of lead zirconate titanate (PZT) and are used as either sensors or actuators. Sensors and actuators are assumed to be collocated, i.e. positioned at the same locations on the top and bottom surfaces of the wing. Actuators are placed on the top and sensors on the bottom surface of the wing structure and are assumed to be perfectly bonded to the wing surface.

The configuration of the wing structure and piezoelectric strips is shown in Figure 7.1, and their dimensions and locations are given in Table 7.1. In the
RESULTS ON THE CONTROL OF THE AEROELASTIC OSCILLATIONS

simulations presented for 2-D case, fifteen pairs of piezoelectric sensors/actuators are chosen which are shown in Figure 7.2 with their numbering scheme.

In order to model the aeroelastic oscillations the transient analysis of ANSYS is used and the time history of the results such as the transverse displacement at the trailing edge of the wing and sensor/actuator voltages is presented.

7.2.2 Control of Oscillations

In this section, a control system is provided to suppress the aeroelastic oscillations subject to the two-dimensional aerodynamic loadings. To obtain the aerodynamic loads, the 2-D aerodynamic model discussed in Chapter 3 is used in a FORTRAN program. At each time step, the unsteady aerodynamic loads calculated in the FORTRAN program are transferred to ANSYS to be applied on the wing structure. The unsteady aerodynamic loads are in the form of distributed loads per unit span which vary along the wing chord (x axis).

The flow velocity is $U_{\infty} = 40$ (m/s) and considered to be along the x axis. The oscillations are initiated by a $5^\circ$ change in the flow angle of attack from the steady zero angle of attack. The uncontrolled oscillations in the form of the transverse (vertical) displacement of the wing are obtained for three points (#1 to 3) located at the wing trailing edge and are demonstrated in Figure 7.3. The coordinates of these points are shown below:

Point #1: $x = w_b$, $y = 0$, $z = 0$

Point #2: $x = w_b$, $y = l_b/2$, $z = 0$  \hspace{1cm} (7.1)

Point #3: $x = w_b$, $y = l_b$, $z = 0$

where $l_b$ is the wing span. It can be observed that the deformations of the three points are very close which implies that the wing has a 2-D motion which is our goal in this section. Hence, the middle point, point #2, can represent the behavior of the trailing edge of the wing for this case.
One can note that the amplitude of the uncontrolled oscillations decreases with time, which is due to the aerodynamic damping. However, applying an appropriate controller can suppress the oscillations in a shorter time which can considerably improve the flight comfort, as will be shown later. The output voltages of the piezoelectric sensors are shown in Figures 7.4 and 7.5. Similar to the previous case, the output voltages of the sensors close to the leading edge are greater than those of the trailing edge.

In order to suppress the uncontrolled oscillations, a PID controller is suggested. In order to obtain the proportional and derivative gains for the controller, the systematic method discussed in Chapter 6 is used which employs the system matrices to calculate the proper feedback gains. Using this systematic method, the values of the proportional and derivative gains for the present case are found as:

\[ K_p = 12.38 \quad , \quad K_d = 0.00992 \]  

(7.2)

Hence, the rounded values of \( K_p = 13, K_d = 0.01 \) are used.
Figure 7.4 Output voltage of sensors in (a) LE (b) the middle row (c) TE
Figure 7.5 Comparison of the output voltage of sensors at different locations on the wing for uncontrolled oscillations subject to unsteady aerodynamic loads

It is worth mentioning that this method calculates the values of the proportional and derivative gains only, and a trial and error method is needed to obtain the value of the integral gain. However, as it will be shown later, the value of the integral gain has a small effect in suppressing the oscillations, and its main effect is in eliminating the steady-state error.

The PID controller is applied at $t = 0.02 \, s$ to suppress the oscillations shown in Figure 7.3. The results are compared for two values of proportional gain $K_P = 8$ and 13 which are shown in Figure 7.6. The values of the derivative and integral gains are considered respectively as $K_D = 0.01$ and $K_I = 10$.

It can be seen that trailing edge displacements for $K_P = 13$ vanishes after 0.3 seconds which confirms that the proposed control has effectively suppressed the oscillations. Here again, increasing the proportional gain increases the damping effect of the control system and reduces the settling time.
RESULTS ON THE CONTROL OF THE AEROELASTIC OSCILLATIONS

The voltages applied to the actuators are shown in Figure 7.7 for the corresponding proportional gains presented in Figure 7.6. The value of the required actuator voltage is relatively high for the first cycle of oscillations, but as the amplitude of oscillations decreases, the value of the actuator voltage also decreases and finally approaches zero.

It is worth noticing that although the values of actuator voltage decreases by time, they are relatively high at the beginning for both cases, especially for $K_p = 13$. This is due to the fact that by constraining the leading edge, the wing structure becomes very stiff and due to the higher values of the shear stress, greater voltages should be applied to the actuators to reduce the amplitude of oscillations. However, as shown in the test results in the previous chapter, lower actuator voltages are required for the case where the wing root is clamped. Nevertheless, for the present case, the voltage actuator can be limited to a reasonable value for application reasons.
Piezoelectric materials have some operating limits for voltage, temperature, etc. which is determined by the chemical composition of the material and manufacturing processes. The maximum operating voltage that the piezoelectric strips can tolerate can be determined by the layer thickness. For high-voltage PZTs with the thickness range of 0.5 to 1.0 mm, the maximum voltage is of the order of 1 to 2 kV/mm. In the present simulation, a voltage limit of 1000 volts is introduced for the actuators and a similar simulation is conducted to compare the results.

The PID gains used are $K_p = 13, K_I = 10, K_D = 0.01$ as before and the results are compared with the no-limit case and are shown in Figure 7.8. The results show that although limiting the actuator voltage decreases the effectiveness of the controller to some extent, this change only exists in the first two cycles. In other words, after $t = 0.3 \text{s}$ both controllers suppress the oscillations completely, which seems evident due to the fact that the actuator voltage is not limited anymore. It can be concluded that although the performance of the controller decreases slightly by limiting the actuator voltage, yet it is still effective.

A comparison between a PD and PID controllers is shown in Figure 7.9 by considering a zero value for integral gain. It is observable that the PD controller is able to damp the oscillations in the same settling time as the PID controller, but there is a small steady state error. In addition, as it is shown in Figure 7.9 (b), the steady-state value of the actuator voltage oscillates around a non-zero voltage which due to limiting the voltage, it is shown as the lower limit of the actuator voltage. However, this steady-state error can be compensated by adding an integral feedback to the controller.
Figure 7.7 Actuator voltages for the case of PID controller for different actuator locations on the wing (a) $K_p = 8$ (b) $K_p = 13$ ($K_I = 10, K_D = 0.01$)
Figure 7.8 (a) Transverse displacement and (b) actuator voltages for the case of PID controller for actuator voltage limitation ($K_p = 13, K_I = 10, K_D = 0.01$)
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Figure 7.9 (a) Comparison of a PD controller ($K_I = 0$) and a PID controller ($K_I = 10$) (b) actuator voltages for the case of PD controller ($K_p = 13, K_I = 0.01$)
Figure 7.10 (a) Comparison of two derivative gains of PID controller subject to unsteady aerodynamic loads (b) actuator voltages ($K_P = 13, K_I = 10, K_D = 0.005$)
The effect of derivative gain is demonstrated in Figure 7.10 for two values of $K_D = 0.005$ and 0.01. The proportional and integral gains are considered as $K_P = 13$ and $K_I = 10$, respectively. It can be observed that by increasing $K_D$, the magnitude of the oscillation is decreased to some extent but its effect is not very considerable. It should be noted that due to the relatively high values of the voltage rate, the values of the derivative gain are relatively very small.

### 7.3 Control of Aeroelastic Oscillations of Wings Based on the 3-D Unsteady Aerodynamic Model

This section presents the simulation results related to control of three-dimensional aeroelastic oscillations of a cantilever wing. The wing structure modeled in ANSYS is a cantilever Aluminum plate which is constrained at the wing root. The piezoelectric strips are made of PZT material and are used as either sensors placed on the top surface of the wing or as actuators placed on the bottom surface. Piezoelectric strips are assumed to be collocated and are perfectly bonded to the wing surface.

![Figure 7.11 Configuration of the wing structure with piezoelectric sensors and actuators](image)
The configuration of the 3-D wing structure and piezoelectric strips is shown in Figure 7.11. Twelve pairs of piezoelectric strips are chosen for the 3-D simulations presented, which are shown in Figure 7.12 with their numbering scheme. The dimensions and locations of the piezoelectric strips and the wing structure are given in Table 7.2.

Table 7.2 Dimensions and location of piezoelectric strips and wing structure

<table>
<thead>
<tr>
<th>Wing structure</th>
<th>Piezoelectric strips</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (span), $l_b$</td>
<td>Length, $l_p$</td>
</tr>
<tr>
<td>3.0 (m)</td>
<td>0.4 (m)</td>
</tr>
<tr>
<td>Width (chord), $w_b$</td>
<td>Width, $w_p$</td>
</tr>
<tr>
<td>1.0 (m)</td>
<td>0.2 (m)</td>
</tr>
<tr>
<td>Thickness, $t_b$</td>
<td>Thickness, $t_p$</td>
</tr>
<tr>
<td>0.04 (m)</td>
<td>0.5 (mm)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Location of piezoelectric strips</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
</tr>
<tr>
<td>0.1 (m)</td>
</tr>
<tr>
<td>$\Delta x_0$</td>
</tr>
<tr>
<td>0.1 (m)</td>
</tr>
<tr>
<td>$y_0$</td>
</tr>
<tr>
<td>0.6 (m)</td>
</tr>
<tr>
<td>$\Delta y_0$</td>
</tr>
<tr>
<td>0.2 (m)</td>
</tr>
</tbody>
</table>
The full transient analysis of ANSYS is used to obtain the time response of the aeroelastic oscillations. The results for control of 3-D aeroelastic oscillations subject to the vertical gust loads are presented and discussed. Two types of sharp-edged and harmonic gusts have been studied in the form of changes of the flow angle of attack which is explained in the following section.

It is assumed that at $t = 0$, the wing is moving in air with a constant velocity when it enters a vertical gust which changes the flow condition to unsteady flow. To calculate the aerodynamic loads, the unsteady 3-D aerodynamic model, discussed in Chapter 4, is employed. At each time step, the aerodynamic loads/pressures are calculated in the FORTRAN program and are transferred to ANSYS code to be applied on the wing structure.

The results related to the transient analysis of the aeroelastic oscillations subject to the sharp-edged and harmonic gust loads are presented in Sections 7.3.2 and 7.3.4, respectively.

### 7.3.1 Gust Model

In order to investigate the effectiveness of the control method, the aeroelastic oscillations of a wing structure subject to vertical gust loads are modeled. The gust model for two types of discrete sharp-edged and harmonic gusts are considered in the form of changes to the angle of attack of the flow.

Figure 7.13 shows a discrete sharp-edged and a harmonic gust (Wright and Cooper 2008). The mathematical model of the sharp-edged gust is of the following general form:

$$
\begin{cases}
\Delta \alpha_g = 0 & t < t_g \\
\Delta \alpha_g \neq 0 & t \geq t_g
\end{cases}
$$

(7.3)

where $\Delta \alpha_g$ represents the changes in the flow angle of attack due to the discrete gust and $t_g$ is the time when the wing enters the sharp-edged gust.
Figure 7.13 (a) Sharp-edged gust (b) Discrete harmonic gust (Wright and Cooper 2008)

The harmonic gust is considered here as a cosine function of time as follows:

$$\Delta \alpha_g = \alpha_{gA} \cos(2\pi f_g t) \quad t \geq t_g$$  \hspace{1cm} (7.4)

where $f_g$ is the gust frequency.

### 7.3.2 Control of 3-D Aerelastic Oscillations Caused by Sharp-edged Gust Loads

In this case, the wing moves with constant velocity $U_\infty = 100$ (m/s) along the $x$ axis. The initial angle of attack at $t = 0$ is $\alpha_0 = 5^\circ$, when the wing enters a sharp-edged gust. The change in angle of attack due to the gust is $\Delta \alpha_g = 0.5^\circ$ for $t > t_g$. For the structural model, values of $\alpha_D = 0.1$ and $\beta_D = 0.0001$ are chosen as the Rayleigh damping coefficients.

The uncontrolled oscillations in the form of the transverse displacement are obtained for three different locations on the wing. These points are located at the
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Figure 7.14 Uncontrolled oscillations of the wing due to sharp-edged gust loads

half-chord along x axis and at three locations, close to the wing root, in the middle of the half-span, and at the wing tip. The transverse displacements of the uncontrolled oscillations are shown in Figure 7.14. It should be noted that the amplitude of the uncontrolled oscillations decreases with time, which is due to the aerodynamic damping.

Figure 7.15 presents the output voltages of the piezoelectric sensors for the uncontrolled wing structure. It can be observed that sensors close to the wing root (#2 and 5) have larger output voltage compared to those located close to the wing tip (#8 and 11). This is because of the greater values of $\Delta \theta$, the difference of slopes, in the wing root area. As discussed in Section 5.2, the output voltage of the piezoelectric sensor is proportionally related to $\Delta \theta$, which is greater closer to the clamped end of a cantilever structure. Similar result was observed in the previous section.
CONTROL OF AEROELASTIC OSCILLATIONS OF WING STRUCTURES USING BONDED PIEZOELECTRIC STRIPS

Figure 7.15 Output voltage of sensors at different locations on the wing for uncontrolled oscillations subject to sharp-edged gust loads

In order to suppress the uncontrolled oscillations, a PD controller is applied at t = 0.1 s. The tip response of the uncontrolled oscillations are compared with the controlled ones for different proportional gain values, $K_p = 10$ and $30$, and results are shown in Figure 7.16. The value of the derivative gain is taken as $K_D = 0.2$. The corresponding applied actuator voltages for the controlled case are illustrated in Figure 7.17. The voltages are shown for four actuators placed at different locations along the wing span. Recall that the numbering scheme of actuators is shown in Figure 7.12.

It can be seen in Figure 7.16 that the tip displacements for the case of $K_p = 30$ have almost vanished after 3 seconds, which confirms that the proposed control has effectively suppressed the oscillations. It can also be noted that increasing the proportional gain increases the damping effect of the control system and reduces the settling time. However, there is a 10% steady-state error for tip displacements.
RESULTS ON THE CONTROL OF THE AEROELASTIC OSCILLATIONS

This is due to the lack of an integral feedback and can be eliminated by adding one to the PD controller.

It can be observed that in Figure 7.17 higher values of actuator voltage are needed at the beginning of the application of control which is mainly due to higher magnitudes of the oscillations and sensor voltage values. However, after a couple of cycles, as the oscillations are suppressed, the actuator voltage decreases. By comparing the voltages of the different actuators, it can be found that the actuators close to the root require higher values of voltage which is mainly due to the concentration of shear strain and stress in that area. It can also be noted that the values of actuator voltage oscillate around a non-zero steady-state value which can be also eliminated by adding an integral feedback to the control system.
Figure 7.17 Actuator voltages for the case of PD controller for different actuator locations on the wing (a) $K_p = 10$ (b) $K_p = 30$ ($K_d = 0.2$)
RESULTS ON THE CONTROL OF THE AEROELASTIC OSCILLATIONS

Figure 7.18 (a) Transverse displacement and (b) actuator voltages for the case of PD controller for actuator voltage limitation ($K_p = 30, K_D = 0.2$)
By examining Figure 7.17, one can observe that in the first few cycles, before $t = 1\, \text{s}$, the value of the actuator voltage is relatively high which is greater than the maximum operating voltage for piezoelectric actuators. Hence, a voltage limit can be introduced in the control scheme, due to the voltage limit that piezoelectric actuators can tolerate. The maximum operating voltage for the actuator is determined by the layer thickness.

For high-voltage PZTs with the thickness range of 0.5 to 1.0 mm, the maximum voltage is of the order of 1 to 2 kV/mm. Here, a limit of 500 volts is applied and the corresponding results are shown in Figure 7.18. By comparing the tip displacement results for the limited-voltage case and the no-limit case, it is evident that the performance of controller decreases by introducing a voltage limit. However, this deterioration in performance occurs in the first few cycles, and later on after $t = 3\, \text{s}$, the controller is able to suppress the oscillations efficiently.
Figure 7.20 Actuator voltages for the case of PID controller for different actuator locations on the wing (a) $K_p = 10$ (b) $K_p = 30$ ($K_i = 10, K_d = 0.2$)
Figure 7.21 (a) Transverse displacement and (b) actuator voltages for the case of PID controller for actuator voltage limitation ($K_p = 30, K_i = 10, K_d = 0.2$)
By adding an integral term to the PD controller with $K_I = 10$, the PID controller is applied at $t = 0.1$ s to reduce the amplitude of oscillations. Figure 7.19 presents a comparison of the tip response for two controllers with $K_P = 10$ and 30. The values of the derivative and integral gains are considered respectively as $K_D = 0.2$ and $K_I = 10$. The applied actuator voltages for two controlled cases are shown in Figure 7.20 for actuators #2, #5, #8, and #11. The numbering scheme of actuators is as shown in Figure 7.12.

One can note that the tip displacements for $K_P = 30$ have vanished after 3 seconds. It is also noticeable that the small steady state error is eliminated and the values of actuator voltage reach zero. Similar simulations are performed by limiting the actuator voltage to 500 volts which are shown in Figure 7.21. The results confirm the previous findings that the limiting the actuator voltage decreases the controller performance to some extent, yet the controller is still effective in damping the oscillations.

### 7.3.3 Effect of Actuators Location

To study the effect of actuator location on the control effectiveness, different scenarios of the actuator configurations are compared. In each scenario, some of the actuators have been deactivated. Table 7.3 summarizes the scenarios and different cases in each scenario.

In the first scenario, the effect of spanwise location is studied. More precisely, the actuators close to the wing root and the wing tip are compared. In case (i), the actuators of the first column (actuators #1, #2, and #3 as indicated in Figure 7.12) are deactivated and the rest of the actuators are active. In case (ii), just the actuators of the first column (#1, #2, and #3) are active and the rest are inactive.

The PID controller gains are considered to be the same ($K_P = 30, K_I = 10, K_D = 0.2$) for both cases of (i) and (ii). The results for these two cases are compared with the case when all the actuators are active and are presented in Figure 7.22. The obtained results show that the actuators close to the wing root
Table 7.3 Different scenarios on the effect of actuator placement

<table>
<thead>
<tr>
<th>Scenario 1: Effect of the wing root actuators</th>
<th>Case (i): Actuators #1, #2, and #3 are inactive, the rest are active.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case (ii): Actuators #1, #2, and #3 are active, the rest are inactive.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario 2: Comparison of the actuators in each column with the constant voltage</th>
<th>Case (a): Only actuators #1, #2, #3 are active.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Case (b): Only actuators #4, #5, #6 are active.</td>
</tr>
<tr>
<td></td>
<td>Case (c): Only actuators #7, #8, #9 are active.</td>
</tr>
<tr>
<td></td>
<td>Case (d): Only actuators #10, #11, #12 are active.</td>
</tr>
</tbody>
</table>

(#1, #2, and #3) are more effective than other actuators. This can be due to the fact that the collocated sensor voltages at these locations have higher values. Similar results have been reported in Mateescu et al. (2010).

To consider the fact that the actuators of the first column have greater voltages than other columns, a second scenario is studied. In the second scenario, the values of actuator voltage are considered constant and equal to the voltages of the actuators of the first column in the first scenario. The simulated results are shown in Figure 7.23 for four different cases. In each case, only one column of actuators is active, and the rest are turned off. For instance, in case (a), only the actuators of the first column (#1, #2, and #3) are active and other columns are inactive. In cases (b), (c), and (d), the actuators of the second, third and forth columns are active, respectively.

It can be observed that as the active actuators are closer to the wing root, their effectiveness improves. In fact, case (a) is the most effective configuration and is able to suppress the oscillations in smaller amount of time. It should be noted that there is a restriction in placing the actuators close to the wing root, since those actuators need relatively higher values of applied voltage which might be beyond the maximum voltage limit of piezoelectric strips.
Figure 7.22 Comparison of the different configurations of actuator placement: first scenario \((K_p = 30, K_i = 10, K_d = 0.2)\) (a) normal view (b) enlarged view
Figure 7.23 Comparison of the different configurations of actuator placement: second scenario \((K_p = 30, K_f = 10, K_D = 0.2)\) (a) normal view (b) enlarged view
7.3.4 Control of 3-D Aeroelastic Oscillations Caused by Harmonic Gust Loads

The mathematical model of the harmonic gust is explained in Section 7.3.1. In this case, the wing velocity is considered to be constant $U_\infty = 100 \text{ (m/s)}$ along the x axis. The initial angle of attack at $t = 0$ is $\alpha_0 = 5^\circ$, when the wing enters a harmonic gust. The change in angle of attack due to the harmonic gust is considered a cosine function of time as described in equation (7.4):

$$\Delta \alpha_g = \alpha_{gA} \cos(2\pi f_g t)$$  \hspace{1cm} (7.5)

where $f_g = 1.5 \text{ (Hz)}$ is the gust frequency.

For the structural model, the values of Rayleigh damping coefficients are considered to be $\alpha_D = 0.1, \beta_D = 0.0001$. The transverse displacements of the wing are shown in Figure 7.24.

![Figure 7.24 Uncontrolled oscillations of the wing due to harmonic gust loads](image)
uncontrolled oscillations are obtained for three different locations on the wing and shown in Figure 7.24. The output voltages of the piezoelectric sensors for the uncontrolled wing structure are presented in Figure 7.25.

To suppress the uncontrolled oscillations, a PID control is applied at $t = 0.1 \text{ s}$. A comparison between the tip response of the uncontrolled and controlled oscillations is shown in Figure 7.26 for two different proportional gain values, $K_p = 15$ and 35. The derivative and integral gains are considered as $K_d = 0.4$, $K_i = 10$, respectively.

It can be observed that the transient oscillations have been suppressed after $t = 2.5 (\text{ sec})$ and the amplitude of tip displacement reaches its steady-state value. It confirms that the PID control can successfully reduce the amplitude of the oscillations. Also, comparing the results for different proportional gains shows...
Figure 7.26 (a) Transverse displacement and (b) actuator voltages for the case of PID controller subject to harmonic gust loads ($K_p = 35, K_i = 10, K_d = 0.4$)
that larger gains \( (K_p = 35) \) can remove the transient oscillations faster, which was also observed in the case of sharp-edged gusts.

Figure 7.26 (b) shows the applied actuator voltages for the controlled case of Figure 7.26 (a). The voltages are shown for four actuators placed at different locations along the wing span based on the numbering scheme shown in Figure 7.12. Actuator voltage has been limited to a maximum of 500 volts similar to the previous case. As the transient response vanishes, the applied actuator voltage is reduced as well, but the steady-state voltage remains constant. Again, the actuators placed close to the wing root require larger voltages due to the concentration of shear strain and stress close to the wing root.
8.1 Summary and Conclusions

The aeroelastic oscillations of the wing structures subject to the unsteady subsonic aerodynamic loads are studied in this thesis and an active feedback control is designed and applied to suppress the oscillations using bonded piezoelectric strips. The following summarizes the work done and the conclusions drawn from the present study.

As a preliminary step, an aerodynamic model for two-dimensional unsteady flows was developed and validated using the existing literature. A numerical panel method using the vortex singularities was employed to calculate the pressure distribution and the aerodynamic loads on the wing. The results of the present model were compared with the pertinent results in the published literature for both cases of the steady and unsteady flows and it was shown that the
developed model is reliable and can accurately describe the steady and unsteady 2-D subsonic flows.

The 2-D aerodynamic model is considered firstly in order to gain confidence in the numerical model and to design a control methodology to suppress the aeroelastic oscillations. However, as mentioned before, the main goal of the present work is to analyze and control the aeroelastic oscillations arising from the 3-D unsteady subsonic aerodynamic loadings which have not been addressed, or addressed insufficiently, by the prior literature.

Next, an aerodynamic model was developed and validated for the more complex case of the three-dimensional unsteady flows past flexible wings executing flexural and torsional oscillations. Vortex ring elements were used in the panel method and the generalized aerodynamic forces were obtained for both cases of the steady and unsteady flows. For the special simple cases where similar results exist in the literature, the results of the present model were compared and validated. It was found that the three-dimensional aerodynamic model can reliably calculate the velocity field around the wing for both the steady and unsteady flows.

The dynamic model and the finite element formulation for the wing structure combined with the bonded piezoelectric strips were presented in this thesis. The constitutive equations of the piezoelectric material were used for the combined finite element model. A coupling was considered between the structural and aerodynamic models in order to simultaneously solve the time-dependent dynamic equations of the elastic structure and the unsteady equations of the aerodynamic flow. This coupling was considered as an interactive computer model between aerodynamic and structural analysis codes.

An active control model was proposed to suppress the aeroelastic oscillations of the wing structure using piezoelectric sensors and actuators. A PID controller was used which employed the output voltage of the sensor strips as feedback to obtain the applied voltage of the actuator strips. To obtain the proper gains for the PID controller, a systematic approach was presented to calculate the feedback
gains by using the system matrices. The presented approach was tested for a sample numerical simulation and showed good agreement between the analytical and the numerical values.

The results of the numerical simulations for control of aeroelastic oscillations caused by 2-D unsteady aerodynamic loads were presented. An appropriate PID control was designed and applied to suppress the oscillations. It was found that the PID control can suppress the oscillations effectively with relatively small gains, hence lower cost. The influence of the feedback gains was studied and it was shown that the proportional gain has a greater role in reducing the amplitude of the oscillations, although the derivative and the integral gains are quite necessary as well.

The numerical results for control of 3-D aeroelastic oscillations subject to the vertical gust loads were presented. An appropriate PID control was designed and applied to suppress the oscillations caused by two types of discrete sharp-edged and harmonic gusts. It was demonstrated that the PID control is effective in controlling aeroelastic oscillations within short time and with relatively small gains. The effects of the feedback gains were investigated for various cases. In addition, an analysis was performed to examine the effects of actuator placement on the wing surface in suppression of oscillations for various scenarios. It was shown that the actuators placed close to the wing root are more effective in reducing the amplitude of the aeroelastic oscillations and are able to suppress the oscillations in less time.

8.2 Recommendations for Future Work

In the future research related to the present study, the following areas of improvement can be potentially addressed:

1- As an important area for further studies, some experiments can be conducted to support the numerical findings presented in this thesis. The experimental analyses such as the wind tunnel tests can be performed to simulate the
aeroelastic oscillations of a trapezoidal wing bonded with piezoelectric strips and also to suppress the oscillations using the proposed PID control model.

2- The theoretical models presented in this thesis can also be improved in the future research. As an improvement to the aerodynamic model, the effect of the wing thickness and the viscous effects can be considered in the aerodynamic model. A finite volume method for the solution of the Navier-Stokes equations can be developed for the unsteady aerodynamics of the oscillating wings in order to improve the accuracy of the computed aerodynamic loads.

3- As an improvement to the structural model, the effects of geometrical nonlinearities can be considered in the model to include the large deflection oscillations. The effect of material nonlinearities and other parameters such as temperature can also be considered in the structural model.

4- As an improvement to the control model, other active control methods can be studied such as LQR, LQG and nonlinear feedback controllers. Although as it was mentioned, these control methods are too complicated and expensive to implement in a high order system, the possibility of applying them in a reduced order model can be studied.

5- As a possible future work, a study can also be performed to investigate the effect of the size of the piezoelectric sensors and actuators and obtain the optimal size of the strips. Also, and optimization analysis can be performed to obtain the optimal location of the piezoelectric strips on the wing surface.
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APPENDIX A

Piezoelectric and Structural Material Properties

Piezoelectric material (PZT):

Stiffness matrix: 
\[
\begin{bmatrix}
14.7 & 8.1 & 8.1 & 0 & 0 & 0 \\
14.7 & 8.1 & 0 & 0 & 0 & 0 \\
13.2 & 0 & 0 & 0 & 0 & 0 \\
3.1 & 0 & 0 & 0 & 0 & 0 \\
3.1 & 0 & 0 & 0 & 0 & 0 \\
3.1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\times 10^{10} \quad (N/m^2)
\]

Piezoelectric stress matrix: 
\[
\begin{bmatrix}
0 & 0 & -3.88 \\
0 & 0 & -3.88 \\
0 & 0 & 13.91 \\
0 & 0 & 0 \\
0 & 10.34 & 0 \\
-10.34 & 0 & 0 \\
\end{bmatrix}
\quad (C/m^2)
\]

Dielectric matrix: 
\[
\begin{bmatrix}
8.003 & 0 & 0 \\
0 & 8.003 & 0 \\
0 & 0 & 2.252 \\
\end{bmatrix}
\times 10^{-9} \quad (F/m)
\]

Density: \( \rho = 7760 \quad (Kg/m^2) \)

Wing structure (Aluminum):

Young’s modulus: \( E = 7.03 \times 10^{10} \quad (Pa) \)

Poisson’s ratio: \( \mu = 0.345 \)

Density: \( \rho = 2690 \quad (Kg/m^2) \)