A ultraviolet complete large $N$ thermal QCD model: renormalization group flow and mesonic spectra

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DEDICATION

This thesis is dedicated to my friends.
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I thank my friends for their support and my collaborators for inspiring discussions. I would especially like to thank my supervisor Prof. Keshav Dasgupta for his advising.

Contribution of Authors:
Fang Chen, Mahammed Mia and Olivier Trottier have made significant contributions to chapter 10.
ABSTRACT

The large $N$ thermal Quantum Chromodynamics (QCD) that we study here is a generalization of $SU(3)$ QCD with two gauge groups at high energy scales, i.e. in the ultraviolet (UV). Both gauge groups are related to $N$ as $SU(N) \times SU(kN)$ with $k < 1$. In our construction, we have a model that has a complete UV description, as well as a low energy, i.e. infrared (IR), behaviour that mimicks that of QCD. Our model has three regions. The IR region is the well known cascading model with extensions to include flavours. The two gauge groups of this theory reduce to a single $SU(kN)$ group which then confines. The UV region uses configurations from type IIB string theory on the conifold to describe the dynamics. The intermediate region smoothly connects the two, i.e. the UV and the IR, so that we have a complete model for all energy scales. All regions are related to the gauge theory by gauge/gravity duality. In particular, we analyze the renormalization group (RG) flow of the gauge coupling in the effective field theory. We see that it asymptotically approaches a constant at high energies and flows toward permanent confinement as energy scale decreases. In the IR, we obtain the mesonic spectrum by compactifying extra dimensions and analyzing the wave functions in Minkowski space-time.
Cette thèse étudie la limite $N \to \infty$ de la théorie chromodynamique quantique avec les deux groupes de jauge $SU(N) \times SU(kN)$ avec $k < 1$. Un modèle est posé pour donner une description complète à la fois dans le régime ultraviolet (UV) et infrarouge (IR). Le régime IR concorde avec celui d’un modèle de cascade, qui réduit de seul groupe de jaude $SU(N)$ puis confine, tandis que le régime UV correspond à la théorie des cordes de type IIB sur conifold. Le régime UV possède alors une dualité de supergravité/jauge s’inscrivant dans la correspondance AdS/CFT. Dans le régime intermédiaire, notre modèle connecte de manière régulière entre IR et UV. Le group de renormalization de la théorie effective est ensuite analysé. Nous comparons les résultats expérimetaux en spectre du méson de notre modèle dans le régime IR.
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At microscopic scales, we observe point-like particles that make up different materials. In the early 20th century, scientists called ‘atoms’ the newly discovered smallest particles. With further tests, they showed that there is structure inside atoms. For example, Rutherford discovered a central charge inside atoms in 1911. In the original atomism, atoms were supposed to be indivisible and could not combine into one. Even though atoms can be further divided into a nucleus and electrons, the name atoms has still been used today. According to the theory of quantum mechanics, electrons do not have a definite location inside atoms, instead they have a probability distribution function and form a so-called electron cloud. Electrons are indivisible particles. Nucleons, on the other hand, are made of protons and neutrons and these can be further divided into quarks. These are the most common particles, scientists already discovered several other particles and postulated more hypothetical particles. The search for fundamental particles continues. The study of subatomic structures is still one of the frontiers of science.

The study of quantum mechanics shows that the quantum effects of particles at small scales becomes important and that quantum field theory is a fairly accurate description of interactions of subatomic particles. Point-like particles are described
as fields, meaning that there is a distribution function for their positions and numbers. The interactions of particles are represented by Feynman diagrams. The decay process of unstable elements can also be represented in this way. These particles are classified by spin, colours and other fundamental properties. However, mass, for example, is not a fundamental property. The true value, or the bare mass, will get modified for a renormalizable theory.

The particles interact with each other via scattering processes. The type of interaction is characterized by the different forces. There are four fundamental forces known in nature: the electromagnetic force, the weak force, the strong force and gravity. The first two types are unified to become the electroweak force. The presence of this type of force implies that the particles are weakly coupled to each other. From the name we already know that the next type, strong force, implies that particles interact strongly. The type of force involved in the interactions between particles depends on the fundamental properties of the particles.

The dynamics of physics is usually characterized by the energy scales. At short distance and high energy scales, certain particles are strongly coupled to each other. A theory that describes the strong interactions (i.e. the interactions of gluons and quarks), namely Quantum Chromodynamics (QCD), emerged in 20th century. Standard perturbative techniques consists of computing loop diagrams to obtain e.g. S-matrix elements and decay rates. This is based on the assumption that particles interact weakly with each other. We need other methods to solve problems involving strong interactions. There is a potential successful theory to understand strong interactions called string theory, which is also theory of gravity. The theory itself demands
extra dimensions to be visible at high energy. String theory also has applications to condensed matter, algebraic geometry and other subjects.

The gauge/gravity duality transfers problems from quantum field theory to string theory. A perturbative string theory with weak couplings has a corresponding quantum field theory description with strong couplings. Since perturbative methods can still be trusted in the string theory, we can use these to understand strong interaction problems in quantum field theory.

Supersymmetry (SUSY) is a another useful tool. This requires an equal number of fermions and bosons in the theory, and the enhanced symmetry simplifies the dynamics of the theory. Both field theories and string theory can be supersymmetric. The worldsheet of the superstring is naturally described by a conformal field thoery. Without supersymmetry, string theory has only bosonic strings whose stable vacuum is not known, so we usually talk about superstring models without mentioning the word superstring. However, we haven’t observed supersymmetric particles in nature. This gives a hint that SUSY structure can perhaps only exist at very high energies, but the real world physics operates at low energies, so we usually extend the theory to high energies and then break the supersymmetry to find the corresponding low energy effective field theory. Other main techniques we used include strong-weak coupling duality and AdS/CFT correspondence, which relates gravity on a certain background space-time to a class of supersymmetric gauge theories with fixed couplings. Remarkably the $\mathcal{N} = 4$ super Yang-Mills theory is also related to a type IIB string theory model by AdS/CFT correspondence.
Combining various theoretical approaches, we construct a string model at high energy and discuss the dynamics at all scales. The theory is asymptotically conformal. As the energy scales gets lower, the conformal invariance is broken. Although in general the non-AdS/non-CFT correspondence is not known, we are able to add a transitional region to connect the UV and the IR. This region has non-trivial fluxes that dominate the dynamics. Our hope is to understand strong interactions using string theory and try to understand unsolved problems in QCD. We will focus on a ultraviolet (UV) complete thermal QCD model in the limit where the number of colours is very large. This thesis reviews the basic ingredients needed to construct this model. Later chapters are built on the knowledge of previous chapters. Chapter 2 introduces notion of supersymmetry. Chapter 3 introduces large $N$ expansion. Chapter 4 discusses how the coupling changes for large $N$ supersymmetric QCD (SQCD). Chapter 5 analyzes how dualities act on the couplings of large $N$ SQCD theories. Chapter 6 uses a series of Seiberg dualities to construct a particular type of $SU(N)$ SQCD. Chapter 7 and 8 introduce string theory and related AdS/CFT correspondence. Chapter 9 gives a string theory interpretation of the SQCD model. Chapter 10 introduces the extensions and chapter 11 is the complete picture of our model. The new result will be discussed in chapter 11.
CHAPTER 2
Supersymmetric Field Theories

2.1 Supersymmetric Fields

Supersymmetry is a type of symmetry that pairs up bosons and fermions. They are two distinct types of particles: fermions obey the Pauli exclusion principle, their respective quantum fields are anti-commuting; bosons can pile up at the same state, they don’t obey the Pauli exclusion principle, and the quantum fields satisfy commutation relations. In supersymmetric gauge theories, the number of supersymmetries is given by $N$. For $N = 1$ supersymmetric theories, there are two sectors, described by chiral or vector superfields. They live on the superspace, which has both commuting coordinates and anti-commuting coordinates. Grassman variables $\theta, \bar{\theta}$ represent anti-commuting coordinates. One basic operation of Grassman variables is integration, defined as follows [3]:

$$\int d\theta^\alpha = \int d\bar{\theta}^{\dot{\alpha}} = 0$$
$$\int d\theta^\alpha \theta^\alpha = \int d\bar{\theta}^{\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} = 1$$

$$d^2 \theta = -\frac{1}{4} d\theta d\theta = -\frac{1}{4} d\theta^\alpha d\theta_{\alpha}$$
$$d^2 \bar{\theta} = -\frac{1}{4} d\bar{\theta} d\bar{\theta} = -\frac{1}{4} d\bar{\theta}^{\dot{\alpha}} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}^{\dot{\alpha}}$$
$$d^4 \theta = d^2 \theta d^2 \bar{\theta}$$
There are two spinor indices on Grassmanian variables, $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$.

A chiral superfield $\Phi(x, \theta)$ satisfies [32]:

$$\bar{D}_{\dot{\alpha}} \Phi = 0 \quad (2.3)$$

where $D_{\dot{\alpha}}$ is defined as:

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\bar{\theta}^{\alpha} \sigma^\mu_{\alpha \dot{\alpha}} \partial \mu \quad (2.4)$$

and $\sigma^\mu$ is the $\mu$'th Pauli matrix.

Written in terms of component fields, $\Phi$ is expressed as:

$$\Phi = A(y) + \sqrt{2} \theta \psi(y) + \theta \theta F(y)$$

$$\quad = A(x) + i\theta \sigma^m \bar{\theta} \partial_m A(x) + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \Box A(x) + \sqrt{2} \theta \psi(x) - \frac{i}{\sqrt{2}} \theta \theta \partial_m \psi(x) \sigma^m \bar{\theta} + \theta \theta F(x) \quad (2.5)$$

where $A(y)$ is a scalar field, $\psi(y)$ is the spinor field and $F(y)$ is an auxiliary field, which has no kinetic term. Its Hermitian conjugate is

$$\Phi^\dagger = A^*(y^\dagger) + \sqrt{2} \theta \bar{\psi}(y^\dagger) + \bar{\theta} \bar{\theta} F^*(y^\dagger)$$

$$\quad = A^*(x) - i\theta \sigma^m \bar{\theta} \partial_m A^*(x) + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \Box A^*(x) + \sqrt{2} \bar{\theta} \psi(x) + \frac{i}{\sqrt{2}} \bar{\theta} \theta \sigma^m \partial_m \bar{\psi}(x) + \bar{\theta} \bar{\theta} F^*(x) \quad (2.6)$$

Vector superfields satisfy the condition:

$$V = V^\dagger \quad (2.7)$$
The Hermitian conjugate of a vector superfield is itself so we call it real. In component form, we find that

\[
V(x, \theta, \bar{\theta}) = C(x) + i\theta \chi(x) - i\bar{\theta} \bar{\chi}(x) + \frac{i}{2} \theta \theta [M(x) + iN(x)] - \frac{i}{2} \bar{\theta} \bar{\theta} [M(x) - iN(x)] \\
- \theta \sigma^m \bar{\theta} v_m(x) + i\theta \bar{\theta} \bar{\lambda}(x) + \frac{i}{2} \bar{\sigma}^m \partial_m \chi(x)] - i\bar{\theta} \theta [\lambda(x) + \frac{i}{2} \sigma^m \partial_m \bar{\chi}(x)] \\
+ \frac{1}{2} \theta \bar{\theta} \bar{\theta} [D(x) + \frac{1}{2} \Box C(x)]
\]

(2.8)

where \(C(x), D(x), M(x), N(x)\) are real scalar fields, \(v_m(x)\) is a real vector field, \(\lambda(x)\) and \(\chi(x)\) are complex fermions. The bar denotes complex conjugate.

In any supersymmetric theory, \(R\) symmetry is a rotation of fields that leaves the Lagrangian invariant. In \(\mathcal{N} = 1\) supersymmetric gauge theories, it acts on chiral superfields as:

\[
R\Phi(\theta, x) = e^{2i\alpha} \Phi(e^{-i\alpha} \theta, x) \\
R\Phi^\dagger(\bar{\theta}, x) = e^{-2i\alpha} \Phi^\dagger(e^{i\alpha} \bar{\theta}, x)
\]

(2.9)

where \(x\) is usual commuting space-time coordinate, \(\alpha\) is the angle of internal rotation and \(n \in \mathbb{Z}\). In terms of component fields, this is

\[
A \rightarrow e^{2i\alpha} A \\
\psi \rightarrow e^{2i(n-\frac{1}{2})\alpha} \psi \\
F \rightarrow e^{2i(n-1)\alpha} F
\]

(2.10)
2.2 Supersymmetric Transformation

From Noether’s theorem, we know that every symmetry leads to an invariant quantity. The supersymmetric transformations are defined in a way such that the superfields and their combinations would remain invariant in the Lagrangian. The infinitesimal supersymmetry transformation for a chiral superfield is

\[
\delta \Phi = i(\xi Q + \bar{\xi} \bar{Q})\Phi \\
\delta \phi = \sqrt{2} \xi \psi \\
\delta \psi = \sqrt{2} \xi F - \sqrt{2} \partial_\mu \phi \sigma^{\mu} \bar{\xi} \\
\delta F = i \sqrt{2} \partial_\mu \psi \sigma^{\mu} \bar{\xi} \tag{2.11}
\]

where \( \xi \) is an anti-commuting parameter and \( Q \) is a differential operator. Once we define the superfields, we can construct a superpotential which is also invariant under supersymmetric transformation. The superpotential could have three chiral superfields or mass term with two chiral super fields, both have dimension four [3].

For vector fields, the gauge transformation is

\[
\delta V = i[\Phi(\theta, \bar{\theta}, x) - \Phi^\dagger(\theta, \bar{\theta}, x)] \\
\tag{2.12}
\]

The following combination of superderivative and vector field is covariant

\[
W_\alpha = -\frac{1}{4} \bar{D}D e^{-V} D_\alpha e^V \\
W_\alpha \rightarrow e^{-i \Lambda} W_\alpha e^{i \Lambda} \tag{2.13}
\]

where \( \Lambda \) is the infinitesimal transformation.

In classical mechanics, the Lagrangian of a system is the difference between the kinetic energy and the potential energy. From that expression we can derive the
equation of motion, and then analyze the motion of objects. In gauge theories, the Lagrangian formalism could give details about the local symmetry of gauge fields. With supersymmetry, the Lagrangian is a combination of superfields which includes kinetic and superpotential terms. The most general form of a Lagrangian that is invariant under supersymmetric transformation is [32]:

\[
L = \frac{1}{16g^2} \text{Tr}(W^\alpha W^\alpha|_{\theta\theta}) + \bar{W}_\alpha \bar{W}^\alpha|_{\bar{\theta}\bar{\theta}} + \Phi^\dagger e^V \Phi|_{\theta\theta\bar{\theta}\bar{\theta}} + \left[\left(\frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k\right|_{\theta\theta} + h.c\right]
\]

(2.14)

where \( g \) is the gauge coupling.

### 2.3 F and D Term SUSY Breaking

Four-dimensional supersymmetric gauge theories are first developed as possible solutions to the hierarchy problem. This problem comes from the difference in observed energy scales. The gap between the weak force and gravity could be explained by the Higgs coupling, which couples more to heavy particles. Strong renormalization due to possible divergences at short distance are avoided if the supersymmetric particles also contribute to the correction in the coupling and the mass, so the two balance each other. However, equal numbers of fermions and bosons have not been observed yet, so supersymmetry should be broken at a scale higher than the ones achieved in any lab. The breaking process can be spontaneous and in that case it could be F-term supersymmetry breaking or D-term supersymmetry breaking. D and F fields are auxiliary fields that appeared in the component form of chiral and vector superfields, see 2.5 and 2.8. The name auxiliary means the fields don’t have kinetic terms.
The D-term equation gives constraints for the scalar fields. It is the coefficient of the term $\theta \bar{\theta}$ in the expansion of $\Phi^\dagger \Phi$. Suppose we add one chiral field with charge $e$ to the system, the D-field with the addition of a Fayet-Iliopoulos D-term is [3]:

$$D = - (\xi + e \phi^\dagger \phi) \quad (2.15)$$

where $\xi$ is a coefficient. This forces the scalar to have a non-zero mass:

$$m^2_\phi = e \xi \quad (2.16)$$

The F-term equation is:

$$F_i^\dagger = - \frac{\partial W(\Phi)}{\partial \phi_i} \quad (2.17)$$

where $W(\Phi)$ is the superpotential.

2.4 $\mathcal{N} = 1$, $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Supersymmetric Gauge Theories

The supersymmetric theories may contain multiple copies of the minimal amount of supersymmetry. The number $\mathcal{N}$ represent the number of copies. All allowed physical states have helicity from $\lambda$ to $\lambda - \mathcal{N}/2$. This can be seen as [1]:

$$\mathcal{N} = 1 : \quad |\lambda\rangle, \ |\lambda - \frac{1}{2}\rangle$$

$$\mathcal{N} = 2 : \quad |\lambda\rangle, \ 2|\lambda - \frac{1}{2}\rangle, \ |\lambda - 1\rangle \quad (2.18)$$

$$\mathcal{N} = 4 : \quad |\lambda\rangle, \ 4|\lambda - \frac{1}{2}\rangle, \ 6|\lambda - 1\rangle, \ 4|\lambda - \frac{3}{2}\rangle, \ |\lambda - 2\rangle$$
$N = 1$ theories have chiral multiplets $(A, \psi)$ and vector multiplets $(A_\mu, \lambda)$ [5]. Without matter, the Lagrangian of an $N = 1$ pure gauge theory is

$$L_{N=1} = \frac{1}{32\pi} \text{Im} \left( \tau \int d^2 \theta \, \text{Tr} W^\alpha W_\alpha \right)$$

$$= \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \lambda + \frac{1}{2} D^2 \right] + \frac{\Theta}{32\pi^2} g^2 \text{Tr} F_{\mu\nu} F^{\mu\nu}$$

(2.19)

With matter, the Lagrangian is modified to additionally include

$$L_m = \int d^2 \theta d^2 \bar{\theta} \phi^\dagger e^V \phi + \int d^2 \theta W(\Phi) + \int d^2 \bar{\theta} W(\Phi)^\dagger$$

(2.20)

$N = 2$ theories have hypermultiplets $(q, \tilde{q}^\dagger, \psi_q, \tilde{\psi}_q^\dagger)$ and vector multiplets $(A_\mu, \phi, \psi, \lambda)$. The general form of the Lagrangian is

$$L_{N=2} = \frac{1}{32\pi} \text{Im} \left( \tau \int d^2 \theta \, \text{Tr} W^\alpha W_\alpha \right) + \int d^2 \theta d^2 \bar{\theta} \, \text{Tr} \phi^\dagger e^{2gV} \phi$$

$$= \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i \lambda \sigma^\mu D_\mu \lambda - i \psi \sigma^\mu D_\mu \bar{\psi} + (D_\mu z) \dagger D^\mu z \right] + \frac{\Theta}{32\pi^2} g^2 \text{Tr} F_{\mu\nu} F^{\mu\nu}$$

$$+ \frac{1}{2} D^2 + f^\dagger f + i \sqrt{2} g z \dagger \{\lambda, \psi\} - i \sqrt{2} g \{\bar{\psi}, \bar{\lambda}\} z + g [z, z^\dagger]$$

(2.21)

where $\Theta$ is the theta angle of the gauge fields.

$N = 4$ theories have the maximal number of supersymmetries and the gauge theory is superconformal.
CHAPTER 3
Large $N$ Expansion

Imposing supersymmetry in the system could avoid divergence in Higgs coupling, but it still doesn’t solve the strong coupling problem. The total energy in the system is usually represented by the Hamiltonian, which is the sum of the kinetic and the potential energy. If a system is given a small perturbation, the new Hamiltonian can be described by the original Hamiltonian plus some corrections, but when the coupling is strong, perturbative methods are not useful. Loop calculations are based on the assumption that particles interact weakly. Each loop gives a small correction to the scattering amplitude. However, the dynamics involving strongly coupled particles such as quarks, cannot be solved using conventional perturbative methods. A way out of this is to change the gauge group of the theory. Let us illustrate this in the following paragraph.

The gauge group of QCD is $SU(3)$. That means quarks have three colours: green, blue and red. The internal symmetry among three colours form the gauge group. To describe how quarks interact, we use double line notation. If the resulting scattering diagrams have no crossing, they are called planar diagrams; if not, they are called the non-planar diagrams. If we extend the theory to large $N$, QCD gets relatively simple, meaning that the contribution of non-planar diagrams to the scattering amplitude are suppressed. We say in this case the non-planar diagrams decouple. The amplitudes have an expansion in order of $1/N$. 
We assume large $N$ QCD model can be continuously deformed into real-world QCD model. In that way, studying large $N$ model is a good starting point to solve problems in QCD. In the following sections, we show explicitly how large $N$ expansion arises in scattering amplitudes.

### 3.1 Perturbative Methods in Quantum Field Theory

In quantum field theory, Feynman diagrams are a useful tool to study the particle interactions. It illustrates the scattering process order by order. Feynman diagrams are drawn in position space. One side is the initial state, the other side is the final state. The arrow indicates direction of propagation. The trajectory is presented by lines, dashed lines, waves or spirals. Usually, lines represent quarks and electrons; dashed lines represent scalars; waves represent photons, W or Z bosons; and spirals represent gluons. The vertices join different legs and represent particle creation and annihilation. Flavour distinguishes different types of quarks. Colours are used to label gluons. Finally, in any given diagrams charge and momentum must be conserved.

For example, consider the electroweak interactions. The one-loop diagram for a photon propagating in space-time is

![Figure 3–1: One-loop diagram for photon propagation. The internal loop represents a fermion.](image)
The intermediate state of scattering events at tree level represent force carriers. Electroweak force carrier are virtual photons, W and Z bosons, the strong force carrier are the gluons. Gluons can interact with quarks or among themselves.

In fact, we can draw many diagrams with different orientations that are all equivalent, so we put a combinatoric factor in front of each diagram count them. For each vertex, the couplings of fields contribute to the amplitude because this is an interaction. To have finite vacuum polarization amplitude in the large $N$ limit, we must have the couplings to be at order of $g/\sqrt{N}$. This combines with combinatoric factors to give a finite amplitude independent of $N$.

The diagrams of gluons can be studied in more details using double line notation. Instead of using a spiral to represent gluons, we use double lines with opposite arrows. The vertex of three gluons appearing in a vacuum polarization diagram is written as $A_\alpha^\beta A_\delta^\gamma \partial_\alpha A_\delta^\rho$.

![Figure 3–2: Double line notation for gluons, $q$ represents a quark, $\bar{q}$ represents an antiquark](image)

If the diagrams have no crossing of lines, it is called the planar diagram. Our large $N$ expansion keeps all the planar diagrams in the theory. Another type of diagrams are non-planar diagrams, they are suppressed in the large $N$ limit. More
precisely, their vertex factors combined with combinatoric factor are at least of order $1/N^2$, so the amplitude of the loop diagram is negligible.

![Figure 3–3: An example of six vertices planar diagram [17]](image)

![Figure 3–4: An example of six vertices non-planar diagram [17]](image)

### 3.2 Mesons at Large $N$

There are three generations of quarks and each generation has two quarks. They interact via the strong force. The force carriers, or the excitation fields that let particles interact, are gluons. Quarks cannot exist individually. The gluons hold quarks together and the force between two quarks increases linearly with their separation. This leads to confinement, *i.e.* bound states of quarks. Baryon are made of three
quarks, mesons are made of a quark and an anti-quark. The meson is a good candidate to test new QCD theories because we know its properties relatively well and we have experimental results to compare with. In the large $N$ limit there are three main properties of mesons [33]:

1) In the large $N$ limit, mesons are free and non-interacting particles.

2) Meson decay amplitudes are of order $1/\sqrt{N}$.

3) Mesons at large $N$ limit are pure states, e.g, $q\bar{q}$.

The scattering amplitude at tree level of mesons is of order $1/N$, so indeed they are free particles at leading order and we do need to worry about correlation of the particles at large separations. These properties will be useful later when we discuss the meson spectrum in our model.
CHAPTER 4
Renormalization Group Flow and Beta Functions

The scattering amplitude is usually calculated by summing up loop diagrams, each gives correction to the tree level amplitude. The coupling between particles indicate how strongly they interact. The change of couplings in a theory with respect to energy scale is called renormalization group flow. To illustrate the basic idea, let us study the action of a scalar with mass $m$ [31].

$$S = \int d^d x [\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2] \quad (4.1)$$

$\phi$ is a complex scalar field, $m$ is the mass and $\lambda$ is the coupling. The coupling constant is fixed at certain energy, but as the energy increases, the fields receive contribution form higher order terms in the scattering amplitude, the flow is called renormalization group (RG) flow. We need some function to describe such change. Define a dimensionless mass $\nu = \frac{m}{\mu}$, $\mu$ is the energy scale.

$$\beta_\nu = \frac{\mu}{\partial_\mu} \frac{\partial \nu}{\partial_\mu} = -\nu + \frac{\partial m}{\partial_\mu} \quad (4.2)$$

This is the beta function for coupling $\nu$, its value changes based on energy scale $\mu$, the second term is the quantum correction for the mass. If we only consider the beta function without quantum corrections, as $\mu \to \infty$, $\nu \to 0$ and vice versa [31].
For supersymmetric quantum chromodynamics (SQCD), the beta function of coupling $g$ is

$$\beta_g = -\frac{g^3}{16\pi^2} \frac{3N - N_f(1 - \gamma_0)}{1 - \frac{g^2N}{8\pi^2}}$$

(4.3)

where $N$ is the number of colours, $N_f$ is the number of flavours and $\gamma_0$ is the anomalous dimension of $g$. Rewrite the beta function in terms of the 't Hooft coupling $\lambda = g^2N$, this is:

$$\beta_\lambda = -\frac{\lambda^2}{8\pi^2} \frac{3 - \frac{N_f}{N}(1 - \gamma_0)}{1 - \frac{\lambda}{8\pi^2}}$$

(4.4)
CHAPTER 5
Duality

After learning that couplings could change when energy scale changes, the study of RG flow could help us to better understand the theory itself. How each type of particle contributes to the coupling could give us information about their properties, and eventually explain the scattering amplitude that could be measured in the lab. One common technique is to make a duality transformation. If the strong couplings are dual to weak couplings, then the perturbative methods are still valid. This chapter introduces Seiberg duality and Strong-weak duality and discusses how they are related. Later this technique will be used to trace couplings when the gauge group changes.

5.1 S-duality

Electro-magnetic duality or strong-weak duality (S duality) is a duality that naturally appears in the theory of Quantum Electrodynamics (QED). Dirac has given an argument why the existence of magnetic monopoles would naturally quantize the electrical charge [25]. If there is a monopole somewhere in the universe, we can choose a circular path around that pole and integrate the electrical field. The amplitude would then remain the same but there would be a phase difference of $2\pi$. The electrical charge $e$ combined with the magnetic charge $b$ needs to satisfy the relation $eb = n$ with $n$ an integer which counts the number of times that the circles wrap around the source. This naturally gives an inverse relation between magnetic and
electrical charge. The magnetic monopoles are the dual particles of the electrons. Together the electro-magnetic force form an $U(1)$ gauge group.

The Yang-Mills theory is an non-abelian gauge theory with for example $SU(N)$ gauge group. The electromagnetic force could be combined with the weak force in Yang-Mills theory. The respective gauge groups are $U(1) \times SU(2)$. In $\mathcal{N} = 4$ super Yang-Mills theory, S-duality applies to the complex coupling $\tau$, which is defined as

$$\tau = \frac{1}{2\pi} \left[ \Theta + i \frac{8\pi^2}{g^2} \right]$$ (5.1)

where $\Theta$ is the theta angle in the gauge theory and $g$ is the gauge coupling. The coupling constant $\tau$ is a true constant. There is a conformal fixed point for every value of $\tau$. S-duality maps $\tau \rightarrow -\frac{1}{\tau}$. This means the $SU(N)$ gauge theory with coupling $\tau$ is dual to $SU(N)/Z_N$ with coupling $-\frac{1}{\tau}$. The S-duality acts as the operation "inverse" in $SL(2,\mathbb{Z})$ group on the moduli space of $\tau$. In general, the $SL(2,\mathbb{Z})$ acts on $\tau$ as the following:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}$$ (5.2)

The fundamental domain of $\tau$ is shown here. The weak coupling region is related to strong coupling region by S-duality. The real part of the coupling is related by translation. Each time we can shift the coupling by one unit if we change theta angle by $2\pi$. Only one unit is identified as the fundamental domain. In this case, $Re \tau \in [-\frac{1}{2}, \frac{1}{2}]$.

### 5.2 An Seiberg Duality Suitable for All Scales

The intuitive picture of Seiberg duality is as follows: consider the RG flow from two different free theories. One finds that they share the same fixed point at low
energy or in the infrared (IR), but they have different description at high energy or ultra-violet (UV). For example, let us consider two theories: $SU(3)$ QCD in the zero-mass limit and the linear sigma model [31]. The first one becomes strongly coupled at some scale $\Lambda$. The second theory is non-renormalizable. It describes mesons at low-momentum. Both theories lead to the same IR physics. We can translate one theory to the other at some low energy scale. Seiberg duality is usually understood as an infrared duality unless the theory is conformal. In that case, the couplings are fixed at all scales, so the Seiberg duality is applicable everywhere. However, we could apply Seiberg duality beyond conformal theories. In supersymmetric QCD theories, Seiberg duality is used to construct a series of dual theories. The RG flow exhibits cascading behaviour.

To clarify this, let us look at one example [31]. Consider SQCD theory $A$ with gauge group $SU(N)$ and its Seiberg dual theory $B$ with gauge group $SU(N_f - N)$, which has negative beta function. They both flow to the same conformal field theory $C$. Name another theory $A'$ with one less flavour than $A$, it has gauge group $SU(N)$,
the dual theory $B'$ has gauge group $SU(N_f - N - 1)$, they flow to conformal field theory $C'$. See Figure 5.2. To change the theory from $A$ to $A'$ one can add a mass term to the superpotential.

$$ W = mQ\tilde{Q} \quad (5.3) $$

where $Q$ is the chiral superfield. In the Seiberg dual theory $B$, the corresponding change is also to add a mass term to the superpotential, then reduces the size of the gauge group by one.

$$ W_{B'} = cMq\tilde{q} + \tilde{m}M_1 \quad (5.4) $$

The $M$ and $q, \tilde{q}$ are summed over all quarks including the extra mass $M_1$ and the corresponding quark $q_1$. $c$ is some proportionality coefficient. The expectation value for one quark is not zero because of this extra mass term, so the massless mesons are baryons reduced by one. The change in one theory induces a corresponding change

Figure 5–2: RG flow of two theories dual to each other
in its Seiberg dual theory.

\[
\frac{\partial W_{B'}}{\partial M_1} = y q_1 \tilde{q}_1 + \tilde{m}
\]  

(5.5)

If the mass term \(\tilde{m}\) is large, even larger than the cut-off scale \(\Lambda\) in the original theory \(A\), then the flow passes the theory \(A'\) and eventually reaches \(C'\). If \(\tilde{m}\) is small, we expect the flow goes the other way, close to theory \(C\) and then flows towards theory \(C'\). They look like two copies of the same trajectory. See Figure 5.2. If we can move the point \(C\) to UV, then they combine into one flow. In fact, in the limit two cut-off scales \(\Lambda\) and \(\Lambda'\) go to infinity, the flow first reaches \(C\) and then flows straight to \(C'\). See Figure 5.2. This shows the Seiberg duality can be applied as the cut-off scale goes to infinity. Thus, it must be a map for all energy scales.
Figure 5–4: RG flow of two theories in the limit of infinite cut-off scale
CHAPTER 6
Cascading SQCD Model

6.1 Beta Functions of KS Model

The Klebanov-Strassler (KS) model is an SQCD model with a cascading feature [31]. There are two coupled gauge couplings. As the energy scale decreases, the change in couplings could be described by a series of Seiberg-dual theories. This is the cascading process. The superpotential has the following form:

\[
W = h \left[ (A_1)_a^\alpha (B_1)_b^\beta (A_2)_b^\alpha (B_2)_a^\beta - (A_1)_a^\alpha (B_2)_b^\beta (A_2)_b^\alpha (B_1)_a^\beta \right] \quad (6.1)
\]

where \( A_i, B_i \) are the chiral fields of each gauge group and \( h \) is the quartic coupling. The KS model has two gauge groups. \( \alpha, \beta \) are the gauge indices of first group, whereas \( a, b \) are the indices of the second gauge group. Define the dimension less coupling as:

\[
\eta = h\mu \quad (6.2)
\]

where \( \mu \) has mass dimension 1. Let us consider a simple case, where the two gauge groups have the same size. The beta functions of the two gauge couplings \( g_1 \) and \( g_2 \) are:

\[
\beta_{g_1} = -\frac{g_1^3}{16\pi^2} \frac{N + 2N\gamma_0}{1 - \frac{g_1^2N}{8\pi^2}} \quad (6.3)
\]

\[
\beta_{g_2} = -\frac{g_2^3}{16\pi^2} \frac{N + 2N\gamma_0}{1 - \frac{g_2^2N}{8\pi^2}} \quad (6.4)
\]
where $N$ is the number of colours and $\gamma_0$ is the anomalous dimension. The anomalous dimension is associated with the RG flow of the dimensionless coupling $\eta$.

$$\beta_\eta = \eta [1 + 2\gamma_0] \quad (6.5)$$

The special case is when $\gamma_0 = -\frac{1}{2}$. We get a fixed point in the theory.

In terms of ’t Hooft coupling $\lambda = g_sN$, the beta functions are:

$$\beta_{\lambda_1} = -\frac{\lambda_1^2}{8\pi^2} \frac{1 + 2\gamma_0}{1 - \frac{\lambda_1}{8\pi^2}} \quad (6.6)$$

$$\beta_{\lambda_2} = -\frac{\lambda_2^2}{8\pi^2} \frac{1 + 2\gamma_0}{1 - \frac{\lambda_2}{8\pi^2}} \quad (6.7)$$

The two couplings seem symmetric and proportional to each other, they have the same order of magnitude. So there must be a line of flow that connects two fixed points together. This line lies on $h = 0$ plane, when the superpotential vanishes, and the couplings move from one fixed point to the other. One of the couplings is zero at the fixed point and the theory is self-dual at this point. The shape of couplings tells us it is infrared stable when $\eta$ is small. We can make an analogy with the rolling of a ball; there is a wall when energy scale is large, and the ball tends to roll down to the bottom. We can also project this plot into the $g_1, g_2$ plane. Similar to the marginal coupling $\rho$, now define parameters in the space of couplings by a marginal coupling for each gauge theory $\rho_{\pm} = \rho_1 \pm \rho_2$. We see the following type of flow: where $\tau_+ = \tau_1 + \tau_2$, $\tau_- = \tau_1 - \tau_2$, $\tau_{1,2}$ are the complex couplings.

This theory is identified with the $\mathcal{N} = 1$ Klebanov-Witten theory. No fixed points exist near $g_1 = g_2 = 0$. In string perturbation theory, we can construct a model that maps to this theory: a stack of D3 branes put at the tip of conifold has
same effective low energy action as Klebanov-Witten theory. If we modify the gauge
groups in this theory, we could break the symmetry between the two gauge groups
and induce cascading. KS model is one example. Suppose one gauge group in KS
model is $SU(kM)$ and the other gauge group is $SU((k - 1)M)$. The beta functions become:

$$
\beta_{g_k} = -\frac{g_1^3}{16\pi^2} \frac{(k + 2)M + 2(k - 1)M\gamma_0}{1 - \frac{g_k^2 kM}{8\pi^2}} \quad (6.8)
$$

$$
\beta_{g_{k-1}} = -\frac{g_2^3}{16\pi^2} \frac{(k - 3)M + 2kM\gamma_0}{1 - \frac{g_{k-1}^2 (k-1)M}{8\pi^2}} \quad (6.9)
$$

In terms of ’t Hooft couplings, the beta functions are

$$
\beta_{\lambda_k} = -\frac{\lambda_k^2}{8\pi^2} \frac{1 + \frac{1}{k} + 2(1 - \frac{1}{k})\gamma_0}{1 - \frac{\lambda_k^2}{8\pi^2}} \quad (6.10)
$$

$$
\beta_{\lambda_{k-1}} = -\frac{\lambda_{k-1}^2}{8\pi^2} \frac{1 - \frac{1}{k-1} + 2(1 + \frac{1}{k-1})\gamma_0}{1 - \frac{\lambda_{k-1}^2}{8\pi^2}} \quad (6.11)
$$

The KS model also has a dual picture in string theory. More details about the string
model are presented in chapter 9.
6.2 Renormalization Group Flow in Cascading Model

The two gauge couplings each have their own flow lines as a function of $\eta$, but we can make a three dimensional plot to track both of them. Suppose the operator has only matter in a representation of one of the two gauge groups, the flow would start along that axis, reach the fixed point of that theory, then we can switch on an infinitesimal value for the other coupling, we again flow towards a fixed point, then flow along the other surface. This is the edge of the whole region of flow.

![3D plot for the RG flow of two gauge couplings.](image)

Figure 6–2: 3D plot for the RG flow of two gauge couplings. $g_k$ and $g_{k-1}$ correspond to gauge group $SU(kN)$ and $SU((k-1)N)$ respectively [31].

Let us start with theory $SU((k+1)M) \times SU(kM)$, $N = kM$. The next configuration would be $SU(kM) \times SU((k-1)M)$. For each step of the flow, we can make a plot like this, the group would differ by $kM$, $k \in Z$. At some energy scale, one theory is an equivalent description of the other theory, as Seiberg duality can translate the flow between the two. It is not hard to see Seiberg duality can be applied to marginal
operators, because they do not receive further corrections from this transition, but other operators also have the proper dual. For example, $SU(10M) \times SU(9M)$ would be dual to $SU(9M) \times SU(8M)$. Note that the second group before the transition has the same size as the first group after the transition, so if we invert the plot after the transition then we can glue the two plots together to get a smooth curve. The point of connection is exactly where the Seiberg duality transformation takes place. In this way, the flow of the two couplings gives us two sides of a square, the vertical axes also matches as one is pointing in the positive direction and its inverse pointing in the negative direction. The flow goes from one fixed point to the other, and when

![Figure 6–3: Joining two corners: one corner is made of coordinates $g_k, g_{k-1}$ and $\eta$; the other corner is made of coordinates $g_{k-2}, g_{k-1}$ and $\tilde{\eta} = 1/\eta$ [31].](image)

we connect the four corners we get the curve of flow on a plane. Divide the plane in eight equal pieces, where each line of division has a different value of $\tau$. The imaginary part of $\tau \to \infty$ lies on the edge of the RG flow region. There is a branch cut when the flow line comes back, this means it flows down to next plane. It makes a multi-planar configuration. The overall scale reduces just as water flows from high altitude to low altitude, this kind of behaviour is called cascading. With well-defined
group structures and a correct interpretation of colours and flavours, this model has the potential to be applied to problems in QCD.

We will derive the above physics from string theory. However before we go into this, let us give a brief review of string theory.
CHAPTER 7
Strings and Branes

7.1 Introduction to Strings

String theory uses strings as basic degrees of freedom. Strings can be either open or closed. The higher dimensional generalization of a string is called a brane. String theory attempts to unify the interactions of the standard model and gravity. The four space-time dimensions cannot incorporate the quantum gravity effects, but if we add extra compact dimensions, then these extra dimensions can be tuned to fit the model. In superstring theory the natural space-time dimension is ten. Six of them are compact dimensions and become invisible at low energies. The other four dimension are Minkowski space-time. The string must have both bosonic and fermionic degrees of freedom. The central charge of the Virasoro algebra does not vanish for just the bosonic modes, so we need fermionic modes. In this way, by itself strings are supersymmetric objects. Closed strings have different modes of oscillation, which gives different particles. They are fundamental objects, so they are indivisible, but open strings can dissolve on branes to give fluxes. They can also scatter, swap fragments and break.
The boundary condition for open strings can either be Dirichlet or Neumann. It corresponds to two types of oscillations. Dirichlet boundary conditions are:

$$\delta X^\mu = 0$$

$$X^\mu|_{\sigma=0} = X_0^\mu$$

$$X^\mu|_{\sigma=\pi} = X_\pi^\mu$$  \hspace{1cm} (7.1)

where $\mu = 1, ..., D - p - 1$ for $Dp$ branes, $\sigma$ is the world sheet space coordinate, $\tau$ is the world sheet time coordinate.

Whereas Neumann boundary conditions are:

$$X_\mu'|_{0,\pi} = 0$$  \hspace{1cm} (7.2)

where $\mu = 1, ..., D - p - 1$ for $Dp$ branes.
The string spectrum is divided into the Ramond (R) sector and the Neuveu-Schwartz (NS) sector. The ground state in the R sector is:

\[ \alpha_n^i |0, k\rangle_R = d_n^i |0, k\rangle_R = 0, \quad \text{for } n > 0 \]  

(7.3)

The ground state in the NS sector is:

\[ \alpha_n^i |0, k\rangle_{NS} = b_n^i |0, k\rangle_{NS} = 0, \quad \text{for } n, r > 0 \]  

(7.4)

and

\[ \alpha_0^\mu |0, k\rangle = \sqrt{2\alpha'} k^\mu |0; k\rangle. \]  

(7.5)

We can imagine a point fixed in space, and as time goes it leaves a straight line along the time line. If it moves in space, we see a curve in this space-time plot. That is called a world line. Branes are higher-dimensional objects and swipe out the analogue of world lines. D\(p\) branes have \(p\) space directions and one time direction in ten dimensions. The open string can end on D branes. This is a special class of branes where the string satisfies Dirichlet boundary conditions. The mass of D branes saturates the BPS bound. The D1 branes have the same dimension as a fundamental string F1. Anti-branes carry an opposite charge and they can annihilate the regular branes.

### 7.2 T duality

The superstring theory has five types, but T duality can translate some types into other types. The massless spectrum of type IIA string theory consists of a
dilaton $\phi$, a graviton $G$, two-forms $B$, fermions $\psi_{-\mu}^{1,2}$, a gaugino $\lambda_\pm$ and odd forms,

$$C_1, \ C_3 \quad (7.6)$$

Type IIB has a dilaton $\phi$, graviton $G$, two-forms $B$, fermions $\psi_{+\mu}^{i=1,2}$, a gaugino $\lambda_-$ and even forms,

$$C_0, \ C_2, \ C_4 \quad (7.7)$$

T-duality maps type IIA to IIB. Suppose we want to make a T-dual transformation along $X^9$. The transformation rules for the NS-NS sector fields are as follows [16]:

$$\tilde{g}_{99} = 1/g_{99}, \quad \tilde{g}_{9\mu} = g_{9\mu}, \quad \tilde{g}_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}B_{9\mu} - g_{9\mu}g_{9\nu} \quad (7.8)$$

The transformation rules for R-R sector fields are

$$\tilde{C}_9 = C, \quad \tilde{C}_\mu = C_\mu, \quad \tilde{C}_{\mu\nu} = C_{\mu\nu}, \quad \tilde{C}_{\mu\nu\rho} = C_{\mu\nu\rho} \quad (7.9)$$

The dilaton is also transformed,

$$\tilde{\phi} = \phi - \frac{1}{2} \log g_{99} \quad (7.10)$$

### 7.3 Calabi-Yau Compactification

The 6 compact dimensions in string theory can be represented as six dimensional Calabi-Yau manifold. The condition for a Calabi-Yau manifold is:

1) Kähler manifold with vanishing first Chern class.

2) Has complex coordinates.
The Hodge numbers characterize the content of the Calabi-Yau manifold and ultimately determine the type and number of forms in the theory. The Hodge diamond for a Calabi-Yau threefold is

\[
\begin{array}{cccc}
  h^{3,3} & h^{3,2} & h^{2,3} \\
  h^{3,1} & h^{2,2} & h^{1,3} & h^{3,0} \\
  h^{2,0} & h^{1,1} & h^{0,2} & h^{2,1} \\
  h^{1,0} & h^{0,1} & h^{0,0} & h^{1,3} \\
  h^{3,0} & h^{2,3} & h^{3,2} & h^{3,3}
\end{array}
\]

(7.11)

All other values are zero except

\[
h^{3,3} = h^{3,0} = h^{0,3} = h^{0,0} = 1, \quad h^{2,2} = h^{1,1}
\]

(7.12)

### 7.4 Chan-Paton Factors

Chan-Paton factors are extra degrees of freedom at the end of strings. Usually these factors represent quarks. One end of the string is associated with the fundamental representation and the other end is associated with the anti-fundamental representation. This creates additional \(N^2\) massless states and the gauge group is \(U(N)\).

### 7.5 Flux

The flux on the brane is like the electromagnetic flux on a surface. It provides a source of energy and prevents the configuration from collapsing. Sometimes, a string
can dissolve into the brane and induce flux on it. Take the one dimensional vector field as an example:

\[ S_{int} = e \int A \]  \hspace{1cm} (7.13)

Maxwell’s equations written in differential forms then become

\[ dF = 0, \quad d \ast F = \frac{\delta S_{int}}{A} \]  \hspace{1cm} (7.14)

We can now generalize this to n dimension, where the field strength \( F_{n+1} = A_n \); as

\[ S_{int} = \mu_p \int A_{p+1}, \quad \mu_p = \int \ast F_{p+2} \]  \hspace{1cm} (7.15)

\[ dF = \ast J_m, \quad d \ast F = \ast J_e \]  \hspace{1cm} (7.16)

where \( J_m \) and \( J_e \) represent the charge densities. They are related by \( S_{int} \) as in (7.14). These are the higher dimensional fluxes in string theory.
CHAPTER 8
AdS/CFT correspondance

The groundbreaking AdS/CFT duality was developed in 1997 by Maldacena. He conjectured that string theory on an AdS geometry corresponds to conformal field theory at the boundary [18]. For $AdS_{d+1}$ space-time in Poincaré coordinates, the metric is

$$ds^2 = \frac{R^2 dx_{d+1}^2 + dt^2}{r^2}, \quad r \geq 0 \quad (8.1)$$

The cover space is CAdS. Poincaré coordinates cover the Poincaré patch of the total space-time. The AdS/CFT correspondence, to be more precisely, is a CAdS/CFT correspondence.

$$ds^2_{CAdS} = \frac{(d\rho^2 + \sin^2 \rho d\Omega_2^2 - dt^2)}{\cos^2 \rho}, \quad r \geq 0 \quad (8.2)$$

The global symmetry is the conformal group $SO(2,4)$, the internal symmetry is an $SO(6)$ which matches with the $R$ symmetry of $\mathcal{N} = 4$ SYM.

The idea is to take $N$ number of D branes. Define $U = \frac{r}{\alpha'}$ and keep it fixed, where $\alpha'$ is the open string Regge slope. Taking the limit $\alpha' \to 0$ and also keeping the mass of the string fixed, we can achieve the decoupling limit. The low energy effective theory is $\mathcal{N} = 4 U(N)$ super Yang-Mills (SYM) theory. Consider the superconformal
point at $r = 0$, when all the branes are on top of each other. The metric becomes:

$$ds^2 = f^{-1/2}dx^2 + f^{1/2}(dr^2 + r^2d\Omega_5^2)$$

$$f = 1 + \frac{4\pi gN\alpha'}{r^4}$$

where $\Omega_5$ is an Einstein manifold. Taking the near horizon limit with $gN \gg 1$ and $f \approx \frac{4\pi gN\alpha'}{r^4}$, the metric becomes

$$ds^2 = \alpha'[\frac{U^2}{\sqrt{4\pi gN}}dx^2 + \sqrt{4\pi gN}\frac{dU^2}{U^2} + \sqrt{4\pi gN}d\Omega_5^2]$$

where $\alpha'$ is an overall factor. The moduli space is independent of coupling, which matches with SYM.

The coupling of SYM is matched with the complexified coupling in type IIB string theory as

$$\frac{1}{g^2_{YM}} + i \frac{\theta}{8\pi^2} = \frac{1}{2\pi} \left( \frac{1}{g} + i \frac{\chi}{2\pi} \right)$$

Finally, the total five-form flux is dual to the size of gauge group:

$$\frac{1}{16\pi^4\alpha'} \int_{S^5} F_5 = N$$

Eventually the fluxes need to match with the RG flow in the non-conformal limit.

### 8.1 Wilson Loops

The Wilson loop is an important gauge invariant operator. For a confining theory, it obeys an area law, whereas for an unconfining theory, it obeys a perimeter law. Suppose the closed path is a square. Then $W \propto e^{cL}$ is the perimeter law and $W \propto e^{cL^2}$ is the area law, with $c$ being some parameter independent of $L$. 

...
8.1.1 Straight Wilson Line in AdS Spacetime

Let us show this explicitly in the following that the expectation of a straight wilson line is one. The expectation value is defined as:

$$\langle W(c) \rangle = e^{-S_{NG}}$$  \hspace{1cm} (8.7)

where $S_{NG}$ in the Nambo-Goto action. The metric is

$$ds^2 = \frac{dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu}{r^2}$$  \hspace{1cm} (8.8)

and the action is

$$S_{NG} = \frac{1}{2\pi\alpha'} \int d^2 \sigma \sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}$$

$$= \frac{1}{2\pi\alpha'} \int dr dx \sqrt{\det g}$$  \hspace{1cm} (8.9)

where $\det g$ is the determinant of the metric. In AdS spacetime, the action measures the minimal surface area of the path encircled by the Wilson operator. In this case, the area is

$$A = \int_{\epsilon}^{\infty} dr \int_{0}^{L} dx \frac{1}{r^2} = 0 + \frac{L}{\epsilon}$$  \hspace{1cm} (8.10)

It has an infinite piece and a finite term. If we add counter terms of order $1/\epsilon$ to regularize it, then it is clear the area $A$ is zero.
8.1.2 Wilson Loop in AdS Space

Similarly, for a circular Wilson loop, we can calculate the expectation value using polar coordinates.

\[
A = \int_{\epsilon}^{1} dr \int_{0}^{2\pi} d\theta \frac{1}{r^2} = 2\pi \left( \frac{1}{\epsilon} - 1 \right) \tag{8.11}
\]

The finite term gives:

\[
e^{-\frac{2\pi}{2\pi\alpha'}} = e^{1/\alpha'} = e^{\sqrt{\lambda}} \tag{8.12}
\]

This leads to

\[
\langle W(c) \rangle = \int e^{\sqrt{X}} \tag{8.13}
\]
CHAPTER 9
String Theory Aspects of the KS Model

9.1 The Corresponding Effective Gauge Theory on the Conifold

We consider a geometry of the form $AdS_5 \times Y^{p,q}$, in particular $T^{1,1}$ as the background for a complete SQCD model. We can also view the geometry as arising from the conifold with base $S^2 \times S^2 \times S^1$. The conifold is defined as:

$$Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0$$  \hspace{1cm} (9.1)

where $Z_i$ are complex coordinates.

Alternatively, the metric of the conifold can be written as

$$ds_c^2 = dr^2 + r^2 ds_{T^{1,1}}^2$$  \hspace{1cm} (9.2)

We see that the global symmetry has two $SU(2)$ groups that rotate flavours into each other and a $U(1)$ [28]

$$A_i \rightarrow A_i e^{i \alpha}, \quad B_j \rightarrow B_j e^{-i \alpha}$$  \hspace{1cm} (9.3)

There are two spurious $U(1)$ symmetries and a $U(1)_R$ symmetry.

Consider a diagonal form for the expectation value of the scalars.

$$\text{diag} \langle A_i \rangle = (a_i^{(1)}, \ldots, a_i^{(N)})$$  \hspace{1cm} (9.4)

$$\text{diag} \langle B_j \rangle = (b_j^{(1)}, \ldots, b_j^{(N)})$$  \hspace{1cm} (9.5)
The F term is

\[ B_1 A_1 B_2 - B_2 A_1 B_1 = 0, \quad A_1 B_j A_2 - A_2 B_j A_1 = 0 \] (9.6)

and the D term is

\[ |a_1^{(r)}|^2 + |a_2^{(r)}|^2 - |b_1^{(r)}|^2 - |b_2^{(r)}|^2 = 0 \] (9.7)

These constrain the possible vacua of the theory. The charges of the fundamental fields are listed as follows:

<table>
<thead>
<tr>
<th>Fields</th>
<th>SU(N + M)</th>
<th>SU(N)</th>
<th>SU(2)</th>
<th>SU(2)</th>
<th>U(1)_{A}</th>
<th>U(1)_{B}</th>
<th>U(1)_{R}</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1, A_2</td>
<td>N + M</td>
<td>N + M</td>
<td>2</td>
<td>1</td>
<td>\frac{2(N+M)N}{1}</td>
<td>\frac{2(N+M)N}{1}</td>
<td>\frac{1}{2}</td>
</tr>
<tr>
<td>B_1, B_2</td>
<td>\bar{N} + \bar{M}</td>
<td>N</td>
<td>1</td>
<td>2</td>
<td>\frac{2(N+M)N}{1}</td>
<td>- \frac{2(N+M)N}{1}</td>
<td>\frac{1}{2}</td>
</tr>
</tbody>
</table>

9.2 Type IIB Model on Conifold

\( T^{1,1} \) has topology of \( S^2 \times S^3 \) as already mentioned above. Define \( Z_i = x_i + iy_i \), the conifold equation (9.1) directly translate to [31]

\[ \sum_i x_i y_i = 0, \quad \sum_i x_i^2 = \sum_i y_i^2 = \frac{r^2}{2} \] (9.8)

The solution is to choose \( x, y \) as the four dimensional vector with length \( \frac{r}{\sqrt{2}} \). The vectors \( x_i \) and \( y_i \) are orthogonal to each other. The \( x_i \) parametrize the \( S^3 \) and the \( y_i \) parametrize the \( S^2 \).

The next step is to show the moduli space is the conifold. In other words, the gravity dual theory of \( \mathcal{N} = 4 \) super Yang-Mills theory could be extracted from the conifold. Not any arbitrary manifold can support this theory. From the D-term
equation, we know the chiral gauge fields need to have the same phases. This eliminates two degrees freedom. Then we are left with six degree of freedom. This hints that we need a six real dimensional manifold corresponding to the six scalar degree of freedom. Observe that this is an abelian theory, so the chiral fields commute. Therefore, the product of these fields can be switched in order:

\[(A_1 B_1)(A_2 B_2) = (A_1 B_2)(A_2 B_1)\]  \hspace{1cm} (9.9)

The conifold has three complex dimensions, which is equivalent to six real dimensions. There is a dual map, so we can still use perturbative methods to study dynamics on the gravity side, then make a dual map to the field theory. This technique avoids the strong coupling problem. Let us consider the basic structure, a stack of \(N\) numbers of D3 branes is put at the tip of the conifold. The conifold is Ricci flat, where the Ricci tensor is defined by

\[R^{\alpha \beta} = R_{\alpha \beta} = \partial_\gamma \Gamma^\gamma_{\beta \alpha} - \partial_\beta \Gamma^\gamma_{\gamma \alpha} + \Gamma^\gamma_{\gamma \rho} \Gamma^\rho_{\beta \alpha} - \Gamma^\gamma_{\beta \rho} \Gamma^\rho_{\gamma \alpha} = 0\]  \hspace{1cm} (9.10)

\(\Gamma\) is the Christoffel symbol. So the Ricci tensor vanishes on the conifold. The base of the conifold is an Einstein manifold \(X_5\), given by:

\[R_{\alpha \beta} = k g_{\alpha \beta}\]  \hspace{1cm} (9.11)

where \(k\) is the constant of proportionality. It is conjectured that type IIB theory on \(AdS \times X_5\) is dual to low-energy limit of the world volume theory on the D3 branes at the singularity, \(i.e. \mathcal{N} = 1\) super Yang-Mills theory. In this case, the theory is also conformal. The gauge group is \(SU(N) \times SU(N)\). The chiral superfields transform
as bifundamentals and anti-bifundamentals. The superpotential has the form

$$W = \lambda \epsilon^{ij} \epsilon_{kl} \text{Tr} A_i B_k A_j B_l$$ (9.12)

The theory has an exact $Z_{2M} R$ symmetry.

The singularity on the conifold is removed by deforming the conifold. It means to blow up one of the base from the back reactions of the D3 branes. The bases together form an Einstein manifold $T^{1,1} = (SU(2) \times SU(2))/U(1)$. Choosing some basis, we can rewrite the metric on $T^{1,1}$ using:

$$a_1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad a_2 = \frac{e^2 - e^4}{\sqrt{2}}, \quad a_3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad a_4 = \frac{e^2 + e^4}{\sqrt{2}}, \quad a_5 = e^5$$ (9.13)

where $e^i, i = 1...4$ are defined as

$$e^1 = -\sin \theta_1 d\phi_1, \quad e^2 = d\theta_1, \quad e^3 = -\cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2,$$

$$e^4 = \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2, \quad e^5 = d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2$$ (9.14)

From this coordinate parametrization, we see clearly the conifold is topologically equivalent to $S^2 \times S^3$. Note that there are two ways to remove singularity on the conifold, either blow-up base $S^3$ to get a deformed conifold or blow up base $S^2$ to get a resolved conifold. If we relax the condition to non-Calabi-Yau manifold, then the conifold can be resolved and deformed at the same time.

9.3 Fractional Branes

Without any fractional branes, the theory is superconformal and the $U(1)_R$ has no anomaly. In the presence of the fractional branes, the integration of the two form NS fluxes over the $S^2$ of the base contributes to the renormalization flow. The precise
connection is as follows [28]:

\[
\frac{1}{g_1^2} + \frac{1}{g_2^2} \sim \text{Im} \tau = e^{-\phi} \tag{9.15}
\]

\[
\frac{1}{g_1^2} - \frac{1}{g_2^2} \sim \text{Im} \left[ \int_{S^2} B_2 - \frac{1}{2} \right] \tag{9.16}
\]

where the period for total fluxes on \( S^2 \) is 1.

D5 branes source R-R three form flux through \( S^3 \) of \( T^{1,1} \). The three form flux has \( M \) units, whereas the five form flux has \( N \) units.

\[
\int_{S^3} F_3 = M, \quad \int_{T^{1,1}} F_5 = N. \tag{9.17}
\]

The radial coordinate on the five dimensional spacetime is dual to the RG scale in the effective field theory. This means the logarithmic dependance of (9.16) would come from:

\[
\int_{S^2} B_2 \sim M e^{\phi} \ln(\frac{r}{r_0}) \tag{9.18}
\]

On the other hand, the field theory also has logarithmic running of coupling constant, so we can match the two sides. The beta functions from the gauge theory side are:

\[
\frac{d}{d\log(\Lambda/\mu)} \frac{8\pi^2}{g_1^2} \sim 3(N + M) - 2N(1 - \gamma) \tag{9.19}
\]

\[
\frac{d}{d\log(\Lambda/\mu)} \frac{8\pi^2}{g_2^2} \sim 3N - 2(N + M)(1 - \gamma) \tag{9.20}
\]

where \( \gamma \) is the dimension of operator \( \text{Tr}A_iB_j \). The difference between the two couplings is

\[
\Delta c = \frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} \sim M \ln(\Lambda/\mu)[3 + 2(1 - \gamma)] \tag{9.21}
\]
This matches precisely with (9.18). If we choose a particular solution $e^\phi = g_s$, the beta functions vanish for $\Delta c$.

From the superpotential we can also analyze the RG flow. For $SU(N + M)$ with $2N$ flavours in the fundamental representation, Seiberg duality transformation changes the gauge group to $SU(2N - (N + M)) = SU(N - M)$, also with $2N$ flavours. $a_i, b_j$ are fundamental and anti-fundamental fields of $SU(N)$.

$$W_1 = \lambda_1 \epsilon^{ik} \epsilon^{jl} \text{Tr} \, M_{ij} M_{kl} \, f + \frac{1}{\mu} \text{Tr} \, M_{ij} a_i b_j$$  \hspace{1cm} (9.22)

$f$ is some function of gauge fields and holomorphic scales $\Lambda$ and $\tilde{\Lambda}$. $\mu$ is the scale where Seiberg duality transformation takes place, the gauge couplings from two different theories can match. Integrate out $M_{ij}$ because it is massive at scale $\mu$, we get the following constraint on the superpotential:

$$2\lambda_1 \epsilon^{ik} \epsilon^{jl} M_{kl} f_1 - \frac{1}{\mu} \text{Tr} \, a_i b_j = 0$$  \hspace{1cm} (9.23)

which means the form of superpotential of the gauge field after transformation will be:

$$W_2 = \lambda_2 \epsilon^{ik} \epsilon^{jl} a_i b_j a_k b_l f_2$$  \hspace{1cm} (9.24)

The holomorphic scale needs to match after transformation. $\Lambda_1$ is the scale for $SU(N + M)$, $\tilde{\Lambda}_1$ is the scale for $SU(N)$.

$$\Lambda \rightarrow \tilde{\Lambda}_2, \quad \tilde{\Lambda} \rightarrow \lambda_2$$  \hspace{1cm} (9.25)
From here, it means the constant of proportionality must be related in this way

\[ \lambda_2 \mu \propto \frac{1}{\lambda_1 \mu} \]  

(9.26)

Because the superpotential has this constant \( \lambda_i \) that needs to be rescaled, the holomorphic scale of the two gauge theories also needs to be rescaled. \( \lambda_1 \) has extra \( M \) colours, \( \lambda_2 \) has less \( M \) colours, we expect the power on it to be negative.

\[ \Lambda_2^{3(N+M)-2N} \tilde{\Lambda}_1^{3(N-M)-2N} \propto \mu^{2N} \]  

(9.27)

\[ \Lambda_1^{3(N+M)-2N} \tilde{\Lambda}_2^{3(N-M)-2N} \lambda_1^M \lambda_2^{-M} \propto \mu^{2N} \]  

(9.28)

The field content of the dual theory after the Seiberg duality transformation is shown in the following table:

<table>
<thead>
<tr>
<th>Fields</th>
<th>SU((N))</th>
<th>SU((N-M))</th>
<th>SU(2)</th>
<th>SU(2)</th>
<th>U(1)(_A)</th>
<th>U(1)(_B)</th>
<th>U(1)(_R)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1, a_2)</td>
<td>(N)</td>
<td>(N-M)</td>
<td>2</td>
<td>1</td>
<td>(\frac{1}{2(N-M)N})</td>
<td>(\frac{1}{2(N-M)N})</td>
<td>(\frac{1}{2})</td>
</tr>
<tr>
<td>(b_1, b_2)</td>
<td>(N)</td>
<td>(\tilde{N}-\tilde{M})</td>
<td>1</td>
<td>2</td>
<td>(\frac{1}{2(N-M)N})</td>
<td>(\frac{1}{2(N-M)N})</td>
<td>(\frac{1}{2})</td>
</tr>
</tbody>
</table>

When the gauge groups shrink to zero, i.e., \(N-nM\) becomes zero after \(n\) times transformation, the flow will stop. It is meaningless to talk about negative gauge group. The flow is solely depend on the reduction of colours. There is no difference for conformal window \(2N_c > N_f > \frac{3}{2}N_c\) and free magnetic window \(\frac{3}{2}N \geq N_f > N_c + 1\).

Close to the tip, the gauge groups break down to SU(\(M\)), in other words, only \(M\) fractional D3 branes are left. In the far infrared, we want to recover the \(\mathcal{N}=1\) super Yang-Mills theory. The \(R\) symmetry is \(\mathbb{Z}_2\). To understand this, consider a
probe D3 brane, the gauge group is $SU(M + 1) \times SU(1)$. $C_i, D_j$ are gauge fields in $M + 1, \bar{M} + \bar{1}$ representation. The non-perturbative superpotential has the form [15]:

$$W = \epsilon^{ij} e^{kl} \text{Tr} \lambda C_i D_j C_k D_l + (M - 1) \left[ \frac{2 \Lambda^{3M+1}}{\epsilon^{ij} e^{kl} C_i D_j C_k D_l} \right] \frac{1}{\Lambda^{|M-1|}}$$

(9.29)

The solution for supersymmetric vacuum is

$$0 = (\lambda - \left[ \frac{2 \Lambda^{3M+1}}{\epsilon^{ij} e^{kl} C_i D_j C_k D_l} \right] \frac{1}{\Lambda^{|M-1|}}) C_i D_j$$

(9.30)

This gives the constraint on the trace of products of fields.

$$(\epsilon^{ij} e^{kl} C_i D_j C_k D_l)^M = \frac{2 \Lambda^{3M+1}}{\Lambda^{M-1}}$$

(9.31)

Analyzing this carefully, we see that the are $M$ branches and the $Z_{2M}$ discrete symmetry rotate $\text{Tr} \epsilon^{ij} e^{kl} C_i D_j C_k D_l$ by $e^{2\pi i / M}$. The symmetry is spontaneously broken to $Z_2$. Finally, we get the low-energy effective superpotential.

The physics of the above graph can be captured by D5 branes collapsed over two-cycle. This changed the gauge group to $SU(N + M) \times SU(N)$. These fractional branes must be located at the singularity. The tension of D5 would grow with radial coordinates $r$ on the conifold, so it is preferable to put the D5 at $r = 0$. They are sources of six form $C_6$ and seven-form flux $F_7$. The dual flux $F_3$ must be perpendicular to $F_7$. The value is proportional to the volume form of the other base $S^3$.

$$F_3 = \star F_7, \quad \frac{1}{4\pi^2\alpha'} \int F_3 = M$$

(9.32)

$M$ counts the number of D5 or the fractional D3 branes. The additional D5 branes breaks the conformal invariance. In other words, if the two gauge groups are not the
same, then the theory is not conformal[31]. The two couplings change as \( g_1^{-2} - g_2^{-2} \) runs logarithmically. Two theories may have different description at high energy. As the energy lowers, the degree of freedom decreases, they may effectively become the same theory. This corresponds to gauge group reducing in size each time by \( M \) unit. The flow is a series of Seiberg duality transformations. Eventually the number of colours would be smaller than \( M \), then the non-perturbative contribution dominates. In this way we recover the confinement at IR.

### 9.4 The Corresponding Process of Cascading in String Theory

The change in coupling was discussed in effective field theory, it also has a string theory picture. Put \( N \) D3, \( M \) D5 at the tip of conifold. Make a T-dual transformation, we have a circle with \( N \) numbers of D4 orthogonal to NS5 in \( x^6 \). One oriented along (12345) direction, the other one placed along (12389) direction. There are \( N \) D4 branes along (1236). The direction \( x^6 \) is compact, it is an \( S^1 \) circle, and the D4 branes wrap around the circle. In type IIB, this is equivalent to putting \( N \) D3 branes at \( \mathcal{N} = 2\mathbb{Z}_2 \) orbifold singularity. The effective low energy field theory is \( \mathcal{N} = 2 \) supersymmetric \( SU(N) \times SU(N) \) gauge theory with bi-fundamental matters.

Moving the two NS5 branes along \( x^6 \), as if the two branes are physically crossing each other, is like flipping the masses of two chiral superfields. The orbifold field theory changes to \( \mathcal{N} = 1 \) supersymmetric field theory. The positions of the NS5 branes determine the gauge couplings. One may note the difference with another model, with two NS5 banes, that has \( N \) number of D4 branes wrapping the full circle. For the present case, when we cross the two NS5 branes, the D4 branes, on the other side of the NS5 branes, regrow. On the field theory side, this maps to
the reduction of gauge group by $M$ units, and consequently the flow of the coupling constant is reversed.

Alternatively, this is can be interpreted as though the two branes are bent towards each other. The T-dual of $M$ D4 branes are fractional D3 branes, which are the D5 branes wrapped on the vanishing two-cycle on $T^{1,1}$. D4 branes induce logarithmic bending along $x^6$. We interpret this as the divergence in $SU(N + M)$ coupling. The NS branes in turn will try to cross along $x^6$ direction. At the crossing point, Seiberg duality transformation takes place on field theory side. Globally, the branes form a DNA-like chain which looks as if they are crossing each other at multiple points. At the crossing point, the other $N$ branes would happen to be wrapped twice, but the newly formed $N + M$ antiD4 branes cancel them, so that in total we have $(N - M)$ D4 branes on one part of the circle and $N$ D4 branes on the other. This is similar to the original configuration but with gauge group changed. The structure then repeats again. This way we reproduce the full cascading dynamics of the KS model from string theory.
CHAPTER 10
Extensions of the KS Model

10.1 Additional D7 Branes

Consider adding D7 branes to the conifold. We choose four directions parallel to the D3 branes, these are Minkowski space-time, and the other four directions on a non-compact Calabi-Yau manifold. There are three form fluxes in the background. The back reaction from the D7 branes gives a supergravity (SUGRA) solution, we observe that the rate of cascading slows down at the IR. Strong coupling problems in field theory becomes weak coupling on string theory side \[27\].

The strings stretched between D3 branes give gluon fields, the flavours come from D7 branes. The D7 branes allow us to study the chiral symmetry at IR. The background is a conifold with \(T^{1,1}\) as a base.

\[
\text{ds}^2 = dr^2 + r^2 \left[ \frac{1}{5} (d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i)^2 + \frac{1}{6} \sum_{i=1}^{2} (d\theta_i^2 + \sin^2 \theta_i d\phi_i^2)^2 \right]
\]  

(10.1)

where \(\psi \in [0, 4\pi]\), \((\theta_1, \phi_1)\) and \((\theta_2, \phi_2)\) are coordinates on \(S^2\). There is a \(U(1)\) fibered over \(S^2 \times S^2\). The angular coordinates are given by:

\[
\begin{align*}
z_1 &= r^{3/2} e^{i/2(\psi - \phi_1 - \phi_2)} \sin(\theta_1/2) \sin(\theta_2/2) \\
z_2 &= r^{3/2} e^{i/2(\psi + \phi_1 + \phi_2)} \cos(\theta_1/2) \cos(\theta_2/2) \\
z_3 &= r^{3/2} e^{i/2(\psi + \phi_1 - \phi_2)} \cos(\theta_1/2) \sin(\theta_2/2) \\
z_4 &= r^{3/2} e^{i/2(\psi - \phi_1 + \phi_2)} \sin(\theta_1/2) \cos(\theta_2/2)
\end{align*}
\]  

(10.2)

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In another parametrization, it is clear that there is an $SO(4) \cong SU(2) \times SU(2)$ rotation symmetry. The total symmetry is $SU(2) \times SU(2) \times U(1)$. The $z_i$ coordinates can also be expressed as:

$$z_1 = w_1 + iw_2, \quad z_2 = w_1 - iw_2, \quad z_3 = -w_3 + iw_4, \quad z_4 = -w_3 - iw_4.$$ 

Or simply $\sum w_i = 0$. This satisfies $\sum |w_i|^2 = \sum |z_i|^2 = r^3$. The RR flux and the metric are defined as follows:

$$ds^2 = h(r)^{-1/2}(dx_\mu dx^\mu + dr^2) + h(r)^{1/2}r^2 ds_T^2, $$

$$g_s F_5 = d^4 x \wedge dh^{-1} + *(d^4 x \wedge dh^{-1})$$

where $h(r) = \frac{L^4}{r^4} = \frac{2\pi g_s N\alpha'^2}{r^4}$.

So far the theory is massless, we want to add matter. As discussed above, D7 branes can have a dual gauge theory with matter. To see the D7 brane embedding, start with a simple solution, along the curve $z_1 = 0$. We then put in $K$ D7 branes, so we expect $SU(K) \times SU(K)$ flavour symmetry. The second $SU(K)$ flavour group with opposed chirality is need to cancel the anomalies.

Table 10–1: The gauge group

<table>
<thead>
<tr>
<th>Field</th>
<th>$SU(N_c) \times SU(N_c)$</th>
<th>$SU(K) \times SU(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q$</td>
<td>$(N, 1)$</td>
<td>$(K, 1)$</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>$(\bar{N}, 1)$</td>
<td>$(1, K)$</td>
</tr>
<tr>
<td>$Q$</td>
<td>$(1, N)$</td>
<td>$(\bar{K}, 1)$</td>
</tr>
<tr>
<td>$\bar{Q}$</td>
<td>$(1, \bar{N})$</td>
<td>$(1, \bar{K})$</td>
</tr>
</tbody>
</table>
Next, we add the masses to the theory. This can be done by breaking the flavour symmetry and the superpotential gets modified to:

\[ W = \mu_1 q \bar{q} + \mu_2 Q \bar{Q} + h q A_1 \bar{Q} + g \bar{q} B_1 \bar{Q} \]  

(10.4)

where \( \mu_1 \) and \( \mu_2 \) are the couplings of the quarks (flavours) that satisfy \( h g z_1 - \mu_1 \mu_2 = 0 \).

The coordinates \( z_i \) are alternatively defined as:

\[ z_1 = A_1 B_1, \quad z_2 = A_2 B_2, \quad z_3 = A_1 B_2, \quad z_4 = A_2 B_1. \]  

(10.5)

This modified KS model is called the Ouyang-Klebanov-Strassler (OKS) model.

10.2 The Three Regions of the UV Complete Model

The KS model has logarithmic running of the gauge couplings at high energy, then the theory encounters Landau-pole at some UV scale. The modified KS model with additional D7 branes does not fix the divergence of gauge couplings. We need to add a UV cap to OKS model. At high energy, the theory becomes conformal and the divergence is removed. However, the transition from a low energy theory to a high energy theory has to be smooth. Therefore, we impose an intermediate energy range, where the two-form fluxes gradually die away. In this way, we get a UV complete model with three regions ranking from small to large radial distance from the conifold singularity:

1) Region 1 has almost the same structure as what is in the KS model. A stack of \( N \) D3 branes are put at the tip of a deformed conifold and the fractional D3 branes are wrapping around the vanishing two-cycle, but there are additional D7 branes that spread from large \( r \) till some minimal scale \( r_{\text{min}} \).
2) Region 2 has a mixture of anti-D5 branes and D7 branes. This region serves as a transition that connects low energy theories to high energy theories. The couplings experience a slow walking stage in region 2.

3) Region 3 has only D7 branes. There are at most twenty-four D7 branes due to constraints derived from F theory. These D7 branes are slightly delocalized. They extend to region 2 and region 1. The theory is superconformal, meaning that there is no running of the couplings and the theory has supersymmetry.

10.3 Addition of BH

The non-AdS space is modified to have a black hole warp factor, which is dual to the gauge theory at non-zero temperature. We expect to have phase transitions at the IR, a superconformal field theory at UV, a slow walking stage in region 2 and cascading in region 1 and finally confinement. The gauge groups goes as $SU(N + M) \times SU(N + M) \rightarrow SU(N + M) \times SU(M) \rightarrow SU(M) \rightarrow 0$ respectively. We also have anti-D5 branes induced from D7 branes. This is because the fluxes on D7 induce extra D5 brane charges. To match with the charges and have a stable configuration, anti-D5 branes are introduced to generate extra D3 bane charges.

The UV beta function is $\beta \sim \frac{1}{\Lambda^2}$ and at intermediate energy it is like $\beta \sim \log \Lambda$. We can describe the RG flow from the axio-dilaton $\tau = C_0 + ie^{-\Phi}$. $C_0$ is the axion, $\phi$ is the dilaton. In this case, the amplitude satisfies

$$\tau = \frac{i}{g_s} + \frac{K}{2\pi i} \log z_i \quad (10.6)$$

Interestingly, this is is related to the D7 branes geometrically. If we make a translation $\tau \rightarrow \tau + K$, we see that the coordinates on D7 branes have an extra rotation
$z_1 \rightarrow z_1 e^{2\pi i}$. This is the correct monodromy if we circle $K$ D7 branes. Each gauge group has two chiral fields, hence effectively contributes $2N$ flavours and D7 branes contribute to the other $N$ flavours.

10.4 Thermal Phase in AdS Space

Let us now discuss the thermal phase for AdS case. The non-AdS case would be discussed in the rest section. The $\mathcal{N} = 4$ SYM theory is conformal, meaning the couplings don’t run for any energy scale. Such a theory at the boundary is dual to a theory on the AdS space. We can study thermodynamics on extensions of AdS/CFT correspondence.

The temperature effect requires putting the theory on a compact space. This is because with conformal invariance, the manifold $S^1$ has fixed circumference, if we compute the partition function on it, we wouldn’t see change with respect to the temperature. On a compact space, the large $N$ theory has a phase transition corresponding to confinement and deconfinement on the gauge theory side. Deconfinement is realized by spontaneous breaking of the centre of gauge group. For large $N$ QCD, we have gauge group $SU(N)$, the chiral symmetry group $Z_N$ is the centre [34]. We can track the symmetry breaking process by looking at the expectation values of temporal Wilson line. Suppose the manifold is $Y \times S^1$ and the $S^1$ has circumference $\beta$. Choose a closed path $C$ and take $\phi$ to be the gauge field in fundamental representation. We find that the Wilson loop becomes:

$$W(c) = \text{Tr} \int_C \phi$$  \hspace{1cm} (10.7)
The gauge transformation of function $g$ obey $g(y, z + \beta) = g(y, z)h$, where $y \in Y$ and $z \in S^1$. This changes the Wilson loop by a factor of $h$:

$$W(c) \rightarrow hW(c) \quad (10.8)$$

$h$ is the root of unity representing $Z_N$, so the Wilson loop is an important measure in the theory. The Wilson loop obeys an area law, demonstrating confinement. If the closed path contains an area $A$, then the expectation value vanishes exponentially with $A$.

Now let us study the thermodynamics on AdS space. The Einstein equation takes the form

$$R_{ij} = -nb^{-2}g_{ij} \quad (10.9)$$

where $b$ is the curvature. The Einstein action and metrics are given below

$$I = -\frac{1}{16\pi G_N} \int d^{n+1}x \sqrt{g} \left( R + \frac{\frac{1}{2}n(n-1)}{b^2} \right) \quad (10.10)$$

$$ds^2_{AdS} = \left( \frac{r^2}{b^2} + 1 \right) dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\Omega^2 \quad (10.11)$$

$$ds^2_{AdS} = \left( \frac{r^2}{b^2} + 1 - \frac{w_n M}{r^{n-2}} \right) dt^2 + \frac{dr^2}{r^2 + 1 - \frac{w_n M}{r^{n-2}}} + r^2 d\Omega^2 \quad (10.12)$$

$$w_n = \frac{16\pi G_N}{(n-1)V_{S^{n-1}}} \quad (10.13)$$

From the metric, the regularized volume of AdS space-time and the Schwarzschild black hole can be calculated. The cut-off is $\Lambda$.

$$V_{AdS} = \int_0^{\beta'} dt \int_0^{\Lambda} dr \int_{S^{n-1}} d\Omega r^{n-1} \quad (10.14)$$
The period for the black hole is $\beta_0$.

$$\beta_0 = \frac{4\pi b^2 r_+}{n r_+^2 + (n - 2)b^2}$$  \hspace{1cm} (10.16)

Taking the difference in the action, the black hole dominates for small $r_+$, whereas the thermal AdS space dominates for large $r_+$. Thus, to go from one configuration to the other, there must be a phase transition.

$$\Delta s = \frac{n}{8\pi G_N} \lim_{\Lambda \to \infty} (V_{BH} - V_{AdS}) = \frac{V_{S^{n-1}}(b^2 r_+^{n-1} - r_+^{n+1})}{4G_N(nr_+^2 + (n - 2)b^2)}$$  \hspace{1cm} (10.17)

The standard way to study thermodynamics is to look at the entropy, partition function and energy of the system. We can check the entropy before the phase transition at small $r_+$, and it indeed satisfies the black hole entropy law:

$$E = \frac{\partial s}{\partial \beta_0} = \frac{(n - 1)V_{S^{n-1}} (b^{-2} r_+^{n} + r_+^{n-2})}{16\pi G_N}$$  \hspace{1cm} (10.18)

$$S = \beta_0 E - s = \frac{r_+^{n-1} V_{S^{n-1}}}{4G_N} = \frac{Area}{4G_N}$$  \hspace{1cm} (10.19)

On the other side, we look at conformal field theory at the boundary on manifold $S^{n-1} \times S^1$. The circumference of $S^{n-1}$ and $S^1$ are 1 and $\beta_0/b$. If $\beta_0$ is small, then $r_+$ must be large, so for an $n - 1$ dimensional hypersphere, the period scales as $\frac{1}{\beta_0^{n-1}}$, and so does the entropy.

As $S^3$ has infinite circumference, it is like $R^3$, the three dimensional real space. On the manifold $R^3 \times S^1$, the circumference of $S^1$ is $\beta_1/b$. The temperature is then
defined as

\[ T = \frac{b}{\beta_1} = \frac{1}{\pi} \tag{10.20} \]

After the phase transition, the Wilson loop is given by the exponential of a regularized area \(-\alpha(D)\) on the world sheet.

\[ \langle W(c) \rangle = \int_D d\mu \ e^{-\alpha(D)} \tag{10.21} \]

This implies the expectation value of wilson loop still obeys an area law if the regularized area scales proportional to the area inside loop. If the closed path is not on the boundary, then the expectation value is zero.

In the later parts of the thesis, we will study the thermodynamics and phase transitions for the non-confomal case using an appropriate gravity dual.
CHAPTER 11
A UV Completed Large $N$ Thermal QCD Model

11.1 Introduction

As we discussed in the earlier sections, the gauge/gravity duality has so far proved to be a powerful technique to solve many strong coupling problems of large $N$ gauge theories, and especially large $N$ QCD, in the planar limit. The application of a gravity dual to understand strongly coupled gauge theory was, in retrospect, the next best thing to do. A simple way to see this would be to consider a particular gauge-theory defined on a $3 + 1$ dimensional slice at a certain energy scale $\Lambda$. Now imagine we stack up all the slices together, described at different energy scales, along an orthogonal direction (call it the “radial” direction $r$). This way we will get a five dimensional space that captures the full dynamics of a given gauge theory from the Ultra-Violet (UV), i.e. large $r$, to the Infra-Red (IR), i.e. small $r$. The “radial” direction would then obviously be the direction along which the energy would change, i.e. the direction of the Renormalisation Group (RG) flow. For a Conformal Field Theory (CFT), the theory does not change along the radial direction\(^1\) and therefore could as well be defined at the boundary of the five-dimensional space. The scale invariance of the underlying gauge theory will restrict the geometry of the

---

\(^1\) Assuming the usual behaviour of the irrelevant operators.
five-dimensional space to the Anti-deSitter (AdS) space[18], although it would be interesting to argue that this is the unique choice\textsuperscript{2}. However, for gauge theories with inherent RG flows, the situation will be different and it would be instructive to study the theories at various $r$ (although we could also restrict ourselves to the boundary again).

The example that we are interested in is large $N$ QCD, which we expect to be asymptotically conformal. It is interesting that we demand conformal behaviour in the UV and not asymptotic freedom. This is because the \'t Hooft coupling $\lambda \equiv g_{YM}^2 N$ approaches a constant in the limit $g_{YM}^2 \to 0$ and $N \to \infty$. This way, the theory is actually asymptotically free in terms of $g_{YM}^2$ but conformal in terms of $\lambda$. Furthermore, we will demand $\lambda$ to be very large throughout the whole RG flow so that the gravity dual can be restricted to its classical limit. in the UV and confining in the far IR. Specific geometries that do the jobs for both zero and non-zero temperatures were presented in [22, 23] although the details of the gauge theories were not presented here.

We will fill up some of the gaps from previous work and argue why we believe our choice of the gravity dual is better suited to study large $N$ thermal QCD.

\textsuperscript{2} Furthermore, a Feynman diagram for any interaction between point-like particles, when \textit{stacked up} as above, would look like an interaction between extended objects, \textit{i.e.} strings! This is basically the essence of using string (or gravity) duals to study gauge theories. It will be informative to make this more precise.
11.2 The Field Theory from the Gravity Dual

The gravity dual of a large $N$ thermal QCD above the deconfinement temperature, described using only a \textit{flavored} Klebanov-Strassler geometry [28] with a black-hole has few ultra-violet (UV) problems. For example, there are Landau poles coming from the flavor branes, and the Wilson loops are generically UV divergent. All these issues could be resolved if we properly augment the Klebanov-Strassler geometry, which we will henceforth call as the Ouyang-Klebanov-Strassler black-hole (OKS-BH) [27, 22, 23] geometry, with a suitable asymptotically Anti-de Sitter (AdS) space. As discussed in [23], this augmentation can only be performed in the presence of an interpolating space and certain number of anti five-brane sources. A summary of the brane configuration discussed here can be found in [23] section 2. The interpolating region, which we called region 2 in [23], can be interpreted alternatively as the \textit{deformation} of the neighbouring geometry once we attach an AdS cap to the OKS-BH geometry. The OKS-BH geometry is in the range $r_h \leq r \leq r_{\min}$ (which we will call as region 1) and the AdS cap is the range $r > r_0$ (which we will call as region 3). Here $r_h$ is the horizon radius. The geometry in the range $r_{\min} \leq r \leq r_0$ is the deformation. Such deformations should be expected for all other UV caps advocated in [22]. This construction was elaborated in some details in [23]. In this paper we will start with a gauge theory interpretation of background.

In the far IR, one may note that the cascading RG flow is never captured by the classical gravity theory. The classical supergravity description would capture only the smooth parts of the RG flow shown in 11.2 at the center of each slices. Note that the circles drawn on each slices should be taken as helices that become more and
more straight as we go to the center of each of the slices. The vertical distances in 11.2 has no physical meaning and only refer to the slices described using appropriate Seiberg dual descriptions. On the other hand, in 11.2 the RG flows all tend to go to zero at some UV scales. This is where the theory become conformal (all scales are chosen with $\alpha' = 1$). Note that, only the strongly coupled parts of figure 11.2 are captured by the classical supergravity description.

For the UV region $r > r_0$ we expect the dual gauge theory to be $SU(N + M) \times SU(N + M)$ with fundamental flavours coming from the seven-branes. This is because addition of $M$ anti five-branes at the junction (i.e. for $r > r_0$) with gauge fluxes on its world-volume tells us that the number of three-branes degrees of freedom are $N + M$, where the $M$ and $N$ factors come from the presence of $M$ five-branes anti-five-branes pairs and $N$ D3-branes. Furthermore, the $SU(N + M) \times SU(N + M)$ gauge theory informs us that the gravity dual is approximately AdS, but has RG flows because of the fundamental flavours. In other words, the two couplings $g_1$ and
$g_2$ of each gauge group would be approximately the same and exhibit a walking RG flow.

At the scale $r = r_0$, we expect one of the gauge group to be Higgsed, so that we are left with $SU(N + M) \times SU(N)$. Now both gauge couplings flow at different rates and give rise to a cascade that is slowed down by the $N_f$ flavours. In the end, at far IR, we expect confinement at zero temperature. The few calculations that we did in [23] regarding (a) the flow of $N$ and $M$ colours, (b) the RG flows, (c) the decay of the three-forms and (d) the behaviour of the dual gravity background all support the gauge theory interpretation that we gave above. What we haven’t been able to demonstrate in [23, 22] is the precise Higgsing that takes us to the cascading picture. From the gravity side, it is clear how this could be interpreted. From the
gauge theory side, we will provide a brief derivation below. But before we dwell on the
details, let us see how the full renormalisation group (RG) flow would look like
with the AdS cap.

11.2.1 Continuous RG Flow from UV to IR

As mentioned above, the gravity dual should give us a RG flow that allows us to
see the UV conformal behaviour and the IR confining behaviour succinctly. However,
there is a subtlety as shown in Figure 11.2. The cascading RG flow in the far IR,
where the theory goes from one Seiberg fixed point to another, is in fact not seen in
the dual gravity side because it runs between weakly coupled theories. Thus, what
we see from the dual gravity side is a smooth RG flow, as depicted at the centre of
each slice in 11.2. More details can be found in [28, 22].

This also means that at any given scale Λ there are in principle an infinite
number of gauge theory descriptions available. Out of which, one of them might be
the most useful description at that scale and is therefore captured by the classical
supergravity analysis at \( r = Λ \).

For example, in the far IR, out of the many available gauge theory descriptions,
it is the confining \( SU(M) \) gauge theory (which is naturally strongly coupled) that
is captured by the classical supergravity solution at small \( r \). As a consequence, we
expect that the definition of the number of colours at any given scale would become
a little ambiguous.

The RG flow in the intermediate region, identified as region 2, is more involved
and will be discussed in details in [24]. However this RG flow connects smoothly to
the RG flow in the AdS cap, called as region 3. The flow in region 3 approaches
conformality where both couplings run at an equal rate as shown in 11.2. The Beta
functions are also easy to compute to first order in $g_sN_f$ from the gravity dual. In
Region 1 the two couplings at a scale $\Lambda$ run in the following way:

\[
\Lambda \frac{\partial}{\partial \Lambda} \left[ \frac{4\pi}{g_1^2} + \frac{4\pi}{g_2^2} \right] = \frac{N_f}{8} \left( \frac{6r^6 + 36a^2r^4}{r^6 + 9a^2r^4} \right) \bigg|_{r=\Lambda}
\]

\[
\Lambda \frac{\partial}{\partial \Lambda} \left[ \frac{4\pi}{g_1^2} - \frac{4\pi}{g_2^2} \right] = 3M \left[ 1 + \frac{3g_sN_f}{4\pi} \log(\Lambda^2 + 9a^2) \right] \bigg|_{r=\Lambda}
\]

where the RHS of both equations is evaluated at $r \equiv \Lambda$ in the gravity picture. The
constant $a$ appearing above is the bare resolution parameter that one may set to
zero\(^3\). In this limit, the RG flow is clearly the NSVZ RG flow [30]. On the other
hand, in region 2, where we still have two couplings, the RG flow is highly non-trivial.
This can be derived from the gravity dual where we see that the three-form fluxes
play an important role in the running of the couplings [24]:

\[
\frac{8\pi^2}{g_1^2} = e^{-\Phi} \left[ \frac{\pi}{2} - \frac{1}{2\pi} \int_{S^2} B_2 \right]
\]

\[
\frac{8\pi^2}{g_2^2} = e^{-\Phi} \left[ \frac{\pi}{2} - \frac{1}{2\pi} \int_{S^2} B_2 \right]
\]  

(11.1)

Finally in region 3, the scenario is somewhat simpler. The two couplings flow approx-
imately at the same rate and the flow is governed by the $N_f$ D7 and anti-D7 pairs
that we keep in region 3 to cancel the Landau poles. These seven-branes are respon-
sible for restoring the $SU(N_f) \times SU(N_f)$ chiral symmetry above the deconfinement

\[^3\text{This is however not so above the deconfinement temperature. As shown recently in [21], even if we demand a vanishing bare resolution parameter, it will get a contribution from the horizon radius } r_h, \text{ such that } a \sim O(r_h). \text{ Of course, on the gauge theory side, the branes are still wrapped on vanishing cycle.}\]
temperature (i.e. when we insert a black-hole with a horizon radius $r_h$ [22, 23]). The running of the coupling, which we call $g_{YM}$, is now:

$$
\Lambda \frac{\partial g_{YM}}{\partial \Lambda} = g_{YM}^3 \sum_{n=1}^{\infty} \frac{D_n}{\Lambda^{3n/2}}
$$

(11.2)

where $D_n$ are all independent of $\Lambda$ and whose precise form will be derived in [24].

The beta functions discussed above can now be succinctly expressed as a continuous flow from UV to IR as shown in fig 2. We have represented this using a slightly unconventional way. We get the complete RG flow by gluing the three regions altogether and using S-duality to transmute strong coupling into weak coupling. Starting from IR regime, once a particular coupling gets strong, a S-duality is performed to reverse the sign of the beta function associated with that coupling. This appears as the sharp edges in the figure above. From the UV region this can be seen in the following way:

The coupling starts as a constant in Region 3, when it gets to the transition point $r_0 = 200$, it has a small plateau region continuing in region 2, then it flows down to Region 1. The RG flow continues after the transition point $r_{\text{min}} = 100$, but since the rate of change is fast more sharp corners appear in region 1. These are the points connected to their S-dual values. Eventually this reaches the smallest energy possible after which we expect linear confinement at low temperatures $^4$. To get

$^4$ We get the complete RG flow by gluing the three regions altogether and using S-duality to transform strong coupling into weak coupling. As before, all scales are chosen with $\alpha' \equiv 1$
a continuous RG flow, we impose a boundary condition that matches up different curves. Since the equations describe a series of flow with different initial conditions, we add a constant to shift the line of flow to be connected with the previous one after S-duality.
11.2.2 Higgsing

In region 3, we have a $SU(N+M) \times SU(N+M)$ gauge group which breaks down to $SU(N+M) \times SU(N)$ by the Higgs mechanism as we enter Region 1. We will study this mechanism in two versions: supersymmetric and non-supersymmetric. Since the purpose is to break the gauge group, we will ignore any fundamental matter fields in the following discussion. Before moving ahead, let us see how we could justify the Higgs mechanism from the gravity perspective. The brane construction that reproduces the gauge theory should be understood on a scale-by-scale basis, so that the full RG flow could be reproduced in the gravity dual. Generically, we expect $N$ D3 branes and $M$ wrapped D5 branes on a vanishing two-cycle of the conifold. Allowing a small resolution factor to the other two-cycle, we can distribute the anti-D5 branes on the resolved sphere such that they wrap the same vanishing two-cycle but are distributed on the other sphere. Similarly the D7 and anti-D7 branes are also distributed over the resolved sphere via the Ouyang embedding [23]. The tachyons between D5 and anti D5 branes or between D7 and anti-D7 branes can be cancelled by switching on appropriate gauge fluxes on the set of anti branes. This phenomena is somewhat similar to the ones in [4]. These gauge fluxes will create bound D3 and bound D5 branes respectively on the two set of brane anti-brane systems. If oriented properly, the system would then be almost BPS in the zero-temperature case when the distance between the two set of branes is large (the multipole forces are heavily suppressed). To stabilize this completely, one may switch on three-form $H_{NS}, H_{RR}$ fluxes on the internal space (the axio-dilaton are already switched on). These $H$-fluxes would not only change the moding of the strings between the branes but also
stabilize the position of the branes, by generating perturbative and non-perturbative superpotential and giving masses to the scalar fields on the branes, along the lines of [9, 12, 2]. Alternatively for short distances, one may dissolve the anti-D5 branes in the D7 anti-D7 system in the way discussed in [23], and then stabilize the seven-brane positions. In either case the physics would be the same.

This configuration is more intuitive from the gravity dual side where the radial coordinate now becomes the scale of the theory. At a given scale we expect $M_\epsilon$ number of wrapped anti-D5 branes where $M_\epsilon = \frac{Me^{\alpha(r-r_0)}}{1+e^{\alpha(r-r_0)}}$ with $r \sim \mathcal{O}(1/\epsilon)$ and $\alpha >> 1$. Then it is easy to see that the resulting gauge group becomes $SU(N + M_\epsilon) \times SU(N + M)$. Clearly, in the limit $\epsilon \to 0$, we recover the conformal gauge group. Therefore, the anti-D5 branes appear to only affect one of the gauge groups in the product. In region 3, where $r >> r_0$, $M_\epsilon \approx M$, this tells us that the RG flow will be mostly due to the flavour seven-branes.

The above construction then instructs us that the Higgsing process generating the cascade should simply be engineered by making some anti-D5 brane degree of freedoms (DOFs) heavy, i.e. by moving the anti-D5 branes away from the $N$ D3 and the $M$ wrapped D5 branes on the resolved sphere as we discussed above.

---

5 The gauge group that actually appears in the far IR is $SU(N + M) \times SU(N) \times U(1)^M$, where the $U(1)$’s are from the massless sector of the anti-D5 branes. However at low energies, once we integrate out the Higgs masses (i.e. the strings between the D5 and the anti-D5 branes), these $U(1)$’s would be decoupled. Furthermore at strong coupling, where we expect the dual gravity description to hold, these $U(1)$’s will never appear. This in turn implies that the anti five-brane degrees of freedom should only be seen at high energies, precisely in the way we predicted in [23].
supersymmetric theory, where the UV completion is done by a $\mathcal{N} = 2$ theory, this process would mean moving the anti-D5 brane DOFs along the Coulomb branch, which in turn implies that the anti-D5 branes’ world-volume scalar multiplets, transforming under a certain subgroup of $SU(N+M)$, will be responsible for the Higgsing mechanism.

For the non-supersymmetric theory, this is rather easy to demonstrate. All we require is that the vev of the Higgs field $\phi$ should only transform under a certain subgroup of the first $SU(N+M)$ group. The Lagrangian is:

$$L = -\frac{1}{2} D^\mu \phi_k D_\mu \phi_k - V(\phi) - \frac{1}{4} F^a_{i\mu\nu} F^{a}_{i\mu\nu}$$ (11.3)

where $i = 1, 2$ refers to each $SU(N+M)$ copy in the product gauge group. $D_\mu \phi_k = \partial_\mu \phi_k - ig_1 A^a_{1\mu} (T^a_1)_{kl} \phi_l$ with $g_1$, $A_{1\mu}$ and $T_1$ being the gauge coupling, gauge field and generators of the first $SU(N+M)$ group respectively.

Now we suppose the potential $V(\phi)$ is minimized at $\langle \phi_i \rangle \equiv v_i$. Then a generator $T^a_1$ is broken if $(T^a_1)_{ij} v_j \neq 0$. To see this, let us define:

$$\tilde{\phi}_i(x) = v_i + H_i(x)$$ (11.4)

where $H_i(x)$ is a real scalar field. The covariant derivative of $\tilde{\phi}_i$ is:

$$D_\mu \tilde{\phi}_i = \partial_\mu H_i(x) - ig_1 A^a_{1\mu} (T^a_1)_{ik} [v_k + H_k(x)]$$ (11.5)
and the kinetic term for \( \phi_k \) becomes:

\[
-\frac{1}{2} D^\mu \phi_k D_\mu \phi_k = -\frac{1}{2} \partial^\mu H_k \partial_\mu H_k - \frac{1}{2} M_k^a M_k^b A_1^{a\mu} A_1^{b\mu} \\
+ M_k^a A_1^{a\mu} \partial^\mu H_k + ig_1 A_1^{a\mu} H_i (T^a_i)_{ij} \partial^\mu H_j \\
+ \frac{1}{2} g_1^2 A_1^{a\mu} A_1^{b\mu} H_i (T^a_i)_{il} (T^b_i)_{lj} H_j \\
+ ig_1 A_1^{a\mu} A_1^{b\mu} M_i^a (T^b_i)_{ij} H_j
\]

(11.6)

where \( M_k^a = ig_1 T^a_{kj} v_j \). It is obvious that \( A^a \) will get massive if \( T^a_{ij} v_j \neq 0 \) and thus the gauge group is broken. The condition \( T^a_{ij} v_j \neq 0 \) may not be too stringent as a requirement. This is because \( T^a_{kj} v_j T^a_{kl} v_i \) are non-negative no matter what linear transformations we do. So it must be zero to preserve the symmetry. However, generically one could also take linear combinations of various \( T \) that leave \( v \) unchanged. This can be done by diagonalizing the mass matrix \( M_k^a M_k^b \), where \( M_k^a \equiv ig_1 T^a_{kj} v_j \), that would involve such linear combinations.

In our case, we only want to break \( M \) of the generators. How this is done depends on the details of the potentials and the specific values of \( N \) and \( M \). From the dual gravity, we expect to see \( M \) anti-D5 branes at \( r \to \infty \) so that the gauge theory is almost conformal. As the radial coordinate decreases, the number of anti-D5 branes become \( M_r \), as given earlier. For \( r << r_0 \) we expect the number of anti-D5 branes to completely vanish so that the gauge group becomes \( SU(N+M) \times SU(N) \) henceforth the cascading behavior begins. In figure 11.2.2 we have plotted the behavior of the function \( f(r) = M_r/M \). From the plot, we see that for \( r << r_0 \), the function vanishes whereas it approaches unity for \( r > r_0 \). The supersymmetric case follows the same
Figure 11–4: A plot of the function \( f(r) \equiv \frac{e^{\alpha(r-r_0)}}{1+e^{\alpha(r-r_0)}} \) for \( r_0 = 5 \) in appropriate units, and for various choices of \( \alpha \).

The general Lagrangian now is:

\[
\mathcal{L} = \int d^4 \theta \, \mathcal{K} \bar{\Phi} e^V \Phi + \int d^2 \theta \left[ \mathcal{W} + \frac{1}{32\pi i} \tau tr f W_\alpha^2 \right] + h.c.
\]

where \((\Phi, W_\alpha = -\frac{1}{4} \bar{D} \bar{D} D_\alpha V)\) are the appropriate \( \mathcal{N} = 1 \) chiral and the vector multiplets with \((\phi_k, A_\mu^a)\) being the complex scalar and the vector fields in their respective multiplets, \( \mathcal{K} \) is a gauge invariant Kähler potential, \( \mathcal{W} \) is a gauge invariant superpotential, \( \tau \equiv \frac{\vartheta}{2\pi} + i \frac{4\pi}{g^2} \) is the complexified gauge coupling with \( \vartheta \)-angle and \( tr f \) is the trace in the fundamental representation. The FI terms don’t appear because they are forbidden by the non-abelian gauge invariance.
The scalar potential obtained by expanding the above Lagrangian in components is a sum of $F^2$ and $D^2$ terms. The $F$ and the $D$-terms are:

\[
F_k = -\frac{\partial \bar{\mathcal{W}}}{\partial \phi_k}, \quad D^a = \bar{\phi}_k (T^a_R)_{kl} \phi_l
\]  

(11.7)

where $T^a_R$ denotes the generator in the $R$ representation. To preserve supersymmetry we must have $F_k = 0$ and $D^a = 0$. Actually in the absence of FI terms whenever $F_k = 0$ has a solution, $D^a = 0$ always has a solution. So we assume we already have a solution that satisfies $F_k = 0$. This solution can break the gauge symmetry as in the non-supersymmetric case. This can be seen in the following way. Write down the relevant kinetic terms of the scalar component of the Higgs multiplet, as:

\[
\int d^4 \theta \, \Phi e^V \Phi = -\left| (\partial_\mu + igA^a_\mu T^a) \phi \right|^2 + \ldots
\]  

(11.8)

which is exactly the same as in the non-supersymmetric case. If the $F$-term solution leads to $M$ generators $T^a$ such that $T^a_{ij} v_j \neq 0$, then the gauge group is broken from $SU(N + M)$ to $SU(N)$. How this happens again depends on the details of the $N, M$ values and the form of the superpotential.

11.3 Phase Transition and Other Applications

Once we have the gauge theory description, it is time to extend our configuration to incorporate temperature. Two immediate scenarios present themselves: the confining theory at low temperatures and the theory above the deconfinement temperature. The process of going from one to another in the gravity dual will appear as the confinement to deconfinement phase transition in the large $N$ thermal QCD.
In [22, 23, 21], the theory above the deconfinement temperature was studied in details. The high temperature phase was understood therein as the one coming from a black-hole with a horizon radius $r_h$ where the temperature was related to $r_h$. The scenario at low temperatures were not discussed in details in [22, 23, 21]. Here, we will study these two phases and discuss their associate phase transition. More elaborations on this will be presented in [20].

Before actually computing the phase transition, let us discuss a couple of issues that may arise in equivalent scenarios dealing with large $N$ thermal QCD. The first issue is the stability at high temperatures. Stability is guaranteed by a positive specific heat $c_v$. A negative specific heat implies instability, which in fact turned out to be the case of many models that study large $N$ thermal QCD without a UV completion [6]. To assess the issue of stability, let us first define the specific heat in terms of the internal energy $E_{int}$, the BH factor $g = 1 - r_h^4/r^4$ and the temperature $T$ in the following way:

$$c_v = \left( \frac{\partial E_{int}}{\partial T} \right)_V, \quad T = \left. \frac{g'}{4\pi\sqrt{h}} \right|_{r_h} \approx \frac{r_h}{\pi L^2}$$

(11.9)

where we have introduced the $AdS_5$ length scale $L$ in anticipation of the AdS cap, and the internal energy is given by the integral of the zeroth component of the stress tensor.

To calculate the heat capacity, we have to know how much energy is encoded in the geometry. As $r \to \infty$, the space-time is approximately $AdS_5 \times T^{1,1}$, where $T^{1,1}$ is the internal space. The internal energy of asymptotically $AdS_5$ space-time can be easily calculated using results from [7]. The total stress-energy tensor $T_{ij}$ is
composed of stress-energy from the medium and the quarks. Quarks can be seen as
excitations of the D7 branes.

At the boundary, only the medium contributes to the stress-energy tensor. Thus, using the background above the deconfinement temperature given in [22, 23, 21], the internal energy, in terms of the string coupling \( g_s \) and the Newton’s constant \( G_N \), becomes:

\[
E_{\text{int}} = \int d^3x \sqrt{g} T_{00} = \frac{\pi^2 r_h^4}{g_s^2 G_N}
\]

(11.10)

which gives the following value for the specific heat:

\[
c_v = + \frac{4\pi^6 L^8}{g_s^2 G_N} T^3
\]

(11.11)

This means that the heat capacity is positive for positive temperatures, showing that the model is stable at high temperatures.

The second issue is slightly tangential to our interest but is nevertheless important enough that we clarify the scenario here. It was proposed recently in an interesting work [19] that the confinement to deconfinement phase transition in the type IIA Sakai-Sugimoto model [29] does not proceed via the standard transition of a solitonic D4-brane to a black D4-brane, as proposed in [34, 26], but via a Gregory-Laflamme transition [13] from a solitonic D4-brane to a certain type IIB Euclideanized D3-brane configuration. In retrospect, this conclusion may not

---

\(^6\) The contribution from \( T_{00} \) and warp factor \( h \) as \( r \to \infty \) is of zeroth order. When the AdS geometry is deformed, the stress-energy receives higher order corrections.
be too surprising because, although the black D4 brane elegantly depicts the five-dimensional deconfined phase, it fails to do so in the four-dimensional case once a certain energy scale is reached. Indeed in this model, there is no reason for integrating out the modes coming from the compact $S^1$ direction. Thus the Euclideanized D3 brane phase should be preferred at high temperatures. Unfortunately however, because the Mandal-Morita [19] picture above the deconfined phase is not a configuration of black D3-branes, the usual computations of transfer coefficients, that rely on the dynamics of black-holes in these spaces, cannot be performed so easily.

This is exactly where our model may have some distinct advantages. Since we are considering configurations of wrapped five-branes and anti five-branes on vanishing two-cycle, the subtlety of Kaluza-Klein (KK) reduction will not appear, and we should be able to go between solitonic D3 and black D3-branes. This would then be the confinement to deconfinement phase transition for our case, which is of course the Hawking-Page [14] transition. In the following we will first take a brief detour to explain the Sakai-Sugimoto limit of our model, before going into the discussion of phase transition in our set-up. More details will appear in [24, 20].

11.3.1 The Sakai-Sugimoto limit

From the above discussion, an interesting question at this stage would be to compare our type IIA dual picture with the Sakai-Sugimoto model [29]. For simplicity, let us only consider the far IR picture where we have D5 branes wrapped on the vanishing two-cycle of the conifold. The vanishing cycle could be parametrized by $(\theta_1, \phi_1)$ and the other two-cycle is along $(\theta_2, \phi_2)$. The $U(1)$ fibration of the conifold is $\psi$ and the radial direction is $r$. The D5 branes have a spacetime stretch along
the usual $x^{0,1,2,3}$ directions. T-dualising along the $\psi$ direction gives us D4-branes stretched between two NS5-branes along the $\psi$ circle. Thus the $\psi$ coordinate is like the $x^4$ coordinate of the Sakai-Sugimoto model. The difference now is that the D4-branes are stretched only along a fraction of the $\psi$ circle and between two orthogonal NS5-branes. The D7 anti-D7 branes become D8 anti-D8-branes along $(x^{0,1,2,3}, r)$ and $P^1_{\theta_1,\phi_1} \times P^1_{\theta_2,\phi_2}$ just like the Sakai-Sugimoto case, but with an instanton configuration on the two spheres that breaks the supersymmetry.

Of course, one might now worry that, since we made $\psi$ non-contractible, the usual issue raised in [19] should appear for our T-dual model too. However, note that the distance between the two NS5-branes could be made arbitrarily small (without changing the size of the $\psi$ circle), so the issue raised in [19] may appear only at very high temperatures! Thus even at arbitrarily high temperature, if we tune the distance between the two NS5-branes appropriately so that the $\psi$ modes are of very high energies, we might still be able to study the deconfined limit using the black D4-branes. Further details and explicit computations on this construction will be reported in [24].

---

7 The tachyons between the D8 anti-D8 branes could now be arranged to be massive in the presence of fluxes so that they do not cancel when the distances between D8 anti-D8 branes become small. The behavior of the D8 anti-D8 branes would then be similar to the Sakai-Sugimoto case.

8 This depends on the choice of the $B_{NS}$ field on the vanishing cycle [8].
11.3.2 Phase Transition

Phase transitions of $SU(N)$ gauge theory can be realized by spontaneous breaking of the center symmetry $Z_N$. In the confined phase, $Z_N$ symmetry is preserved and its associated order parameter, a temporal Wilson loop, is zero (i.e. $\langle W \rangle = 0$). In the deconfined phase, $Z_N$ symmetry is spontaneously broken with $\langle W \rangle \neq 0$. In [23], we computed $\langle W \rangle$ using the gravity description and showed that OKS-BH geometry with large black holes give $\langle W \rangle \neq 0$ while the OKS geometry without black holes give $\langle W \rangle = 0$. This indicates that extremal geometry is dual to confined phase while non-extremal geometry corresponds to deconfining phase.

Here we will obtain the critical temperature for confinement/deconfinement transition by computing the free energy of extremal and non-extremal geometries and identifying it with the free energy of the gauge theory. We start with the on-shell type IIB supergravity action with appropriate Gibbons-Hawking boundary terms and counter terms:

$$ S = \beta E_{\text{free}} = S_{\text{IIB}} + S_{\text{GH}} + S_{\text{counter}} \quad (11.12) $$

where $E_{\text{free}}$ is the free energy, $S_{\text{IIB}}$ is the ten dimensional type IIB Euclidean supergravity action including localized sources [9][11], $S_{\text{GH}}$ is the Gibbons-Hawking surface term [10] and $S_{\text{counter}}$ is the counter term necessary to renormalize the action [7][22][20]. Just like the case for AdS gravity discussed by Hawking and Page [14] and subsequently by Witten [34], the above action gives rise to both extremal and non-extremal metric and both geometries can incorporate non-zero temperature of the dual gauge theory in the following way: Wick rotate $t \rightarrow i\tau, \tau \in (0, \beta)$ and
identify temperature $T$ as $T = 1/\beta$. At a fixed temperature of the gauge theory, we have two geometries — extremal and non-extremal — and the geometry with smaller on-shell action will be preferred. The free energy of the gauge theory will then be given by the free energy of the geometry obtained through (11.24). Denoting the on-shell value of the action for the extremal geometry with $S_1$ and the non-extremal geometry with $S_2$, we compute the action difference in the absence of D7 branes and localized sources, i.e., $N_f = 0$ and the axio-dilaton $\tau$ is a constant (i.e. without fundamental matter), as [20]:

$$\Delta S = S_2 - S_1 = \frac{g_s M^2 \beta V_8}{2 \kappa_{10}^2 N} \lim_{\kappa \to \infty} \left[ \frac{r_h^4}{32} \log \left( \frac{\mathcal{R}}{r_h} \right) - \frac{5d r_h^4}{128} \right]$$

(11.13)

where $V_8$ is the volume of $R^3 \times T^1.1$, $T^1.1$ being the base of the conifold with approximate radius $L = (g_s N)^{1/4} \sqrt{\alpha'}$, $N, M$ are number of D3 and D5 branes, $\mathcal{R}$ is the boundary value of $r$, and $r_h$ is the black hole horizon radius. Here $d > 0$ is a constant independent of $N, M, g_s$ and depends on the boundary values of derivatives of the metric [20]. In obtaining (11.13), we have only kept terms up to linear order in $g_s M^2/N$ which is valid for $N \gg g_s M^2$ and the exact form of $S_{\text{counter}}, S_{\text{GH}}$ is presented in [20]. Furthermore in the limit $\mathcal{R} \to \infty$ in (11.13), we are dropping $\mathcal{R}^{-n}, n > 0$ terms.

The critical temperature is obtained by evaluating the critical horizon $r_h^c$ for which $\Delta S(r_h^c) = 0$ and the result is [20]:

$$r_h^c = \mathcal{R} \exp \left( \frac{-5d}{4} \right), \quad T_c = \frac{1 + \mathcal{O} \left( \frac{g_s M^2}{N} \right)}{\pi \exp \left( \frac{5d}{4} \right) (g_s N)^{1/4} \sqrt{\alpha'}}$$

(11.14)
where we have used the scaling \( R = L = (g_s N)^{1/4} \sqrt{\alpha'} \to \infty \). For \( T > T_c \), \( \Delta S < 0 \), i.e. the black hole geometry has lower free energy and thus preferred, while for \( T < T_c \), \( \Delta S > 0 \), i.e. the extremal geometry is preferred. For extremal geometry, one readily gets an entropy \( s = -\frac{\partial E_{\text{free}}}{\partial T} = 0 \), while for the black hole geometry:

\[
s \sim N^2 T^3 \left[ 1 + \frac{g_s M^2 b}{N} \log(L T) \right] \tag{11.15}
\]

at lowest order in \( g_s M^2 / N \) and \( b > 0 \) is a constant independent of \( N, M, g_s \). Observe that when \( M = 0 \), \( \Delta S = 0 \), \( \forall r_h \) i.e. extremal and non-extremal action is equivalent for all temperatures of the boundary gauge theory. This is consistent with the field theory picture because the \( M = 0 \) limit gives an \( AdS_5 \times T^{1,1} \) geometry which describes a conformal theory. A conformal theory on \( S^1 \times R^3 \) with circumference \( \beta \) for \( S^1 \) has no phase transition since the value of \( \beta \) can be scaled away by conformal invariance [34] i.e. the vacuum phase is equivalent to the thermal phase.

Even with \( M \neq 0 \), when we do not have any D7 branes i.e. we do not have any matter in the fundamental representation\(^9\) the confinement to deconfinement phase transition for the gauge theory mimics the first order transition in pure glue theory and is described by a Hawking-Page transition in the dual geometry.

Observe that in deriving (11.13), we defined the boundary \( r = R \to \infty \), but did not explicitly add a UV geometry. By adding counter terms \( S_{\text{counter}} \) to the on-shell

\(^9\) The field theory has bi-fundamental fields \( A_i, B_j \) and in the far IR can be equivalently described by pure glue \( SU(M) \) theory. If \( T_c \) is very small, the confined phase consists of glue balls and the deconfining phase consists of free gluons of \( SU(M) \). If \( T_c \) is large, the deconfining phase is best described by \( A_i, B_j \) fields.
action, we subtracted the terms in $S_{\text{IIB}} + S_{\text{GH}}$ that diverge at the boundary $r = R$, which is effectively choosing a particular UV completion. Explicitly adding an AdS UV cap would require taking account of the localized sources in the bulk in addition to the fluxes, and the exact on-shell action for a UV complete geometry is not known.

However, the UV completion resulting from our regularization already gives us a first order phase transition with an exact result for the critical temperature and thus is already insightful. Furthermore, since confinement is an IR phenomenon, the critical temperature may not be extremely sensitive to the details of the UV completion and thus the $T_c$ in (11.14) can even be relevant for the UV complete geometry.

### 11.4 Meson Spectrum

Having given a comparison with the meson spectrum, we now want to compute meson spectrum. To get the vector meson field, we first need to translate our type IIB theory into type IIA theory, so we make a T-duality transformation along $\psi$ direction\(^\text{10}\). Since we have a non-trivial B field on the brane, the pull back metric

\(^{10}\) See previous section, where we have described how the T-dual of the type IIB background leads to the modified Sakai-Sugimoto model.
on the D8 brane also has components from B field:

\[
\begin{align*}
  ds_{IIA}^2 &= \frac{1}{\sqrt{h(r)}}(-g_1 dt^2 + dx^2 + dy^2 + dz^2) + \frac{\sqrt{h(r)}}{g_2 h_1} dr^2 + \frac{1}{r^2 \sqrt{h(r)h_1}} d\psi^2 + \frac{2b_{\psi\theta_1}}{r^2 h_1 \sqrt{h}} d\psi d\theta_1 \\
  &\quad + \frac{2b_{\psi\theta_2}}{h_1 \sqrt{h} r^2} d\psi d\theta_2 + \left(\sqrt{h(r)} r^2 h_2 + \frac{b_{\psi_1}}{h_1 \sqrt{h} r^2}\right) d\theta_1^2 + \left(\sqrt{h(r)} r^2 h_5 \cos \psi + \frac{2b_{\psi_1} b_{\psi_2}}{h_1 \sqrt{h} r^2}\right) d\theta_1 d\theta_2 \\
  &\quad - h_5 \sin \psi \sin \theta_2 d\theta_1 d\phi_2 + h_5 \sin \psi \sin \theta_1 d\theta_2 d\phi_1 + \sqrt{h(r)} r^2 (2h_1 \cos \theta_1 \cos \theta_2 - h_5 \cos \psi \sin \theta_1 \sin \theta_2) d\phi_1^2 \\
  &\quad - 2h_1 \cos \theta_1 \cos \theta_2 d\phi_1 d\phi_2 + \sqrt{h(r)} r^2 (h_2 \sin^2 \theta_1 + h_1 \cos^2 \theta_1 - h_1 \cos^2 \theta_2) d\phi_1^2 \\
  &\quad + \left(\sqrt{h(r)} r^2 h_4 h_3 + \frac{b_{\psi_2}}{r^2 h_1 \sqrt{h(r)}}\right) d\theta_2^2 + \sqrt{h(r)} r^2 (h_4 \sin^2 \theta_2 + h_1 \cos^2 \theta_2 - h_1 \cos^2 \theta_1) d\phi_2^2 \\
  &= (11.16)
\end{align*}
\]

where the type IIA NS B-fields are:

\[
\begin{align*}
  B_{\psi_1} &= \cos \theta_1, \quad B_{\psi_2} = \cos \theta_2, \quad B_{\theta_1 \phi_1} = \bar{b}_{\theta_1 \phi_1} - \cos \theta_1 \bar{b}_{\psi_1}, \\
  B_{\theta_1 \phi_2} &= \bar{b}_{\theta_1 \phi_2} - \cos \theta_2 \bar{b}_{\psi_1}, \quad B_{\theta_2 \phi_1} = \bar{b}_{\theta_2 \phi_1} - \cos \theta_1 \bar{b}_{\psi_2}, \quad B_{\theta_2 \phi_2} = \bar{b}_{\theta_2 \phi_2} - \cos \theta_2 \bar{b}_{\psi_2}, \\
  \bar{\phi} &= \phi - \frac{1}{2} \ln g_{\psi \psi} \\
  &= (11.17)
\end{align*}
\]

and \(b_{ij}\) are the type IIB NS B-fields.

We can calculate the DBI action on the D8 brane by taking constant slices in the internal directions, and considering the dynamics of the brane only along \(x = 0, 1, 2, 3, \psi\) directions:

\[
S_{\text{IADS}} \propto \int d^4 x d\psi e^{-\bar{\phi}} \sqrt{G + B} \\
\]

where \(G_{\psi \psi} = g_{\psi \psi} + g_{rr} r'(\psi)^2\) and all other components \(G_{ij} = g_{ij}\). The action can also be written in the following form:
$$S \propto \int d^4x \psi e^{-\Phi} \sqrt{-\det A_1 \sqrt{\det B_1}}$$  (11.19)

where $A_1$ is the block of Minkowski space-time, $B_1$ is the block of internal 5D manifold. At low energy limit, $g_1 = g_2 = 1$, so we can simplify the metric, and then expand the action.

Since the energy is conserved, we get the Hamiltonian as a constant with respect to $\psi$:

$$H = \frac{\partial}{\partial \psi}(r' \frac{\partial L}{\partial r'} - L) = 0$$  (11.20)

$$H \bigg|_{r'(1)=0, r(1)=r_h} = C$$

Next, we look at the gauge fields on the D8 brane. Our aim is to change the coordinates of embedding, and look at the $r$ direction in order to get the eigenfunctions concretely. This gives us:

$$S_{\text{HAD8}} = -T \int dx^9 e^{-\Phi} \sqrt{G + B + 2\pi \alpha' F} \propto \tilde{T} \int d^4x dr e^{-\Phi} \sqrt{G + B + 2\pi \alpha' F}$$  (11.21)

where $\tilde{T}$ is the effective tension. Note that $\psi'$ is negligible away from the singularity of D8 and anti-D8 pair. In the following calculation, we take the limit when it goes to zero. Since the internal directions will be integrated out, we can focus on the gauge fields in $x = 0, 1, 2, 3, r$ directions and express the action as:

$$S = -\tilde{T}(2\pi \alpha')^2 \int d^4x dr \left[ \frac{r h(r)^{-5/4} \eta^{\mu\nu} \eta^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}}{h(r)^{1/4}} + \frac{r}{h(r)^{1/4}} \eta^{\mu\nu} F_{\mu r} F_{\nu r} \right]$$  (11.22)

This action can be rewritten in terms of a gauge field $\Psi$ and the vector meson field $B$. More precisely, the field strength $F_{\mu\nu}$ along $\mu, \nu = 0, 1, 2, 3$ directions can be
split into $\Psi(r)$ multiplied by the field strength $F_{\mu\nu}(x^\mu)$ as a function of Minkowski coordinates; the field strength $F_{\mu r}$ can be split into $\partial\Psi(r)$ multiplied by the vector meson field $B(x^\mu)$ as a function of Minkowski coordinates and other fields [29].

\[
F_{\mu\nu}(x^\mu, r) = \sum_n F^{(n)}_{\mu\nu}(x^\mu)\Psi_n(r)
\]

\[
F_{\mu r}(x^\mu, r) = \sum_n (\partial_\mu \phi^{(n)}(x^\mu)\phi_n(r) - B^{(n)}_{\mu}(x^\mu)\partial\Psi_n(r))
\]  

Plugging (11.23) into (11.22), the action becomes:

\[
S = -\frac{T}{2}(2\pi\alpha')^2 \int d^4xdr \sum_{m,n} rh(r)^{-5/4} F^{(m)}_{\mu\nu} F^{(n)\mu\nu}\Psi_m\Psi_n + \frac{2r}{h(r)^{1/4}} B^{(m)}_{\mu} B^{(n)\mu}\partial\Psi_m\partial\Psi_n
\]

\[
= -\frac{T}{2}(2\pi\alpha')^2 \int d^4x \sum_n \frac{1}{4} F^{(n)}_{\mu\nu} F^{(n)\mu\nu} + \frac{1}{2} \lambda_n M_{KK}^2 B^{(n)}_{\mu} B^{(n)\mu}
\]

(11.24)

where $M_{KK}$ is the Klein-Gordon mass on the compact direction and $\lambda_n$ is the eigen value of the wave function. If we choose $M_{KK}$ on the compact direction to be normalized to 1, then we get the normalization condition for the gauge fields:

\[
-\frac{T}{2}(2\pi\alpha')^2 \int dr \sum_{m,n} rh(r)^{-5/4} \Psi_m\Psi_n = \frac{1}{4} \delta_{nm}
\]

(11.25)

\[
-\frac{T}{2}(2\pi\alpha')^2 \int dr \sum_{m,n} \left(\frac{r}{h(r)^{1/4}}\right) \partial\Psi_m\partial\Psi_n = \frac{1}{4} \lambda_n \delta_{nm}
\]

(11.26)

Matching the two normalization conditions:

\[
\int dr \left(\frac{r}{h(r)^{1/4}}\right) \partial\Psi_m\partial\Psi_n = \int dr rh(r)^{-5/4} \lambda_n \Psi_m\Psi_n
\]

(11.27)
we get the eigenfunction for vector mesons:

\[ \partial_r \left[ \left( \frac{r}{h(r)^{1/4}} \right) \partial_r \Psi_n \right] + \lambda_n r h(r)^{-5/4} \Psi_n = 0 \tag{11.28} \]

whose solution will determine the mesonic spectra.

The odd modes of \( \Psi \) satisfy \( \partial \Psi = 0 \); the even modes satisfy \( \Psi = 0 \). The ratio of \( \lambda \) is independent of cut-off scale. It is also independent of choice of coefficients and \( M_{KK} \). The ratios are as follows:

<table>
<thead>
<tr>
<th>( \frac{\lambda_2}{\lambda_1} )</th>
<th>Experimental</th>
<th>Sukai-Sugimoto Model</th>
<th>large N QCD Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\lambda_2}{\lambda_1} )</td>
<td>2.51</td>
<td>2.4</td>
<td>1.7</td>
</tr>
</tbody>
</table>

The difference with the experimental value could be due to the lack of higher order corrections. Ideally we need to know all the back-reactions on metric, but the calculations only includes leading order or first order in \( g_s m \). The details on the solution etc. will be discussed in our following up paper "five non-easy pieces part I ".

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CHAPTER 12
Summary

In this thesis, we discuss supersymmetric gauge theories and their gravity dual theories. In particular, the gauge/gravity duality transfers strong coupling problems into weak coupling systems where perturbative methods are still valid. We also mention other dualities like Seiberg duality, T-duality and AdS/CFT correspondence. Our string model can reproduce the RG flow, meson spectrum and many other features of real-world thermal QCD although the gauge theory is with large $N$ expansion.

On the field theory side, the large $N$ expansion of gauge groups suppresses non-planar diagrams in the scattering event, therefore simplifies the theory. At high energy, we have approximately a conformal field theory at the boundary. The heat capacity for this space is positive, therefore the model is stable at high energy. When we discuss the high temperature effect, supersymmetry is automatically broken. The theory then undergoes phase-transition that can be explicitly shown in the IR.
References


