THE CLASSIFICATION OF Ruled SURFACES
AND RANK 2 VECTOR BUNDLES OVER A
CURVE OF GENUS 0 OR 1

by

Joël Malard

A thesis submitted to the Faculty of Graduate Studies and
Research at McGill University, in partial fulfilment of the
requirements for the degree of Master of Science.

Department of Mathematics
McGill University
Montreal
LA CLASSIFICATION DES SURFACES RÉGLEES ET DES FIBRÉS VECTORIELS DE RANG 2 SUR UNE COURBE DE GENRE 0 OU 1

RÉSUMÉ

Le premier chapitre contient la définition des fibrés vectoriels sur une variété algébrique \( X \), ainsi que l'énoncé de certaines propriétés de ces fibrés. Les fibrés projectifs sont définis au deuxième chapitre, où l'on montre comment relier les classifications de ces deux types de fibrés lorsque \( X \) est une courbe lisse. Les fibrés vectoriels et les fibrés projectifs sont soit décomposables, soit indécomposables, et leur classification se fait selon cette distinction. Dans le troisième chapitre, la courbe \( X \) étant lisse et complète, on énonce le théorème de dualité de Serre et le théorème de Riemann-Roch pour les fibrés vectoriels sur \( X \). Le troisième chapitre se termine par la classification des fibrés vectoriels de rang 2 et des fibrés projectifs de fibre \( \mathbb{P}^1 \), décomposables sur \( X \), de genre 0 ou 1. Les fibres indécomposables sont traitées à l'aide des représentants normalisés de fibrés projectifs de fibre \( \mathbb{P}^1 \), indécomposables.

Département de Mathématiques

Université McGill

Montréal
THE CLASSIFICATION OF RULED SURFACES
AND RANK 2 VECTOR BUNDLES OVER A
CURVE OF GENUS 0 OR 1

ABSTRACT

In the first chapter, we define vector bundles over an algebraic variety \( X \) and state some of their properties. \( \mathbb{P}^n \)-bundles over \( X \) are defined in the second chapter. We also show there how to relate the classifications of these two types of fibre bundles when \( X \) is a non-singular curve. Vector bundles over \( X \) and \( \mathbb{P}^n \)-bundles over \( X \) are either decomposable or indecomposable, and their classification is made accordingly. In the third chapter, we introduce Serre's duality theorem and the Riemann-Roch theorem for vector bundles over \( X \), the curve \( X \) being complete and nonsingular. The classification of rank 2 vector bundles and \( \mathbb{P}^1 \)-bundles over \( X \) of genus zero or one is given in the decomposable case at the end of this chapter. The indecomposable case is treated in the last chapter by using normalized representatives of indecomposable \( \mathbb{P}^1 \)-bundles over \( X \).

Department of Mathematics
McGill University
Montreal
ACKNOWLEDGEMENTS

I wish to thank my thesis director Professor P.K. Russell, who suggested to me the title of this thesis, for his explanations and advice and for the financial support he gave me through his research fund. I also thank Professor W.J. Whitely for the financial help he gave me through his research fund.

I thank Mrs. Esther Massa for typing the manuscript with so much care.

A special thought to Jeanne who never discouraged of seeing this thesis done.

Finally my thanks go to my parents with gratitude.
A ma chère Jeanne
TABLE OF CONTENTS

INTRODUCTION 1

CHAPTER 1. VECTOR BUNDLES. 4

1.1 Definition of Vector Bundles over X 4

1.2 Morphisms of Vector Bundles 8

1.3 The Sheaf of Sections of a Vector Bundle 11

1.4 Operations on Vector Bundles 15

1.4.1 Sub-vector-bundles and quotient vector bundles 15

1.4.2 The direct sum 18

1.4.3 The tensor product 20

1.4.4 Hom(E,F) and the dual E* of E 21

1.4.5 The exterior power \( \Lambda^k(E) \) and \( \det(E) \) 24

1.4.6 The canonical isomorphism: \( \mathcal{E} \otimes F \cong \text{Hom}(E,F) \) 26

1.4.7 Another canonical isomorphism 27

CHAPTER 2. \( \mathbb{P}^1 \)-BUNDLES AND RULED SURFACES 29

2.1 \( \mathbb{P}^n \)-bundles over an Algebraic Variety X 29

2.2 Sections of a \( \mathbb{P}^n \)-bundle over a Nonsingular Curve 32

2.3 The Projectivization of a Vector Bundle 34

2.4 Sub-Line-Bundles and Sections of \( \mathbb{P}^n \)-bundles 37

2.5 About Ruled Surfaces 41

CHAPTER 3. DECOMPOSABLE BUNDLES 46

3.1 A First Proposition 46

3.2 Some Definitions about Divisors 48

3.3 The Isomorphism: \( \text{Cl}(X) \cong \Delta(X) \) 49

3.4 The Canonical Divisor Class on X 52
3.5 Serre's Duality and the Riemann-Roch Theorem for Complete Nonsingular Curves

3.6 The Classification of Decomposable Bundles

CHAPTER 4. INDECOMPOSABLE BUNDLES

4.1 Extensions of Line Bundles

4.2 Normalized Representatives

4.3 Classification of Indecomposable Bundles

SUMMARY

BIBLIOGRAPHY
In the first part of this thesis we define vector bundles and $\mathbb{F}^n$-bundles over an algebraic variety $X$ and give some of the properties of these fibre bundles when $X$ is a nonsingular curve. The ground field $k$ is a fixed algebraically closed field of arbitrary characteristic. When the curve $X$ is complete and nonsingular, $\mathbb{P}^1$-bundles over $X$ are precisely the (geometrically) ruled surfaces over $X$, which play a central role in the classification of algebraic surfaces (cf. [Chamberlain 1]).

The purpose of this work is to present the classification of rank two vector bundles and of $\mathbb{P}^1$-bundles over a complete nonsingular curve of genus zero or one. Our treatment is based on a well known article by M.F. Atiyah, [2], published in 1957. The restriction to $\mathbb{P}^1$-bundles allows some simplifications of the exposition.

A first article on the subject by M.F. Atiyah [1] gave the classification of ruled surfaces over an algebraic curve of genus at most two over the field of complex numbers. In order to obtain his classification, M.F. Atiyah used holomorphic $\mathbb{P}^1$-bundles and the proof is hard. What was needed at that time was a suitable cohomology for algebraic varieties. This cohomology was first constructed by J.P. Serre [1], and an alternative construction was then given by A. Grothendieck [2]. In 1956, A. Grothendieck published an article [1] in which it was shown that every holomorphic vector bundle over the Riemann Sphere is decomposable into a direct sum of line bundles. The proof given in this article is valid for any algebraically closed ground field. At about the same time, J.P. Serre [2] showed that the classifications of algebraic vector bundles and of analytic vector bundles
over a projective algebraic variety are the same. Using these developments, M.F. Atiyah [1] obtained a classification of vector bundles and of $F^m$-bundles over an elliptic curve. The classification of vector bundles over a curve of arbitrary genus was given in 1965 by A.N. Tjurin.

Our aims are not so ambitious. In fact, all the proofs written here are elementary. In the first chapter we establish the correspondence between vector bundles over $X$ and locally free sheaves on $X$ of the same rank. This will give us a way to define various operations on vector bundles that we will need later on. This chapter ends by the construction of two canonical isomorphisms from which we will deduce that isomorphism classes of line bundles over a complete non-singular curve $X$ form a group, denoted $\Delta(X)$. This group is simply the Picard group of $X$. In the second chapter we will show that $F^n$-bundles over the curve $X$ correspond to equivalence classes of rank $n+1$ vector bundles. This can be done by considering the projectivizations of rank $n+1$ vector bundles over $X$. Every local section of a $F^n$-bundle over the non-singular curve $X$ is the restriction of a unique global section, so there is a one-to-one correspondence between the sub-line-bundles of any given vector bundle over $X$ and the sections of its projectivization. One can easily deduce from this that every vector bundle over the curve $X$ has a sub-line-bundle. This result is only valid for curves.

After introducing the notions of decomposable bundles and indecomposable bundles, we will state some well-known facts from the theory of algebraic surfaces, to show that $F^1$-bundles and ruled surfaces over complete non-singular curves are the same.

After establishing a preliminary result for decomposable vector bundles over $X$, we will show that the group $\Delta(X)$ constructed in the first chapter
is the divisor class group of $X$. We will state Serre's duality theorem and the Riemann-Roch theorem for vector bundles over $X$. At the end of this third chapter we will give the classification of decomposable rank two vector bundles and $\mathfrak{p}$-bundles over the complete nonsingular curve $X$ of genus zero or one. In the last chapter we will show that for any two line bundles $L$ and $M$ over $X$, the extensions of $M$ by $L$ form, modulo a certain equivalence relation, a group isomorphic to $H^1(X,M \otimes X L^{-1})$, where $M$ and $L$ are the sheaves of sections of $M$ and $L$. We will complete the classification of rank two vector bundles and of $\mathfrak{p}$-bundles over $X$ of genus zero or one, by using normalized representatives of indecomposable $\mathfrak{p}$-bundles over $X$. Before we proceed, note that in the following pages algebraic varieties are always reduced and irreducible.
Chapter 1

VECTOR BUNDLES

In this chapter, $X$ will denote an arbitrary algebraic variety defined over an algebraically closed field $k$. We start by defining vector bundles of finite rank over $X$, and morphisms between them. The next step is to show how these geometric objects are related to locally free sheaves of finite rank over $X$.

A vector bundle over $X$ may be viewed as a family of $k$-vector spaces parametrized by the variety $X$. And the basic operations on vector spaces, such as the direct sum and the tensor product, extend naturally to vector bundles over $X$. We conclude this chapter by showing that isomorphism classes of rank 1 vector bundles over $X$ form a group with respect to the tensor product.

1.1 DEFINITION OF VECTOR BUNDLES OVER $X$

Definition 1.1: Let $r$ be a non-negative integer. A rank $r$ vector bundle over $X$ is an object $E \xrightarrow{\pi} X$, where $E$ is an algebraic variety called the bundle space, and $\pi$ is an epimorphism called the projection of the bundle, for which we can find:

1. an open covering $U = \{U_i\}_{i \in I}$ of $X$, indexed by a set $I$,
2. for each $i \in I$, an isomorphism $\phi_i : E_i \cong \pi^{-1}(U_i) \conarrow U_i \times k^r$

satisfying the following two properties:

(a) for each $i \in I$, the diagram
is commutative (here \( p_1 \) is the projection onto the first factor).

(b) For every pair \( i,j \in I \), the automorphism:

\[
\phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times k^r \rightarrow (U_i \cap U_j) \times k^r
\]

is linear on the fibres, i.e., we have:

\[
\forall p \in U_i \cap U_j, \forall u \in k^r; \quad \phi_i \circ \phi_j^{-1}(p, u) = (p, g_{ij}(p) \cdot u),
\]

where \( g_{ij} \) is an invertible \( r \times r \) matrix with entries in \( \Gamma(U_i \cap U_j, \mathcal{O}_X) \).

Accordingly, the morphisms \( \phi_i \circ \phi_j^{-1} \) will also be written as \( (\text{Id}, g_{ij}) \).

The collection of isomorphisms \( \{\phi_i\}_{i \in I} \) is called a trivialization of the vector bundle \( E \xrightarrow{\pi} X \) over the covering \( \mathcal{U} \) of \( X \). Note that the existence of such a trivialization implies that the bundle space \( E \) is an algebraic variety which is non-singular when \( X \) is non-singular.

Let \( p \) be a point in some \( U_i \in \mathcal{U} \). The morphism \( \phi_i \) restricts to an isomorphism between the fibre \( \pi^{-1}(p) \) of \( E \) over \( X \) and \( k^r \). This isomorphism may be viewed as giving coordinates on \( \pi^{-1}(p) \). Now, if \( p \) lies also in some other \( U_j \), the morphism \( \phi_j \) gives another isomorphism between \( \pi^{-1}(p) \) and \( k^r \), while \( \phi_i \circ \phi_j^{-1} \) gives us the change of coordinates. Condition (b) says that this change of coordinates respects the vector space structure of \( k^r \), and is given by a matrix \( g_{ij}(p) \).
The matrices \( g_{ij} \) have the following two properties, which will be referred to as (TM):

\[
\begin{align*}
(1) & \quad \forall i,j,k \in I, \quad \forall p \in U_i \cap U_j \cap U_k; \\
& \quad g_{ij}(p) \cdot g_{jk}(p) \cdot g_{ki}(p) = \text{Id}_r; \\
(2) & \quad \forall i \in I, \quad \forall p \in U_i; \quad g_{ii}(p) = \text{Id}_r,
\end{align*}
\]

where \( \text{Id}_r \) is the identity \( r \times r \) matrix.

The collection \( \{g_{ij}\}_{i,j \in I} \) is called a system of transition matrices for the vector bundle \( \pi: E \to X \) over the covering \( U \) of \( X \).

Note that if \( V \) is an open covering of \( X \), finer than the given covering \( U \), then any trivialization \( \{\phi_i\}_{i \in I} \) of \( E \to X \) over \( U \) induces a trivialization of \( E \to X \) over \( V \). Since \( X \) is an algebraic variety, it is quasicompact. Taking \( V \) to be a finite subcovering of \( U \), we see that the vector bundle \( E \to X \) trivializes over a finite covering of \( X \). In the particular case where this finite covering can be chosen to be \( \{X\} \), we say that \( E \to X \) is a trivial rank \( r \) vector bundle over \( X \).

Now, suppose we are given a collection \( \{h_{ij}\}_{i,j \in I} \) of invertible \( r \times r \) matrices \( h_{ij} \) with entries in \( \Gamma(U_i \cap U_j, \mathcal{O}_X) \), satisfying condition (TM) over the open covering \( U = \{U_i\}_{i \in I} \) of \( X \). We construct a rank \( r \) vector bundle \( F \to X \) having \( \{h_{ij}\}_{i,j \in I} \) for system of transition matrices over \( U \) as follows:

Let \( \Sigma \) be the disjoint union of all open sets \( U_i \times k^r \), and take \( F \) to be the quotient space of \( \Sigma \) by the equivalence relation \( \sim \) defined by:

\[
\begin{align*}
\forall i,j \in I, \quad \forall (p,\hat{u}) \in U_i \times k^r, \quad \forall (q,\hat{v}) \in U_j \times k^r; \\
(p,\hat{u}) \sim (q,\hat{v}) \iff (p = q, \text{ and } \hat{u} = h_{ij}(p)\hat{v}).
\end{align*}
\]
There is no loss of generality in assuming that the covering $\mathcal{U}$ of $X$ is by affine open sets. For every $i \in I$, let $F_i$ be the subset of $F$ corresponding to $U_i \times k^r$ under the canonical projection of $\Sigma$ onto $F$. We have for each $i \in I$, an isomorphism

$$\psi_i : F_i \rightarrow U_i \times k^r$$

which makes $F_i$ into an affine open set. Then there is a unique structure of prevariety on $F$, which makes all the $F_i$ into affine open subsets of $F$. It is then well known that $F$ is a variety if and only if for every pair $i, j \in I$, we have,

1. the intersection $F_i \cap F_j$ is an affine open subset of $F_i$ and $F_j$,

2. the homomorphism

$$\Delta_{ij} : \Gamma(F_i \cap F_j, O_{F_i}) \rightarrow \Gamma(F_i \cap F_j, O_{F_j})$$

defined by $f \Theta g \mapsto f \cdot g$, is onto.

For every pair $i, j \in I$, we have an isomorphism

$$\psi_{ij} : F_i \cap F_j \rightarrow (U_i \cap U_j) \times k^r.$$

Since $X$ is a variety, all the intersections $U_i \cap U_j$ are affine, and the first condition is satisfied. On the other hand, the isomorphisms $\psi_i$ and $\psi_{ij}$ induce isomorphisms of rings

$$\psi_i^* : \Gamma(F_i, O_{F_i}) \rightarrow \Gamma(U_i, O_{U_i})[X_1, \ldots, X_r],$$

$$\psi_{ij}^* : \Gamma(F_i \cap F_j, O_{F_i \cap F_j}) \rightarrow \Gamma(U_i \cap U_j, O_{U_i \cap U_j})[X_1, \ldots, X_r],$$

where $X_1, \ldots, X_r$ are coordinates on $k^r$. But then the homomorphism $\Delta_{ij}$ is onto if and only if the corresponding homomorphism

$$\Delta_{ij}^* : \Gamma(F_i \cap F_j, O_{F_i \cap F_j}) \rightarrow \Gamma(U_i \cap U_j, O_{U_i \cap U_j})[X_1, \ldots, X_r]$$

is onto.
\[ \Gamma(U_i \cap U_j, \mathcal{O}_{U_i}) \otimes_k (U_i \cap U_j, \mathcal{O}_{U_j}) \rightarrow \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}) \]

induced by \( \psi_i^*, \psi_j^* \), and \( \psi_{ij}^* \) is onto. Again, this follows from \( X \) being a variety, and we see that \( F \) is a variety.

The projection \( \sigma : F \rightarrow X \) is simply the one induced on \( F \) by all the projections onto the first factor \( U_i \times k^r \rightarrow U_i \).

### 1.2 MORPHISMS OF VECTOR BUNDLES

**Definition 1.2:** A morphism of vector bundles over \( X \), say from \( E \xrightarrow{\pi} X \) to \( E' \xrightarrow{\pi'} X \), is a morphism of algebraic varieties \( f : E \rightarrow E' \) satisfying the following two conditions:

(a') \( f \) sends fibres to fibres, i.e., the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\pi \downarrow & & \pi' \downarrow \\
X & & X
\end{array}
\]

commutes,

(b') \( f \) is linear on the fibres. More precisely, let \( U = \{U_i\}_{i \in I} \) be any open covering of \( X \), over which both vector bundles trivialize, say with, for each \( i \in I \), the trivializations

\[
\phi_i : E_i = \pi^{-1}(U_i) \rightarrow U_i \times k^r \quad \text{and} \quad \phi'_i : E'_i = \pi'^{-1}(U_i) \rightarrow U_i \times k^{r'}.
\]

Then condition (b') is that for each \( i \in I \), the morphism

\[
\phi'_i \circ f \circ \phi^{-1}_i : U_i \times k^r \rightarrow U_i \times k^{r'}
\]

is of the form:
$\forall p \in U_i, \quad u \in k^r; \quad \phi'_1 \circ f \circ \phi'^{-1}_1(p, u) = (p, f_1(p) \cdot u)$,

where $f_1$ is an $r' \times r$ matrix with coefficients in $\Gamma(U_i, O_X)$.

Clearly, there is no loss of generality in specifying in the definition of morphisms of vector bundles over $X$, that the trivializations $\{\phi_i\}_{i \in I}$ and $\{\phi'_i\}_{i \in I}$ of the two vector bundles are given over the same open covering of $X$. It is also clear that the notion of morphism between two vector bundles over $X$ is independent of the choice of trivializations.

Now let $E \xrightarrow{\pi} X$, and $E' \xrightarrow{\pi'} X$ be rank $r$, and $r'$ vector bundles, with respective trivializations $\{\phi_i\}_{i \in I}$ and $\{\phi'_i\}_{i \in I}$ over a common open covering $U = \{U_i\}_{i \in I}$ of $X$, and with corresponding systems of transition matrices $\{g_{ij}\}_{i,j \in I}$ and $\{g'_{ij}\}_{i,j \in I}$ over $U$. Let also $f$ be a morphism from $E \xrightarrow{\pi} X$ to $E' \xrightarrow{\pi'} X$. The definition of the transition matrices $\{g_{ij}\}_{i,j \in I}$ and $\{g'_{ij}\}_{i,j \in I}$ corresponds to the commutative diagrams:

\[\begin{array}{ccc}
\text{Res } \phi_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^r \\
E_j \supseteq E_i \cap E_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^r \\
\downarrow & & \\
E_i \supseteq E_i \cap E_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^r \\
\text{Res } \phi'_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^{r'} \\
E'_j \supseteq E'_i \cap E'_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^{r'} \\
\downarrow & & \\
E'_i \supseteq E'_i \cap E'_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^{r'} \\
\text{Res } \phi'_j & \xrightarrow{\sim} & (U_i \cap U_j) \times k^{r'}
\end{array}\]
And since the mapping \( f \) sends each intersection \( E_i \cap E_j \) to the corresponding intersection \( E'_i \cap E'_j \), these diagrams give a commutative diagram:

\[
\begin{array}{ccc}
(U_i \cap U_j) \times k^r & \xrightarrow{(Id, f_j)} & (U_i \cap U_j) \times k^{r'} \\
(Id, g_{ij}) \downarrow & & \downarrow (Id, g_{ij}') \\
(U_i \cap U_j) \times k^r & \xrightarrow{(Id, f_i)} & (U_i \cap U_j) \times k^{r'}
\end{array}
\]

where we write \((Id, f_i)\) for \( \phi_i^{-1} \cdot f \cdot \phi_i^{-1} \).

So the matrices \( f_i \) induced by the morphism \( f \) satisfy:

\[
\text{(MV)} \quad \forall i, j \in I ; \quad g_{ij}^{-1} \cdot f_j = f_i \cdot g_{ij} \quad \text{over} \quad U_i \cap U_j.
\]

Conversely any collection \( \{ f_i \}_{i \in I} \) of \( r' \times r \) matrices \( f_i \) with entries in \( \Gamma(U_i, \mathcal{O}_X) \) satisfying these equalities \( \text{(MV)} \), defines a unique morphism \( f \) of vector bundles from \( E \xrightarrow{\pi} X \) to \( E' \xrightarrow{\pi'} X \) by setting:

\[
\forall i \in I ; \quad \text{Res}_{E_i}(f) = \phi_i^{-1} \circ (Id, f_i) \circ \phi_i.
\]

We say that the matrices \( f_i \) represent the morphism \( f \) over the covering \( U \) (with respect to the trivializations \( \{ \phi_i \}_{i \in I} \) and \( \{ \phi_i' \}_{i \in I} \)).

The zero morphism \( 0 \) from \( E \xrightarrow{\pi} X \) to \( E' \xrightarrow{\pi'} X \) is the morphism represented by zero matrices.

The condition \( \text{(MV)} \) gives us a local criterion for two vector bundles over \( X \) to be isomorphic. Namely:

Two vector bundles over \( X \), \( E \xrightarrow{\pi} X \), and \( E' \xrightarrow{\pi'} X \) are isomorphic if and only if they have the same rank \( r \), and given respective systems of
transition matrices \( \{g_{ij}\}_{i,j \in I} \) and \( \{g'_{ij}\}_{i,j \in I} \) over a common open covering \( U = \{U_i\}_{i \in I} \) of \( X \), we can find a collection \( \{f_i\}_{i \in I} \) of invertible \( r \times r \) matrices \( f_i \) with entries in \( \Gamma(U_i, O_X) \) satisfying the condition:

\[
V_{i,j} \in I, \quad g_{ij} = f_i \cdot g'_{ij} \cdot f_j^{-1} \quad \text{over } U_i \cap U_j.
\]

In particular, a vector bundle over \( X \) is determined up to isomorphism by any one of its systems of transition matrices.

1.3 THE SHEAF OF SECTIONS OF A VECTOR BUNDLE

Definition 1.3: Let \( E \to X \) be a rank \( r \) vector bundle over \( X \), and let \( U \) be an open subset of \( X \).

A local section of \( E \to X \) over \( U \) is a morphism \( s : U \to E \) such that \( \pi \circ s = \text{Id}_U \).

A global section of \( E \to X \) is a local section of \( E \to X \) whose domain \( U \) is the whole of \( X \).

When no confusion is likely to arise, global sections are simply called sections.

Now, let \( E \to X \) be a rank \( r \) vector bundle with trivialization \( \{\phi_i\}_{i \in I} \) over some open covering \( U = \{U_i\}_{i \in I} \) of \( X \), and corresponding transition matrices \( \{g_{ij}\}_{i,j \in I} \). Given a local section \( s \) of \( E \to X \) over some open subset \( U \) of \( X \), for every index \( i \in I \), the morphism

\[
\phi_i \circ s : U \cap U_i \to (U \cap U_i) \times k^r
\]

is of the form

\[
\forall p \in U \cap U_i; \quad \phi_i \circ s(p) = (p, s_i(p)),
\]
where \( s_i \) is an \( r \times 1 \) matrix with entries in \( \Gamma(U \cap U_i, \mathcal{O}_X) \).

For every pair \( i, j \in I \), we have the equality:

\[
\phi_i \circ s = \phi_i \circ \phi_j^{-1} \circ \phi_j \circ s \quad \text{over } U \cap U_i \cap U_j.
\]

Equivalently, we have:

\[
(V): \quad \forall i, j \in I; \quad s_i = s_{ij} \cdot s_j \quad \text{over } U \cap U_i \cap U_j.
\]

Conversely, given a collection \( \{s_i\}_{i \in I} \) of \( r \times 1 \) matrices \( s_i \) with entries in \( \Gamma(U \cap U_i, \mathcal{O}_X) \) satisfying the equalities in \( (S) \), we construct a local section \( s \) of \( E \rightarrow X \) over \( U \) by setting for every \( i \in I \),

\[
\text{Res}_{U \cap U_i} (s) = \phi_i^{-1} \circ \text{Id} \circ s_i.
\]

The matrices \( s_i \) are called the representatives of the local section \( s \) over the covering \( \{U \cap U_i\}_{i \in I} \) of \( U \), with respect to the trivialization \( \{\phi_i\}_{i \in I} \) of \( E \rightarrow X \).

Note that every vector bundle over \( X \) has a global section represented by zero matrices with respect to any trivialization. This section is called the zero section of the vector bundle.

Now take \( E \rightarrow X \) to be a vector bundle over \( X \) with given trivialization over some open covering \( \{U_i\}_{i \in I} \) of \( X \); and let \( \{g_{ij}\}_{i, j \in I} \) be the corresponding system of transition matrices. Given any two local sections \( s \) and \( t \) of \( E \rightarrow X \) defined over the same open subset \( U \) of \( X \), and respectively represented over \( U \) by collections \( \{s_i\}_{i \in I} \) and \( \{t_i\}_{i \in I} \), we define their sum \( s + t \) to be the local section of \( E \rightarrow X \) over \( U \) which is represented by the matrices \( \{s_i + t_i\}_{i \in I} \). Similarly, if \( f \) is any regular function on \( U \), i.e. \( f \in \Gamma(U, \mathcal{O}_X) \), the collection \( \{f \cdot s_i\}_{i \in I} \) represents a unique section of \( E \rightarrow X \) over \( U \), and we call this section \( fs \).
These two operations of addition and scalar multiplication, give to the set of local sections of \( E \to X \) over \( U \), a structure of \( \Gamma(U, \mathcal{O}_X) \)-module. In this way, we obtain a presheaf of functions defined by local properties. So this presheaf is in fact a sheaf of \( \mathcal{O}_X \)-modules that we call the sheaf of sections of the vector bundle \( E \to X \), and denote as \( \mathcal{E} \). Recall that if \( F \) and \( G \) are coherent sheaves on \( X \), their direct sum \( F \oplus G \) is defined to be the presheaf \( U \mapsto \Gamma(U, F) \oplus \Gamma(U, G) \), and is itself a coherent sheaf on \( X \). Then if \( \mathcal{O}^r_X \) denotes the direct sum of \( r \) copies of \( \mathcal{O}_X \), we have for every \( i \in I \), and every open subset \( U \) of \( U_i \) an isomorphism of \( \Gamma(U, \mathcal{O}_X) \)-modules

\[
\Gamma(U, \mathcal{E}_i|_{U_i}) \cong \Gamma(U, \mathcal{O}^r_X) = \Gamma(U, \mathcal{O}^r_X),
\]

where \( \mathcal{E}_i|_{U_i} \) denotes the restriction of \( \mathcal{E}_i \) to \( U_i \). This is because over every open set \( U_i \), the vector bundle \( E \to X \) is trivial (i.e. it is isomorphic to \( U_i \times \mathbb{A}^r \)). The sheaf \( \mathcal{E} \) being locally isomorphic to the sheaf \( \mathcal{O}^r_X \), it is a locally free sheaf of rank \( r \) on \( X \).

Now, suppose \( E' \to X \) is another rank \( r \) vector bundle over \( X \), and let \( f : E \to E' \) be a morphism from \( E \to X \) to \( E' \to X \). Then if \( s \) is any local section of \( E \to X \) over some open subset of \( X \), \( f \circ s \) is a local section of \( E' \to X \) over the same open subset, and \( f \) induces a morphism of sheaves of \( \mathcal{O}_X \)-modules from \( \mathcal{E} \) to \( \mathcal{E}' \).

On the other hand, let \( F \) be a locally free sheaf of rank \( r \) on \( X \). There exist an open covering \( V = \{ V_j \} \) of \( X \) and for each \( j \in J \), an isomorphism \( \Theta_j \) of \( \mathcal{O}_X \)-modules:

\[
\Theta_j : F|_{V_j} \to \mathcal{O}^r|_{V_j}.
\]
Then for any pair $i, j \in J$, we have an automorphism $\rho_{ij}$ of the 
$\Gamma(V_i \cap V_j, \mathcal{O}_X^r)$-module $\Gamma(V_i \cap V_j, \mathcal{O}_X)$ induced by $\Theta_i(V_j)$ and $\Theta_j(V_i)^{-1}$, 
making the diagram:

\[
\begin{array}{ccc}
\Gamma(V_i \cap V_j, \mathcal{O}_X^r) & \xrightarrow{\rho_{ij}} & \Gamma(V_i \cap V_j, \mathcal{O}_X^r) \\
\downarrow & & \downarrow \\
\Gamma(V_i \cap V_j, \mathcal{O}_X) & \xrightarrow{\Theta_i^{-1}(V_j)} & \Gamma(V_i \cap V_j, \mathcal{O}_X)
\end{array}
\]

commutative. Since $\Gamma(V_i \cap V_j, \mathcal{O}_X^r)$ is a free $\Gamma(V_i \cap V_j, \mathcal{O}_X)$-module of rank $r$, $\rho_{ij}$ corresponds to an invertible $r \times r$ matrix $h_{ij}$ with entries in $\Gamma(V_i \cap V_j, \mathcal{O}_X)$. The collection $\{h_{ij}^{-1}, i, j \in J\}$ satisfies the condition (TM). So it is a system of transition matrices for some rank $r$ vector bundle over $X$ constructed using the procedure given in 1.1. Call this vector bundle $F$, the projection onto $X$ being understood. It will be referred to as the vector bundle associated to the locally free sheaf $F$.

Given another locally free sheaf $F'$ of rank $r$ on $X$, let $F'$ be its associated rank $r$ vector bundle over $X$. We may assume that $F'$ trivializes over the same open covering of $X$ as $F$, by a collection $\{0'_j\}_{j \in J}$. So let $\{h'_{ij}, i, j \in J\}$ be the system of transition matrices for $F'$ over $V$, induced by $\{0'_j\}_{j \in J}$. Then for any morphism of sheaves of $\mathcal{O}_X$-modules $\psi: F \rightarrow F'$, and for each $j \in J$, the homomorphism

\[
\Theta_j'^{-1} \circ \psi \circ \Theta_j^{-1} : \mathcal{O}_X^r|_{V_j} \rightarrow \mathcal{O}_X^r|_{V_j}
\]

corresponds to an $r \times r$ matrix $\psi_j$ with entries in $\Gamma(V_j, \mathcal{O}_X)$. The
collection \( \{ \psi_j \}_{j \in J} \) satisfies

\[(\text{MV}): \forall i, j \in J; \quad h_{ij}' \cdot \psi_j = \psi_i \cdot h_{ij} \quad \text{over } V_i \cap V_j.\]

So the matrices \( \psi_i \) represent a unique morphism of vector bundles from \( F \) to \( F' \).

These two processes \((E \xrightarrow{\pi} X) \xrightarrow{\pi} E \quad \text{and} \quad F \xrightarrow{\pi} F\) are functors, and clearly define an equivalence between the category of rank \( r \) vector bundles over \( X \) and the category of locally free sheaves of rank \( r \) on \( X \).

1.4 OPERATIONS ON VECTOR BUNDLES

1.4.1 Sub-Vector-Bundles and Quotient Vector Bundles

We will use this equivalence of categories to define the basic operations on vector bundles over \( X \). But in order to simplify the notation, let us agree to write \( E \) for a vector bundle \( E \xrightarrow{\pi} X \) over \( X \), the projection onto \( X \) being understood. Then, if \( p \) is any point on \( X \), the fibre \( \pi^{-1}(p) \) of \( E \) over \( p \) will be denoted as \( E_p \).

We now give the following definition, which will be of importance in the next chapters:

**Definition 1.4:** Let \( E', E, \) and \( E'' \) be vector bundles over \( X \), of respective ranks \( r', r, \) and \( r'' \). And let \( f : E' \rightarrow E \) and \( g : E \rightarrow E'' \) be morphisms of vector bundles over \( X \). We say that the sequence

\[
\begin{array}{c}
0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0 \\
\end{array}
\]

is exact, to mean that the corresponding sequence of locally free sheaves

\[
\begin{array}{c}
0 \rightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \rightarrow 0 \\
\end{array}
\]
is exact. When the sequence (*) is exact, it is also called an extension of $E''$ by $E'$.

Since kernels and cokernels of homomorphisms of vector spaces are defined independently of basis, it makes sense to say for a point $p$ on $X$ that the restricted sequence of fibres over $p$

$$
(\ast\ast): \quad 0 \rightarrow E'_p \rightarrow E|_p \rightarrow E''|_p \rightarrow 0
$$
is exact. For a fine enough covering $U = \{U_i\}_{i \in I}$ of $X$, the restricted sheaves $E'|_{U_i}$, $E|_{U_i}$, $E''|_{U_i}$ are free and the sequence of sheaves [*] is exact precisely when for each $i \in I$ the restricted sequence

$$
0 \rightarrow E'_i \rightarrow E|_{U_i} \rightarrow E''|_{U_i} \rightarrow 0
$$
is exact. Since morphisms of free sheaves can be viewed as matrices with regular functions as entries, this will happen precisely when for every point $p$ on $X$, the sequence $(\ast\ast)_p$ is exact.

Now suppose we are just given an injection of vector bundles $f : E' \rightarrow F$. Let it be represented over some open covering $U = \{U_i\}_{i \in I}$ of $X$, by a collection $\{f_i\}_{i \in I}$ of $r \times r'$ matrices. For each index $i \in I$, and for every point $p \in U_i$, the vector space $f_i(p)(k^{r'})$ is defined in $k^r$ by a system of $r - r'$ equations

$$(e): \quad a_{i;1}(p) z_1 + \ldots + a_{i;r'}(p) z_r = 0,$$

where $z_1, \ldots, z_r$ are coordinates on $k^r$, and the $a_{i;\ell}$ are regular functions on $U_i$. For each $i \in I$ let $g_i$ be the matrix $(a_{i;\ell})$. Then
the collection \( \{ f_i \}_{i \in I} \) determines a surjective morphism \( \pi \) from \( E \) onto a rank \( (r - r' + 1) \) vector bundle \( E'' \) over \( X \) such that the sequence of vector bundles

\[
0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0
\]

is exact. In particular \( F'' \equiv F/F' \), and we have

**Proposition 1.** Any collection of vector bundles over \( X \), \( \{ E_i \} \rightarrow F \), induces a corresponding sequence of vector bundles over \( X \),

\[
0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0
\]

But in order to prove this, we need to introduce the concept of vector bundles over a scheme.

One point should be noted here. Namely, an injection of sheaves of \( O_X \)-modules \( \phi : F' \rightarrow F \) corresponds to an injection of vector bundles over \( X \), \( \{ E_i \} \rightarrow F \). It follows immediately when the quotient sheaf \( F/F' \) is not locally free on \( X \). An example: take \( X \) to be the projective line \( \mathbb{P}^1 \), and choose any point \( \eta \) on \( \mathbb{P}^1 \). The ideal sheaf \( I_{\eta} \) of \( \eta \) on \( \mathbb{P}^1 \) is defined \( \mathcal{F}(U, q) = \text{Hom}(\mathcal{O}_X, U) \), \( \mathcal{F}(\{q\}) = 0 \) if \( q \in U \),

and is locally free of rank one on \( X \), as will be shown in ": 0.3. We have an inclusion of sheaves of \( \mathcal{O}_X \)-modules \( \mathcal{I}_\eta \rightarrow \mathcal{O}_X \), and the quotient sheaf \( F = \mathcal{O}_X/\mathcal{I}_\eta \) has the property that for any open subset \( U \subset X \), \( \mathcal{F}(U, I) \) is zero if \( \eta \notin U \), and \( I(1) \) if \( \eta \in U \). The sheaf \( I \) is called a skyscraper sheaf of \( k \) units at the point \( \eta \) (cf. [P. Katz, Cor. p. 226]).

In particular \( F \) is not locally free, and the injection \( \{ E_i \} \rightarrow F \) does not correspond to an injection of the corresponding vector bundles over \( X \).

We may now define sub-vector-bundles of \( F \) vector bundles over \( X ").
space of $E$ in such a way that the inclusion is also a morphism $i$ of vector bundles over $X$. If

$$0 \rightarrow E' \xrightarrow{i} E \rightarrow E'' \rightarrow 0$$

is the exact sequence given in Proposition 1.1, the vector bundle $E''$ is called the quotient vector bundle of $E$ by $E'$.

Every injection of vector bundles over $X$, $E' \rightarrow E$, defines a sub-vector-bundle of $E$. Two such injections $E' \rightarrow E$, and $E'' \rightarrow E$ are said to define isomorphic sub-vector-bundles of $E$ if and only if there exists an isomorphism $\phi : E' \rightarrow E''$ of vector bundles over $X$ making the diagram:

$$\begin{array}{ccc}
E' & \xrightarrow{\phi} & E'' \\
\downarrow & & \downarrow \\
E & \xleftarrow{} & 
\end{array}$$

commutative.

We may now define operations on vector bundles. So take two arbitrary vector bundles $E$ and $F$ over $X$, of respective ranks $r$ and $s$, and fix $\{g_{ij}\}_{i,j \in I}$ and $\{h_{ij}\}_{i,j \in I}$ to be their respective systems of transition matrices over an open covering $U = \{U_i\}_{i \in I}$ of $X$ over which both vector bundles trivialize.

1.4.2 The Direct Sum

The sheaf of $\mathcal{O}_X$-modules $E \oplus F$ defined to be the presheaf

$$U \mapsto \Gamma(U, E) \oplus \Gamma(U, F),$$

is locally free of rank $(r + s)$ over $X$. So we define:
Definition 1.6: The direct sum of $E$ and $F$, denoted $E \oplus F$, is the rank $(r + s)$ vector bundle over $X$ associated to the locally free sheaf $E \otimes F$.

The collection $\{g_{ij} \otimes h_{ij}\}_{i,j \in I}$ forms a system of transition matrices for the direct sum $E \oplus F$ over the covering $U$, where $g_{ij} \otimes h_{ij}$ stands for the invertible $(r + s) \times (r + s)$ matrix

\[
\begin{pmatrix}
g_{ij} & 0 \\
\end{pmatrix}
\begin{pmatrix}
h_{ij}
\end{pmatrix}
\]

Thus, there is an exact sequence of vector bundles over $X$

\[
(*)': \quad 0 \longrightarrow E \overset{i}{\longrightarrow} E \oplus F \overset{p}{\longrightarrow} F \overset{j}{\longrightarrow} 0,
\]

and an injection of vector bundles over $X$, $j: F \longrightarrow E \oplus F$ such that $p \circ j$ is the identify on $F$. So the exact sequence $(*)'$ is split.

As in the case of vector spaces, the converse is also true. Namely, if an exact sequence

\[
0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0
\]

of vector bundles over $X$ splits, then $G$ is isomorphic to $E \oplus F$. In such a case, the corresponding injections $E \longrightarrow G$ and $F \longrightarrow G$ define sub-vector bundles of $G$, of complementary ranks, and whose intersection is the image of the zero section of $G$. By considering small enough open subsets of $X$ over which all three vector bundles $E$, $F$, and $G$ are trivial, it is not hard to see that the converse is also true, since over such open
sets the matter reduces to a question of vector spaces. So the following result holds:

**Proposition 1.2:** A rank $t$ vector bundle $G$ over $X$ is the direct sum of two of its sub-vector-bundles $E$ and $F$ of respective ranks $r$ and $s$, if and only if $r + s = t$, and the intersection of $E$ and $F$ in $G$ is the image of the zero section of $G$.

1.4.3 The Tensor Product

The tensor product of two sheaves of $\mathcal{O}_X$-modules $E$ and $F$ is defined to be the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, E) \otimes \Gamma(U, \mathcal{O}_X) \Gamma(U, F),$$

and is denoted $E \otimes_{\mathcal{O}_X} F$.

Returning to our given vector bundles $E$ and $F$, let $U$ be any open subset of $X$ over which both $E$ and $F$ are trivial. The restricted sheaves $E|_U$ and $F|_U$ are free sheaves of $\mathcal{O}_X|_U$-modules, and the homomorphisms of $\Gamma(U, \mathcal{O}_X)$-modules

$$\Gamma(U, E) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, F) \cong \Gamma(U, \mathcal{O}_X^r) \otimes_{\Gamma(U, \mathcal{O}_X)} \Gamma(U, \mathcal{O}_X^s)$$

$$\cong \Gamma(U, \mathcal{O}_X^{rs})$$

induce an isomorphism $(E \otimes_{\mathcal{O}_X} F)|_U \cong \mathcal{O}_X^{rs}|_U$.

Taking $U$ to be any open subset $U_i$ in our given covering $U$ of $X$, we see by a previous argument that the sheaf $E \otimes_{\mathcal{O}_X} F$ is locally free of rank $rs$ on $X$. We set:
Definition 1.7: The tensor product \( E \otimes F \) of the vector bundles \( E \) and \( F \) over \( X \), is the vector bundle over \( X \) associated to the locally free sheaf \( E \otimes \mathcal{O}_X^r F \) of rank \( rs \) on \( X \).

Using an isomorphism \( k^r \otimes_k k^s \cong k^{rs} \), we can write the transition matrices of \( E \otimes F \) over \( U \) as \( g_{ij} \otimes h_{ij} \), where for each point \( p \in U \cap U_j \) and for every pair \( u, v \in k^r, v' \in k^s \),

\[
(g_{ij} \otimes h_{ij})(p)(u \otimes v) = (g_{ij}(p)u) \otimes (h_{ij}(p)v).
\]

1.4.4 \( \text{Hom}(E,F) \) and the Dual \( E^* \) of \( E \)

Let \( E \) and \( F \) be any sheaves of \( \mathcal{O}_X \)-modules. The set of morphisms from \( E \) to \( F \) has a natural structure of \( \Gamma(X,\mathcal{O}_X) \)-module. We call this module \( \text{Hom}_{\mathcal{O}_X}(E,F) \). Then the presheaf

\[
\text{Hom}_{\mathcal{O}_X}(E,F) \|_{U \rightarrow \text{Hom}_{\mathcal{O}_X}(E \|_U, F \|_U)} \]

is a sheaf of \( \mathcal{O}_X \)-modules that we denote \( \text{Hom}_{\mathcal{O}_X}(E,F) \). It is the sheaf of homomorphisms from \( E \) to \( F \), and is locally free when \( E \) and \( F \) are locally free. This is so because when \( E \) and \( F \) are free sheaves of respective ranks \( r \) and \( s \), a morphism from \( E \) to \( F \) may be viewed as an \( s \times r \) matrix with entries in \( \Gamma(X,\mathcal{O}_X) \). We set:

Definition 1.8: The vector bundle of morphisms from \( E \) to \( F \), denoted \( \text{Hom}(E,F) \), is the vector bundle over \( X \) associated to the locally free sheaf \( \text{Hom}_{\mathcal{O}_X}(E,F) \).

The dual vector bundle \( E^* \) of \( E \) is \( \text{Hom}(E, \mathcal{O}_X) \), where \( I \) is the trivial line bundle over \( X \) defined by the condition \( I \cong \mathcal{O}_X \).

Note that the sections of \( \text{Hom}(E,F) \) over \( X \) may be viewed as morphisms of vector bundles over \( X \) from \( E \) to \( F \).
In the category of sheaves of $\mathcal{O}_X$-modules the functors $\text{Hom}_{\mathcal{O}_X}(E, -)$ and $\text{Hom}_{\mathcal{O}_X}(-, E)$ are only left exact in general. However, if $H$ is a locally free sheaf on $X$, then the functors $\text{Hom}_{\mathcal{O}_X}(H, -)$ and $\text{Hom}_{\mathcal{O}_X}(-, H)$ are exact in the category of locally free sheaves on $X$. To see this, let

\[
\begin{array}{cccc}
0 & \rightarrow & E & \xrightarrow{f} \mathcal{G} & \xrightarrow{g} \mathcal{F} & \rightarrow & 0
\end{array}
\]

be an exact sequence of locally free sheaves on $X$. In order to show that the functor $\text{Hom}_{\mathcal{O}_X}(\mathcal{H}, -)$ is exact it is sufficient to show that the morphism

\[
f^{\ast} = \text{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{H}) \rightarrow \text{Hom}_{\mathcal{O}_X}(E, \mathcal{H})
\]

is surjective. Covering the space $X$ by open subsets $U$ over which the restricted sheaves $E|_U$, $\mathcal{G}|_U$, and $\mathcal{H}|_U$ are free, we reduce to the case where $E$, $F$, and $\mathcal{G}$ are free on $X$. The exact sequence \([\ast]\) is then split. Choosing an isomorphism $\mathcal{G} \cong \mathcal{E} \oplus \mathcal{F}$, we can lift any morphism $\phi : E \rightarrow \mathcal{H}$ to a morphism $\psi : \mathcal{G} \rightarrow \mathcal{H}$ with $f^{\ast}(\psi) = \psi \circ f = \phi$. So we have:

Lemma 1.1: In the category of locally free sheaves on $X$, the functor $\text{Hom}_{\mathcal{O}_X}(\mathcal{H}, -)$ is exact.

By a similar argument the functor $\text{Hom}_{\mathcal{O}_X}(\mathcal{H}, -)$ is also exact on locally free sheaves on $X$.

One should remark that the functors $\text{Hom}_{\mathcal{O}_X}(\mathcal{H}, -)$ and $\text{Hom}_{\mathcal{O}_X}(-, \mathcal{H})$ from the category of locally free sheaves on $X$ to the category of abelian groups are not exact in general, but only left exact.

We now compute the transition matrices for the vector bundle $E^{\ast}$ over the covering $U$ of $X$. So let $(\phi_i)_{i \in I}$ be a collection of isomorphisms $\phi_i : \mathcal{E}|_{U_i} \rightarrow \mathcal{O}_X^{\ast}|_{U_i}$, which induces the given transition matrices.
Given any $i \in I$, an element of $\Gamma(U_i, \underline{\text{Hom}}_{O_X}(E, O_X))$ is a morphism of sheaves $\Lambda : E|_{U_i} \longrightarrow O_X|_{U_i}$.

Composing it with $\phi_i^{-1}$ we get a morphism

$$\Lambda \circ \phi_i^{-1} : O_X|_{U_i} \longrightarrow O_X|_{U_i};$$

and for any open subset $V$ of $U_i$, we have a homomorphism of $\Gamma(V, O_X)$-modules

$$\Lambda \circ \phi_i^{-1}(V) : \Gamma(V, O_X)^T \equiv \Gamma(V, O_X^T) \longrightarrow \Gamma(V, O_X).$$

This homomorphism is represented by a $r \times 1$ matrix $(\Lambda^T)_{\ell=1}^r$ with entries in $\Gamma(V, O_X)$, the correspondence being given by:

$$\forall f = (f_\ell)_{\ell=1}^r \in \Gamma(V, O_X)^T,$$

$$[(\Lambda \circ \phi_i^{-1}(V))(f)] = (\Lambda^T)_{\ell=1}^r \cdot (f_\ell)_{\ell=1}^r.$$

Since $\Lambda \circ \phi_i^{-1}$ is a morphism of sheaves, we find that the matrix $(\Lambda^T)_{\ell=1}^r$ has entries in $\Gamma(U_i, O_X)$, and depends only on the set $U_i$.

The assignment $\Lambda \longmapsto (\Lambda^T)_{\ell=1}^r$ defines an isomorphism

$$\Theta_i : \Gamma(U_i, \underline{\text{Hom}}_{O_X}(E, O_X)) \equiv \Gamma(U_i, O_X)^T$$

and the collection $\{\Theta_i \}_{i \in I}$ is a trivialization of $E^*$ over the covering $\mathcal{U}$ of $X$.

Given any $i,j \in I$, let $\lambda_i, \lambda_j \in \Gamma(U_i \cap U_j, O_X)^T$ with $\lambda_i = (\lambda_i^\ell)_{\ell=1}^r$ and $\lambda_j = (\lambda_j^\ell)_{\ell=1}^r$. Then $\Theta_i^{-1}(\lambda_i)$ and $\Theta_j^{-1}(\lambda_j)$ agree as morphisms of sheaves of $O_X$-modules over $U_i \cap U_j$ if and only if they induce the same homomorphism of $\Gamma(U_i \cap U_j, O_X)$-module from $\Gamma(U_i \cap U_j, E)$ into $\Gamma(U_i \cap U_j, O_X)$, i.e., if and only if:
\[ \forall \xi \in \Gamma(U_i \cap U_j, E), \]
\[ \Theta^{-1}(\lambda_j)(\xi) = (\lambda_j^{(1)})^T_{l=1} \cdots r \cdot \phi_j(\xi) \]
\[ = (\lambda^{(1)})^T_{l=1} \cdots r \cdot \phi(\xi) \]
\[ = \Theta^{-1}(\lambda_i)(\xi), \]

and this condition is equivalent to the equality
\[ (\lambda_i^{(1)})^r_{l=1} = (g^{-1}_{ij})^T \cdot (\lambda_j^{(1)})^r_{l=1}. \]

So the collection \(((g^{-1}_{ij})^T_{i,j \in I})\) forms a system of transition matrices for \(E^*\) over the covering \(U_i\).

1.4.5 The Exterior Power \(\Lambda^p(E)\) and \(\det(E)\)

Let \(F\) be a sheaf of \(\mathcal{O}_X\)-modules. For every non-negative integer \(p\), we have a presheaf on \(X\):
\[ U \mapsto \Lambda^p(\Gamma(U, F)), \]
where \(\Lambda^p(\Gamma(U, F))\) is the \(p\)th exterior power of the \(\Gamma(U, \mathcal{O}_X)\)-module \(\Gamma(U, F)\) (for the definition of exterior powers of modules see [Hartshorne, p. 127, Ex. 5.16] on which this paragraph is based).

Then, the \(p\)th exterior power of \(F\), denoted \(\Lambda^p(F)\), is the sheaf associated to this presheaf. If for some open subset \(U\) of \(X\) the \(\Gamma(U, \mathcal{O}_X)\)-module \(\Gamma(U, F)\) is free of rank \(r\), then \(\Lambda^p(\Gamma(U, F))\) is a free \(\Gamma(U, \mathcal{O}_X)\)-module whose rank is the binomial coefficient \(\binom{r}{p}\). In particular \(\Lambda^p(F)\) is locally free on \(X\) whenever \(F\) is locally free. So we set:

**Definition 1.9**: The \(p\)th exterior power of the rank \(r\) vector bundle \(E\) is the vector bundle over \(X\) associated to the locally free sheaf \(\Lambda^p(E)\), and is denoted \(\Lambda^p(E)\). When \(p\) is the rank \(r\) of \(E\), \(\Lambda^p(E)\) is a rank 1 vector bundle, or line bundle, and is denoted \(\det(E)\).
Fixing an isomorphism $\Lambda^p(k^r) \cong k^{p}$, we can write the transition matrices of $\Lambda^p(E)$ over $U$ as $\Lambda^p(g_{ij})$, where for each point $q \in U_i \cap U_j$, and for every $u_1, u_2, \ldots, u_p \in k^r$,

$$\Lambda^p(g_{ij})(q)(u_1 \wedge u_2 \wedge \ldots \wedge u_p) = (g_{ij}(q)u_1)(g_{ij}(q)u_2) \wedge \ldots \wedge (g_{ij}(q)u_p).$$

In particular, $\{\det(g_{ij})\}_{i,j \in I}$ is a system of transition matrices for $\det(E)$ over the covering $U$.

If $g : E \longrightarrow F$ is a morphism of vector bundles over $X$, represented over the covering $U$ by matrices $\{g_i\}_{i \in I}$, then the matrices $\{\Lambda^p(g_i)\}_{i \in I}$ will define a morphism of vector bundles over $X$, $\Lambda^p(g) : \Lambda^p(E) \longrightarrow \Lambda^p(F)$, which will be surjective if the original $g$ is surjective.

Now, let $0 \longrightarrow E' \longrightarrow G \longrightarrow F \longrightarrow 0$ be an exact sequence of vector bundles over $X$, where $G$ has rank $t = r + s$. Let also $g$ and $f$ be respectively represented by collections $\{g_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ over $U$.

Given $p \in \mathbb{N}$, we have a surjective morphism $\Lambda^p(g) : \Lambda^p(G) \longrightarrow \Lambda^p(F)$, which corresponds by Proposition 1.1 to an exact sequence

$$0 \longrightarrow E' \xrightarrow{f^*} \Lambda^p(G) \xrightarrow{\Lambda^p(g)} \Lambda^p(F) \longrightarrow 0,$$

where $E'$ may be viewed as a sub-vector-bundle of $\Lambda^p(G)$.

Now, for any index $i \in I$, and any point $q \in U_i$, an element $\omega$ of $\Lambda^p(k^r)$ will satisfy the equality

$$\Lambda^p(g_i)(\omega) = 0,$$

precisely when there exist $v_1 \in k^r$ and $u_1, \ldots, u_{p-1} \in k^r$ such that $\omega$ may be written as $f_i(q)v_1 \wedge u_1 \wedge \ldots \wedge u_{p-1}$. Using this we can construct a surjective morphism $g' : E_1 \longrightarrow E \otimes \Lambda^{p-1}(F)$, and so, an exact sequence:

$$0 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E \otimes \Lambda^{p-1}(F) \longrightarrow 0.$$
By repeating this process, we obtain a sequence of sub-vector-bundles
\( \Lambda^p(G) \supseteq E_1 \supseteq E_2 \ldots \supseteq E_p \), such that
\[
\Lambda^p(G)/E_1 \cong \Lambda^p(F),
\]
\[
\forall l = 1, \ldots, p-1 \quad E_l/E_{l+1} \cong \Lambda^l(E) \otimes \Lambda^{p-l}(F),
\]
and
\[ E_p \cong \Lambda^p(E). \]

Then taking \( p = t \), we find an isomorphism \( \Lambda^t(G) \cong \Lambda^t(E) \otimes \Lambda^s(F) \).

**Proposition 1.3:** If \( 0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0 \) is an exact sequence
of vector bundles over \( X \), then there is an isomorphism
\[
\det(G) \cong \det(E) \otimes \det(F).
\]

1.4.6 **The Canonical Isomorphism:** \( E^* \otimes F \cong \text{Hom}(E,F) \)

Let \( A \) be a commutative ring. For any two \( A \)-modules \( M \) and \( N \), the canonical homomorphism
\[
\Delta(M,N) : M \otimes_A N \longrightarrow \text{Hom}_A(M,N)
\]
is defined by setting:
\[
\forall \phi \in M^*, \forall n \in N, \forall m \in M ; \quad \Delta(M,N)(\phi \otimes n)(m) = \phi(m) \cdot n
\]
When \( M \) and \( N \) are finitely generated free \( A \)-modules, \( \Delta(M,N) \) is an isomorphism.

For any sheaves of \( O_X \)-modules \( E \) and \( F \), and for each open subset \( U \)
of \( X \), one constructs, using the same formula, a canonical homomorphism
\[
\Gamma(U,E^* \otimes F) \longrightarrow \text{Hom}_{O_X}(E|_U^c, F|_U^c).
\]
Since these homomorphisms are defined independently of the choice of local generators, we thus obtain a canonical homomorphism of sheaves of \( O_X \)-modules
\[ \Delta(E,F) : E^* \otimes_{\mathcal{O}_X} F \rightarrow \text{Hom}_{\mathcal{O}_X}(E,F). \]

Now, for every open subset \( U \) of \( X \) over which the given sheaves of sections \( E \) and \( F \) are free, the canonical homomorphism \( \Delta(\mathcal{E}|_U, \mathcal{F}|_U) \) is an isomorphism. Covering \( X \) by such open sets \( U \), we see that \( \Delta(E,F) \) is an isomorphism and we proved

**Proposition 1.4:** For any locally free sheaves \( E \) and \( F \) on \( X \) there is a canonical isomorphism from \( E^* \otimes_{\mathcal{O}_X} F \) to \( \text{Hom}_{\mathcal{O}_X}(E,F) \). As a consequence, the vector bundles \( E^* \otimes F \) and \( \text{Hom}(E,F) \) are isomorphic over \( X \).

In fact all properties that hold for direct sums, tensor products, and duals of vector spaces, and do not depend upon the choice of a basis, also hold for direct sums, tensor products, and duals of vector bundles over \( X \). This is as one should expect, since vector spaces are vector bundles over a point.

In particular we have canonical isomorphisms

1. \( (E \otimes F) \otimes G \cong (E \otimes G) \otimes (F \otimes G) \)

2. \( (E \otimes F) \otimes G \cong E \otimes (F \otimes G) \)

3. \( I \otimes E \cong E \), where

\( I \) is the trivial line bundle over \( X \).

As a direct consequence of Proposition 1.4 and Lemma 1.1, we get

**Lemma 1.2:** For any locally free sheaf \( \mathcal{H} \) on \( X \), the functors \( \mathcal{H} \otimes_{\mathcal{O}_X} - \) and \( - \otimes_{\mathcal{O}_X} \mathcal{H} \) are exact in the category of locally free sheaves on \( X \).

1.4.7 **Another Canonical Isomorphism**

Suppose \( L \) is any line bundle over \( X \). Its sheaf of sections \( L \) has rank one, and is said to be invertible. This is because, as we will see,
there exists a locally free sheaf $M$ with the property that $L \otimes_{\mathcal{O}_X} M \cong \mathcal{O}_X$. 

In order to show this we prove:

**Proposition 1.5:** For any invertible sheaf $L$ on $X$, there is an isomorphism:

$$\text{Hom}_{\mathcal{O}_X}(L,L) \cong \mathcal{O}_X.$$ 

**Proof:** Take a trivialization $\{\Phi_i\}_{i \in I}$ of $L$ over $U$, and let $\{a_{ij}\}_{i,j \in I}$ be the corresponding system of $1 \times 1$ transition matrices of $L$. 

Given any open subset $U$ of $X$, let $\psi$ be an element of $\Gamma(U, \text{Hom}_{\mathcal{O}_X}(L,L))$. 

Then for every $i \in I$, $\text{Res}(\psi) : L|_{U \cap U_i} \longrightarrow L|_{U \cap U_i}$ is represented by a unique regular function $f_{\psi,i} \in \Gamma(U \cap U_i, \mathcal{O}_X)$; and for every pair $i,j \in I$ we have over $U \cap U_i \cap U_j$, the equality:

$$a_{ij} f_{\psi,i} = f_{\psi,j} a_{ij},$$

which gives $f_{\psi,i} = f_{\psi,j}$ over $U \cap U_i \cap U_j$.

So $\psi$ corresponds to a unique element of $\Gamma(U, \mathcal{O}_X)$. And from this correspondence we get an isomorphism of sheaves of $\mathcal{O}_X$-modules from $\text{Hom}_{\mathcal{O}_X}(L,L)$ onto $\mathcal{O}_X$. ☐

In particular by Proposition 1.5, we have $L^* \otimes_{\mathcal{O}_X} L \cong \text{Hom}_{\mathcal{O}_X}(L,L) \cong \mathcal{O}_X$. 

So isomorphism classes of invertible sheaves on $X$ form a group with respect to tensor product, with $\mathcal{O}_X$ as identity. This group is called the Picard group of $X$ and is denoted $\text{Pic}(X)$. The corresponding group for isomorphism classes of line bundles over $X$ will be denoted $\Delta(X)$. 


Chapter 2

\( F^n \)-BUNDLES AND RULED SURFACES

We start this chapter by defining \( F^n \)-bundles over an algebraic variety \( X \). This definition is the same as the one of vector bundles over \( X \), except that \( k^n \) is replaced by a projective space \( F^n \). In section 2.2, we give an elementary proof of the well known fact that every local section of a \( F^n \)-bundle over a nonsingular curve is the restriction of a unique global section. From there on the variety \( X \) will be a nonsingular curve.

In section 2.3, we will show that \( F^n \)-bundles over \( X \) can be viewed as equivalence classes of rank \( n + 1 \) vector bundles over \( X \). In section 2.4, we will reinterpret sections of \( F^n \)-bundles over \( X \) in terms of rank \( n + 1 \) vector bundles over \( X \). To conclude this chapter, we will state some results from the theory of algebraic surfaces to show that \( F^1 \)-bundles and ruled surfaces over the complete nonsingular curve \( X \) are one and the same.

The only outlined proof in section 2.5 comes from [Shafarevich (1), V, 1], and is included here for completeness.

2.1 \( F^n \)-BUNDLES OVER AN ALGEBRAIC VARIETY \( X \)

Definition 2.1: A \( F^n \)-bundle over \( X \) is an object \( P \xrightarrow{\pi} X \), where \( P \) is an algebraic variety and \( \pi \) is an epimorphism, for which we can find:

1. an open covering \( U = \{U_i\}_{i \in I} \) of \( X \),
2. for each \( i \in I \), an isomorphism \( \phi_i : P_i = \pi^{-1}(U_i) \cong U_i \times F^n \),
such that for every \( i \in I \), the diagram

\[
\begin{array}{ccc}
\phi_i & \sim & U_i \times \mathbb{P}^n \\
P_i \downarrow & \swarrow \pi \\
U_i & \sim & P_1
\end{array}
\]

commutes.

Note that for every pair \( i, j \in I \), the automorphism \( \phi_i \circ \phi_j \leftarrow (U_i \cap U_j) \times \mathbb{P}^n \sim (U_i \cap U_j) \times \mathbb{P}^n \) induces a morphism

\[
\theta_{ij} : U_i \cap U_j \longrightarrow \text{PGL}_n(k),
\]
given by

\[
\forall p \in U_i \cap U_j, \forall u \in \mathbb{P}^n; \quad \phi_i \circ \phi_j^{-1}(p, u) = (p, \theta_{ij}(p)u).
\]

This is because the group of automorphisms of \( \mathbb{P}^n \) is simply the group \( \text{PGL}_n(k) \) of fractional transformations of \( \mathbb{P}^n \) (cf. [Hartshorne, II, 7, p. 151]). So the condition (b) of the Definition 1.1 of vector bundles over \( X \) is redundant in the case of \( \mathbb{P}^n \)-bundles.

The collection \( \{\theta_{ij}\}_{i, j \in I} \) is called a system of transition functions for the \( \mathbb{P}^n \)-bundle \( P \longrightarrow X \) over \( J \), and satisfies the condition (TF):

1. \( \forall i, j, \ell \in I, \ \forall p \in U_i \cap U_j \cap U_\ell; \ 
\theta_{ij}(p) \circ \theta_{\ell j}(p) \circ \theta_{\ell i}(p) = \text{Id}_{\mathbb{P}^n}, \)

2. \( \forall i \in I, \forall p \in U_i; \quad \theta_{ij}(p) = \text{Id}_{\mathbb{P}^n}. \)

Except for transition functions, all the terminology of vector bundles over \( X \) extends to \( \mathbb{P}^n \)-bundles over \( X \). For example the collection \( \{\phi_i\}_{i \in I} \)
is called a trivialization of the $\mathbb{P}^n$-bundle $P \to X$ over $U$, and the space $P$ is called its bundle space.

Also, if $\{\theta_{ij}\}_{i,j \in I}$ is any collection of morphisms

$$\theta_{ij} : U_i \cap U_j \to \text{PGL}_n(k)$$

satisfying condition (TF) over some open covering $U = \{U_i\}_{i \in I}$ of $X$, we can construct as in the case of vector bundles, a $\mathbb{P}^n$-bundle over $X$ having the given $\theta_{ij}$ for transition functions over the covering $U$ of $X$.

Note that the Definition 2.1 implies that the bundle space $P$ of a $\mathbb{P}^n$-bundle over $X$ is an algebraic variety birational to $X \times \mathbb{P}^n$, i.e., the function $t : \text{R}(P)$ of $P$ is isomorphic to $K(X \times \mathbb{P}^n)$.

We only define isomorphisms of $\mathbb{P}^n$-bundles over $X$. This goes as follows:

**Definition 2.2:** An isomorphism of $\mathbb{P}^n$-bundles over $X$, say from $P \to X$ onto $P' \to X$, is an isomorphism of algebraic varieties $f : P \to P'$ which respects the projections, i.e., such that $\pi' \circ f = \pi$.

Let the $\mathbb{P}^n$-bundles $P \to X$ and $P' \to X$ have respective trivializations $\{\phi_i\}_{i \in I}$ and $\{\phi_i'\}_{i \in I}$ over some open covering $U = \{U_i\}_{i \in I}$ of $X$, and let $f : P \to P'$ be as above. Then for every $i \in I$, $f$ restricts to an isomorphism between $\pi^{-1}(U_i)$ and $\pi'^{-1}(U_i)$, and the isomorphism

$$\phi_i' \circ f \circ \phi_i^{-1} : U_i \times \mathbb{P}^n \to U_i \times \mathbb{P}^n$$

does of the form

$$\forall p \in U_i, \forall u \in \mathbb{P}^n; \quad \phi_i' \circ f \circ \phi_i^{-1}(p,u) = (p, f_i(p)u),$$

where $f_i : U_i \to \text{PGL}_n(k)$ is a morphism.

As in the case of vector bundles over $X$, the isomorphism $f : P \to P'$
over $X$ is uniquely determined by the collection of morphisms $\{f_i\}_{i \in I}$ and the given trivializations of the $\mathbb{P}^n$-bundles $P \xrightarrow{\pi} X$ and $P' \xrightarrow{\pi'} X$ over $U$. Also, any two $\mathbb{P}^n$-bundles over $X$ are isomorphic if and only if they have respective systems of transition functions $\{\theta_{ij}\}_{i,j \in I}$ and $\{\theta'_{ij}\}_{i,j \in I}$ over some common open covering $U = \{U_i\}_{i \in I}$ of $X$, for which we can find a collection $\{f_i\}_{i \in I}$ of morphisms $f_i : U_i \rightarrow \mathbb{P}^n(k)$ such that

$$\forall i,j \in I, \quad \theta'_{ij} = f_i \circ \theta_{ij} \circ f_{ij}^{-1} \text{ on } U_i \cap U_j.$$

Local sections of $\mathbb{P}^n$-bundles over $X$ are defined in the same way that local sections of vector bundles over $X$ are defined. And if $P \xrightarrow{\pi} X$ is a $\mathbb{P}^n$-bundle with fixed trivialization $\{\phi_i\}_{i \in I}$ over an open covering $U = \{U_i\}_{i \in I}$ of $X$, then for any open subset $U$ of $X$ the local sections of $P \xrightarrow{\pi} X$ over $U$ are in one-to-one correspondence with the collections $\{s_i\}_{i \in I}$ of morphisms $s_i : U \cap U_i \rightarrow \mathbb{P}^n$ satisfying for every pair $i,j \in I$ the equality

$$s_i = \theta_{ij} \circ s_j \text{ over } U \cap U_i \cap U_j,$$

where $\{\theta_{ij}\}_{i,j \in I}$ is the system of transition functions for $P \xrightarrow{\pi} X$ corresponding to $\{\phi_i\}_{i \in I}$.

### 2.2 SECTIONS OF A $\mathbb{P}^n$-BUNDLE OVER A NONSINGULAR CURVE

When the variety $X$ is a nonsingular curve, the local ring $\mathcal{O}_{p,X}$ of any point $p$ on $X$ is a discrete valuation ring. So we can choose for each point $p$ on $X$, a uniformizing parameter $t_p$ of $X$ at $p$. Then every element $f$ of the "function field $K(X)$ of $X$ is of the form

$$f = u t_p^n,$$

where $n_p$ is an integer uniquely determined by $f$ and $p$, and where $u$
is a unit in $\mathcal{O}_p, X$.

The integer $n_p$ is called the order of $f$ at $p$, and is denoted $\text{ord}_p(f)$.

From this we deduce the following well known fact:

Lemma 2.1: Any rational map $f$ from $X$ into $\mathbb{F}^n$ is a morphism.

Proof: Let us agree to write $[z]$ the image in $\mathbb{F}^n$ of any point $z \in \mathbb{K}^{n+1}$ under the canonical projection $\mathbb{K}^{n+1} \to \mathbb{K}^{n+1}/\mathbb{K}\{0\} \cong \mathbb{F}^n$.

Now take rational functions $f_0, f_1, \ldots, f_n$, regular on some open subset $U$ of $X$, and such that the restriction of $f$ to $U$ is the morphism given by

$$p \mapsto [f_0(p), f_1(p), \ldots, f_n(p)].$$

Given any point $q$ on $X$, set

$$\alpha = \min_{q \in U} \{\text{ord}_q(f_i) \mid i = 0, \ldots, n\}.$$

Then $f': p \mapsto [t_q^{-\alpha}(p) f_0(p), \ldots, t_q^{-\alpha}(p) f_n(p)]$ is a well defined morphism into $\mathbb{F}^n$ whose domain is a neighborhood $V$ of $q$. Since $f$ and $f'$ agree as morphisms on $U \cap V$, they also agree as rational maps from $X$ into $\mathbb{F}^n$. The map $f$ being regular at every point of $X$ is a morphism, as required. \( \Box \)

We now prove the following

Proposition 2.1: Let $P$ denote any $\mathbb{F}^n$-bundle over the nonsingular curve $X$. Then any local section of $P$ is the restriction of a unique global section of $P$. In particular, every $\mathbb{F}^n$-bundle over the curve $X$ has a global section.
Proof: Let $U$ be an open subset of $X$ and let $s: U \rightarrow P$ be a local section of $P$ over $U$. Take also $\{\theta_{ij}\}_{i,j \in I}$ to be a system of transition functions for $P$ over some open covering $U = \{U_i\}_{i \in I}$ of $X$.

The local section $s$ is represented by a collection $\{s_i\}_{i \in I}$ of morphisms $s_i: U_i \cap U \rightarrow \mathbb{P}^n$ satisfying for every pair $i,j \in I$ the equality

$$s_i = \theta_{ij} \circ s_j \quad \text{on} \quad U_i \cap U_j.$$

For every $i \in I$, there exists by Lemma 2.1 a unique morphism $S_i: U_i \rightarrow \mathbb{P}^n$ whose restriction to $U_i \cap U$ is $s_i$. For every pair $i,j \in I$, the morphisms $S_i$ and $\theta_{ij} \circ S_j$ agree on $U_i \cap U_j$. Since $S_i$ and $\theta_{ij} \circ S_j$ are both defined on $U_i \cap U_j$, they also agree on $U_i \cap U_j$. So the collection $\{S_i\}_{i \in I}$ defines a global section $S$ of $P$ whose restriction to $U$ is the given local section $s$. The open set $U$ is dense in $X$, so the global section $S$ is uniquely determined by its restriction to $U$. $\Box$

2.3 THE PROJECTIVIZATION OF A VECTOR BUNDLE

From now on $X$ will denote a nonsingular curve. We are going to relate $\mathbb{P}^n$-bundles and vector bundles over $X$. So take $E$ to be any rank $n+1$ vector bundle over $X$ and let $\{g_{ij}\}_{i,j \in I}$ be a system of transition matrices for $E$ over some open covering $U = \{U_i\}_{i \in I}$ of $X$.

The matrices $g_{ij}$ may be viewed as morphisms

$$g_{ij}: U_i \cap U_j \rightarrow GL_{n+1}(k).$$

Denote by $p_r$ the canonical projection from $GL_{n+1}(k)$ onto $PGL_n(k)$, and for every pair $i,j \in I$ let $\tilde{g}_{ij}$ be the morphism $p_r \circ g_{ij}$ into
PGL\(_n(k)\). The collection \(\{g_{ij}\}_{i,j \in I}\) satisfies condition (TF) of 2.1, so it is a system of transition functions for some \(\mathbb{F}^n\)-bundle over \(X\) that we call the projectivization of \(E\) and denote \(\text{Proj}(E)\).

In terms of the bundle space of \(E\), the projectivization is obtained by first removing the zero section thus obtaining an open subset \(E'\) of \(E\). Then for each \(i \in I\), we have an isomorphism \(E' \to E_i \to U_i \times (k^{n+1} - \{0\})\) such that for every pair \(i,j \in I\), the induced automorphism of 

\[(U_i \cap U_j) \times (k^{n+1} - \{0\}) \text{ is linear on the second factor. So if we take the coordinates of } k^{n+1} \text{ to be homogeneous coordinates of the projective space } \mathbb{P}^n,\]

we obtain a collection \(\{U_i \times \mathbb{P}^n\}\) of open subsets of \(X \times \mathbb{P}^n\), which patch together in such a way that they form a \(\mathbb{P}^n\)-bundle over \(X\). This \(\mathbb{P}^n\)-bundle is \(\text{Proj}(E)\).

Now, let \(P\) be any \(\mathbb{P}^n\)-bundle over the curve \(X\), with transition functions \(\{\tilde{g}_{ij}\}_{i,j \in I}\) over some open covering \(U = \{U_i\}_{i \in I}\) of \(X\). Take \(U\) to be a finite covering with \(I = \{0,1,\ldots,h\}\). Then after possibly replacing \(U\) by a finer covering of \(X\), we may assume that for every \(j = 0,1,\ldots,h\) there exists a morphism

\[\tilde{g}_{oj} : U_j \to GL_{n+1}(k)\]

such that

\[\tilde{g}_{oj} = p_r \circ g_{oj} : U_0 \cap U_j \to PGL_n(k)\]

is equal to \(\tilde{g}_{oj}\).

For every pair \(i,j \in I\), set

\[g_{ij} = g_{oi}^{-1} \circ g_{oj}\]

on \(U_i \cap U_j\).

Then \(\{g_{ij}\}_{i,j \in I}\) is a system of transition matrices for some rank \(n+1\) vector bundle over \(X\). In particular, we see that every \(\mathbb{P}^n\)-bundle over the curve \(X\) is the projectivization of some rank \(n+1\)-vector bundle over \(X\).
The question we now ask is: when do two vector bundles over $X$ of the same rank have isomorphic projectivizations? So let $E$ and $E'$ be two rank $n+1$ vector bundles over $X$, with respective systems of transition matrices $\{g_{ij}\}_{i,j \in I}$ and $\{g'_{ij}\}_{i,j \in I}$ over some open covering $U = \{U_i\}_{i \in I}$ of $X$. Let also $\{\tilde{g}_{ij}\}$ and $\{\tilde{g'}_{ij}\}$ be the systems of transition functions induced by $\{g_{ij}\}_{i,j \in I}$ and $\{g'_{ij}\}_{i,j \in I}$.

The two vector bundles $E$ and $E'$ have isomorphic projectivizations if and only if we can find a collection $\{f_i\}_{i \in I}$ of morphisms $\tilde{f}_i : U_i \longrightarrow \text{PGL}_n(k)$ such that for every pair $i,j \in I$, the equality $g'_{ij} = \tilde{f}_i \circ \tilde{g}_{ij} \circ \tilde{f}_j^{-1}$ holds over $U_i \cap U_j$. By possibly refining the covering $U$, we may suppose once more, that for each $i \in I$ the morphism $\tilde{f}_i$ has a lifting $f_i : U_i \longrightarrow \text{GL}_{n+1}(k)$. Then for all $p \in U_i \cap U_j$, there exists $\lambda_{ij}(p) \in k^*$ such that

$$\lambda_{ij}(p) \cdot g_{ij}'(p) = f_i(p) \cdot g_{ij}(p) \cdot f_j(p)^{-1}.$$ 

The assignment $p \mapsto \lambda_{ij}(p)$ defines a morphism $\lambda_{ij} : U_i \cap U_j \longrightarrow \text{GL}_1(k) = k^*$. Also, for all $i,j,l \in I$ and each $p \in U_i \cap U_j \cap U_l$,

$$\lambda_{ij}(p) \cdot \lambda_{jl}(p) \cdot \lambda_{li}(p) \cdot \text{Id}_{n+1} = \lambda_{ij}(p) \cdot g_{ij}'(p) \cdot \lambda_{jl}(p) \cdot g_{jl}'(p) \cdot \lambda_{li}(p) \cdot g_{li}'(p).$$

Thus $\lambda_{ij} \cdot \lambda_{jl} \cdot \lambda_{li} = \text{Id}_1$ over $U_i \cap U_j \cap U_l$ and similarly for every $i \in I$, $\lambda_{ii} = \text{Id}_1$ over $U_i$. So $\{\lambda_{ij}\}_{i,j \in I}$ is a system of transition matrices for some line bundle $L$ over $X$. Hence $E$ and $E'$ have isomorphic projectivizations if and only if there exists a line bundle $L$ over $X$ for which $E$ and $E' \otimes L$ are isomorphic.
To sum up

**Theorem 2.1:** For any given $n$, the group $\Delta(X)$ of isomorphism classes of line bundles over $X$ acts via tensor product on the set of isomorphism classes of rank $n+1$ vector bundles over $X$, and the $\mathbb{P}^n$-bundles over $X$ are (up to isomorphism) in one-to-one correspondence with the orbits of this action of $\Delta(X)$. Any two elements of the same orbit are said to be projectively equivalent.

### 2.4 Sub-line-bundles and Sections of $\mathbb{P}^n$-bundles

Theorem 2.1 gives an alternative definition of $\mathbb{P}^n$-bundles over $X$ in terms of rank $n+1$ vector bundles over $X$. Sections of $\mathbb{P}^n$-bundles may then be reinterpreted as follows:

**Proposition 2.2:** Let $E$ be a rank $n+1$ vector bundle over the non-singular curve $X$ and let $P$ be the projectivization of $E$.

There is a one-to-one correspondence between sub-line-bundles of $E$ and sections of $P$.

**Proof:** Suppose $L$ is a sub-line-bundle of $E$ defined by an injection $f : L \to E$. Let $s$ be a non-zero local section of $L$ and let $U$ be an open subset of $X$ over which $s$ is defined and never vanishes. The local section $f \circ s$ of $E$ over $U$ induces in the obvious way, a local section $S$ of $P$ over $U$. By Proposition 2.1, this local section is in fact a global section of $P$. Now suppose that the morphism $f$ is represented over some open covering $U = \{U_i\}_{i \in I}$ of $X$ by the collection $\{f_i\}_{i \in I}$ and let $\{s_i\}_{i \in I}$ be the corresponding representatives of $s$.
over \( U \) \( = \{ U \cap U_i \}_{i \in I} \). The representatives for the section \( S \) over \( U \) are the morphisms written as \([f_i, s_i]\), in particular the section \( S \) is independent of the choice of a local section of \( L \). It is also clear that in this way, distinct sub-line-bundles of \( E \) are associated to distinct sections of \( P \).

Conversely, suppose that \( \tilde{S} \) is a section of \( P \) and let \( \{ \phi_{ij} \} \) \( i, j \in I \) be a system of transition functions for \( P \) over some open covering \( U = \{ U_i \}_{i \in I} \) of \( X \). After possibly refining the covering \( U \) we may assume that \( E \) has a system of transition matrices \( \{ g_{ij} \} \) \( j, i \in I \) over \( U \), and that the section \( S \) is represented over \( U \) by a collection \( \{ [s_{10}, \ldots, s_{in}] \}_{i \in I} \), where the regular functions \( s_{10}, \ldots, s_{in} \) on \( U_i \) never vanish simultaneously. On each intersection \( U_i \cap U_j \), we have the equality

\[
[s_{10}, \ldots, s_{in}] = \phi_{ij} \circ [s_{j0}, \ldots, s_{jn}].
\]

So there exists for every pair \( i, j \in I \), a non-vanishing regular function \( \lambda_{ij} \) on \( U_i \cap U_j \) such that the equality

\[
\lambda_{ij} \begin{pmatrix} s_{10} \\ \vdots \\ s_{in} \end{pmatrix} = g_{ij} \begin{pmatrix} s_{j0} \\ \vdots \\ s_{jn} \end{pmatrix}
\]

holds on \( U_i \cap U_j \). Clearly the functions \( \lambda_{ij} \) satisfy the condition (TM), so they are transition matrices for some line bundle \( L \), and the matrices

\[
f_i = \begin{pmatrix} s_{10} \\ \vdots \\ s_{in} \end{pmatrix}
\]

represent a morphism \( f \) from \( L \) into \( E \), which is injective since for
each \( i \in I \) the matrix \( f_i \) is never zero on \( U_i \). The sub-line-bundle of \( E \) defined by \( f \) induces the given section \( S \) of \( P \), and we have established the required correspondence. □

Combining Proposition 2.1 and Proposition 2.2, we get

**Proposition 2.3:** Any vector bundle \( E \) over the nonsingular curve \( X \) has a sub-line-bundle. In particular, all rank 2 vector bundles over \( X \) can be written as extensions of line bundles.

This is because the projectivization \( P \) of \( E \) clearly has a local section. Remark, however, that the proof of Proposition 2.1 rests on the assumption that \( X \) is a curve. When applied to surfaces, Proposition 2.1 is no longer true and, for example, the tangent sheaf on \( \mathbb{P}^2 \) does not have a sub-line-bundle (cf. [R. Ganong and P. Russell, 2.1]).

Note that if \( f : L \rightarrow E \) is an injection defining a proper sub-line-bundle of a rank \( n \) vector bundle \( E \), then there exists by Proposition 1.1 an exact sequence

\[
0 \rightarrow L \xrightarrow{f} E \rightarrow F \rightarrow 0,
\]

where \( F \) is a rank \( n - 1 \) vector bundle such that according to Proposition 1.3, \( \det(E) \cong L \otimes \det(F) \). Tensoring this exact sequence with \( L^* \), we obtain an exact sequence

\[
0 \rightarrow L \otimes E \rightarrow F \otimes L^* \rightarrow 0.
\]

So every \( \mathbb{P}^n \)-bundle over \( X \) is the projectivization of a rank \( n+1 \) vector bundle over \( X \) which can be written as an extension of a rank \( n \) vector bundle by the trivial line bundle \( 1 \).
When an exact sequence of vector bundles
\[ 0 \rightarrow G \rightarrow E \rightarrow F \rightarrow 0 \]
splits, the vector bundle \( E \) is said to be written as a trivial extension of \( F \) by \( G \). We saw in paragraph 1.4.2 that \( E \) is then a direct sum of \( F \) and \( G \). Note, however, that there exist rank 2 vector bundles \( E \) which are direct sums of line bundles, but for which we can find a non-splitting exact sequence as above (for example see [Hartshorne, V, 2.15.1]).

By distributivity of the tensor product over the direct sum, a vector bundle \( E \) over \( X \) will be a direct sum of two of its proper subvector bundles if and only if the same is true for all vector bundles projectively equivalent to \( E \). So the following definition is justified:

**Definition 2.3:** A vector bundle \( E \) over \( X \) is said to be decomposable if it can be written as a direct sum of proper sub-vector-bundles, otherwise \( E \) is said to be indecomposable. Accordingly, a \( \mathbb{P}^n \)-bundle over \( X \) is said to be decomposable or indecomposable if it is, respectively, the projectivization of a decomposable or indecomposable vector bundle over \( X \).

In classifying rank 2 vector bundles and \( \mathbb{P}^1 \)-bundles over a nonsingular complete curve, we will distinguish between two cases, namely the decomposable bundles and the indecomposable bundles. The decomposable \( \mathbb{P}^1 \)-bundles are easy to classify and will serve as a model for the indecomposable case. What one needs to know then is that \( \mathbb{P}^1 \)-bundles and (geometrically) ruled surfaces over a complete nonsingular curve are one and the same.
2.5 ABOUT RULED SURFACES

It can be shown that when $X$ is a complete nonsingular curve, the bundle space of any $\mathbb{P}^1$-bundle over $X$ is in fact complete (cf. [J.P. Serre (3), 4]). We then take the definition in [Shafarevich (1), V, 1], which makes $\mathbb{P}^1$-bundles over $X$ into ruled surfaces over $X$.

**Definition 2.4:** A (geometrically) ruled surface over a complete nonsingular curve $X$ is an object $V \rightarrow X$, such that $V$ is a complete nonsingular surface birational to $X \times \mathbb{P}^1$ over $X$, and where $\pi$ is an epimorphism such that for every point $p \in X$, the fibre $\pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$.

Explicitly, we say that $V$ is birational to $X \times \mathbb{P}^1$ over $X$, when there exists an open subset $W$ of $V$ and an embedding $\phi : W \rightarrow X \times \mathbb{P}^1$ such that the diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\phi} & X \times \mathbb{P}^1 \\
\downarrow{\pi} & & \downarrow{\pi}
\end{array}
\]

commutes. The corresponding birational transformation from $V$ to $X \times \mathbb{P}^1$ is the closure in $V \times X \times \mathbb{P}^1$ of the graph of $\phi$.

Note that in the language of schemes the curve $X$ has a non-closed point $\xi$, called a generic point of $X$, whose closure is $X$. Then by a theorem of M. Noether, the condition that $\pi^{-1}(\xi)$ be isomorphic to $\mathbb{P}^1$ already implies that $V$ is birational to $X \times \mathbb{P}^1$ over $X$ (cf. [Shafarevich (1), I, 3 and IV, 1]).
Recall that for a point $q$ lying on a complete nonsingular surface $V$, a blowing up of $q$ on $V$, is a surjective morphism $\sigma : V' \longrightarrow V$ from a complete nonsingular surface $V'$ onto $V$ such that:

1. $\sigma^{-1}(q) = L$ is isomorphic to $\mathbb{P}^1$,
2. $\sigma$ restricts to an isomorphism from $V' - L$ onto $V - \{q\}$.

The morphism $\sigma$ is also called a blowing down of the curve $L$ on $V'$.

Any point on $V$ can be blown up, but the blowing down of a curve $L$ imposes conditions on $L$.

It is well known that any birational transformation of complete nonsingular surfaces can be factored into a finite sequence of blowing up of points, and blowing down of curves (cf. [Hartshorne, V, 5]). There is a more precise result in the case of ruled surfaces over $X$.

Suppose $V \longrightarrow X$ is a ruled surface over $X$ and let $q$ be any point on $V$ with, say, $p = \pi(q)$. Then by blowing up the point $q$, we transform the fibre $\pi^{-1}(p)$ into a line $L$ which can be blown down to a point $q''$ on a complete nonsingular surface $V''$. The projection $\pi$ induces a projection $\pi'' : V'' \longrightarrow X$ that makes $V''$ into a ruled surface over $X$. The ruled surface $V'' \longrightarrow X$ is said to be obtained from $V \longrightarrow X$ by an elementary transformation with center at $q$, denoted $\text{elm}_q : V \longrightarrow V''$.

Now, all ruled surfaces over $X$ are obtained from the ruled surface $p_1 : X \times \mathbb{P}^1 \longrightarrow X$ by a finite sequence of elementary transformations.

In [Shafarevich (1), V, 1] we find the following result, given there as Proposition 1:
Proposition 2.3: Every ruled surface obtained by an elementary transformation from a \( \mathbb{P}^1 \)-bundle over \( X \), is itself a \( \mathbb{P}^1 \)-bundle over \( X \).

The proof goes as follows:

Let \( V \xrightarrow{\pi} X \), \( V'' \xrightarrow{\pi''} X \), \( q \), and \( p = \pi(q) \) be as above, and suppose \( V \xrightarrow{\pi} X \) is a \( \mathbb{P}^1 \)-bundle. To show that \( V'' \xrightarrow{\pi''} X \) is a \( \mathbb{P}^1 \)-bundle, it is sufficient to show that every point \( p' \) on \( X \) has an open neighborhood \( U \) for which there exists an isomorphism \( \pi''^{-1}(U) \xrightarrow{\sim} U \times \mathbb{P}^1 \) over \( U \). This is clear for any point \( p' \) different from \( p \).

For the point \( p \) itself, let \( t_p \) be a uniformizing parameter of \( X \) at \( p \), and take \( U \) to be an open neighborhood of \( p \) where \( t_p \) has no zero or pole except at \( p \). Restrict \( U \), if necessary, so that there exists an isomorphism \( \phi: \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{P}^1 \).

Then for each point \( \alpha \) on \( \mathbb{P}^1 \), construct a local section \( C_\alpha \) of \( V \xrightarrow{\pi} X \) over \( U \) as follows:

Fix homogeneous coordinates on \( \mathbb{P}^1 \), such that \( p_2 \circ \phi(q) = [0, 1] \).

Then for every \( \alpha \in k \), set

\[
C_{[\alpha, 1]} = \phi^{-1}\{ (p', [at_p(p'), 1]) \mid p' \in U \},
\]

and

\[
C_{[1, 0]} = \phi^{-1}\{ (p', [1, 0]) \mid p' \in U \}.
\]

The local sections \( C_\alpha \) are transformed by \( \text{elm}_q \) into local sections \( C'_\alpha \) of \( V'' \xrightarrow{\pi''} X \) over \( U \) (their proper transforms by \( \text{elm}_q \)) according to:
And the assignment $\xi \in C_{\alpha} \mapsto (\pi''(\xi), \alpha)$ defines an isomorphism $\pi''^{-1}(U) \sim U \times \mathbb{R}^1$. \qed

Thus we have
Corollary 2.3: Every ruled surface over a complete nonsingular curve $X$ is a $\mathbb{P}^1$-bundle over $X$.

Now using this correspondence between ruled surfaces and $\mathbb{P}^1$-bundles over $X$, we also find in [Shafarevich (1), V, 7] the following construction for decomposable $\mathbb{P}^1$-bundles over $X$.

First note that by Proposition 1.2 and Proposition 2.2, a $\mathbb{P}^1$-bundle over $X$ is decomposable if and only if it has two non-intersecting sections. So let $X \rightarrow X'$ and $X \rightarrow X''$ be two embeddings of the curve $X$ in some projective space $\mathbb{P}^n$ such that $X'$ and $X''$ do not intersect. Then take an isomorphism $\phi : X' \rightarrow X''$ such that we can join each point $p \in X'$ to the point $\phi(p)$ by a line in $\mathbb{P}^n$ such a way that no two of these lines intersect. The resulting ruled surface is called a scroll, and clearly is a decomposable $\mathbb{P}^1$-bundle over $X$. It is also clear that every decomposable $\mathbb{P}^1$-bundle over $X$ can be constructed in this way.
Chapter 3

DECOMPOSABLE BUNDLES

In the first section of this chapter we show how to relate the classification of decomposable $\mathbb{P}^1$-bundles and rank 2 vector bundles over $X$ to the structure of the group $\Delta(X)$. Since the curve $X$ is nonsingular, $\Delta(X)$ is isomorphic to the divisor class group $\text{Cl}(X)$ of $X$. After recalling, in section 3.2, the basic definitions for divisors, we will construct, in section 3.3, an isomorphism between $\Delta(X)$ and $\text{Cl}(X)$.

In section 3.4 we define the canonical divisor class on $X$, in order to state, in section 3.5, the Serre's duality theorem and the Riemann-Roch theorem for divisors on a complete nonsingular curve. The extension of this last theorem to vector bundles of arbitrary rank follows as a corollary. Finally, in section 3.6 we give a classification of decomposable $\mathbb{P}^1$-bundles and rank 2 vector bundles over a complete nonsingular curve $X$ of genus zero or one.

3.1 A FIRST PROPOSITION

The classification of $\mathbb{P}^1$-bundles and rank 2 vector bundles over $X$ is much simpler in the decomposable case. This is because of the well known result:

Proposition 3.1: Let $E$ be a rank 2 vector bundle over the complete curve $X$, and suppose $L_1$, $L_2$, $M_1$, and $M_2$ are line bundles over $X$ with:

$$L_1 \otimes L_2 \cong E \quad \text{and} \quad M_1 \otimes M_2 \cong E.$$
Then after possibly interchanging the indices, we have

\[ L_1 \cong M_1 \quad \text{and} \quad L_2 \cong M_2. \]

**Proof:** Let \( i_m : L_m \to E, \quad j_m : M_m \to E \quad (m = 1, 2) \) be the canonical injections, and let

\[ p_m : E \to L_m, \quad q_m : E \to M_m \quad (m = 1, 2) \]

be the canonical projections.

For \( m = 1, 2 \) we have

\[ p_m i_m = \text{Id}_{L_m}, \quad q_m j_m = \text{Id}_{M_m}, \]

and

\[ i_1 p_1 + i_2 p_2 = \text{Id}_E = j_1 q_1 + j_2 q_2. \]

This gives \( \text{Id}_{L_1} = p_1 i_1 = p_1 j_1 q_1 i_1 + p_1 j_2 q_2 i_1 \), and hence, one of \( p_1 j_1 q_1 i_1 \) or \( p_1 j_2 q_2 i_1 \) must be a non-zero endomorphism of \( L_1 \). Suppose for example that \( p_1 j_1 q_1 i_1 \) is non-zero. Since \( X \) is complete, we have isomorphisms:

\[ \text{Hom}(L_1, L_1) \cong \Gamma(X, \mathcal{O}_X) \cong k. \]

Hence, \( p_1 j_1 q_1 i_1 \) is an automorphism of \( L_1 \). The fibres of \( L_1 \) and \( M_1 \) are isomorphic to \( k \), so it is not hard to see that both \( p_1 j_1 : M_1 \to L_1 \), and \( q_1 i_1 : L_1 \to M_1 \) are isomorphisms of line bundles.

The other isomorphism \( L_2 \cong M_2 \) follows right away. \( \square \)

So the set of isomorphism classes of decomposable rank 2 vector bundles over \( X \) is in bijection with the set of non-ordered pairs of elements of \( \Delta(X) \).

Now, for any decomposable \( \mathbb{P}^1 \)-bundle \( P \) over \( X \) we can always find a line bundle \( L \) over \( X \), such that \( I \oplus L \) has \( P \) for projectivization. And for any vector bundle of the type \( I \oplus M \), it is clear that there will exist a line bundle \( H \) with \( (I \oplus M) \otimes H \cong (I \oplus L) \), precisely when \( M \cong L \), or
M \cong L^*$. So if we let the cyclic group of order 2, denoted \( \langle t \rangle \), act on \( \Delta(X) \) by \((-1) \cdot L = L^*\), we see that the set of isomorphism classes of decomposable \( \mathbb{F}^1 \)-bundles over \( X \) is in bijection with the set \( \Delta(X)/\langle t \rangle \) of orbits of \( \Delta(X) \) under the action of \( \langle t \rangle \).

### 3.2 SOME DEFINITIONS ABOUT DIVISORS

In order to compute the group \( \Delta(X) \), we will need to study divisors on \( X \). So let us make a quick review of the basic notions for divisors.

The group of divisors on the curve \( X \) is the free abelian group generated by the points of \( X \) and is denoted \( \text{Div}(X) \). Divisors on \( X \) are written as \( D = \sum_{p \in X} n_p \cdot p \), where all but finitely many of the integers \( n_p \) are zero.

When all the coefficients \( n_p \) are non-negative, the divisor \( D \) is said to be effective and we write \( D \geq 0 \). Then for any two divisors \( D \) and \( D' \) on \( X \), the expression \( D \geq D' \) stands for \( D - D' \geq 0 \). Let \( D = \sum_{p \in X} n_p \cdot p \) be a divisor on \( X \). The degree of \( D \) is defined to be the integer \( \deg(D) = \sum_{p \in X} n_p \). Sending each divisor on \( X \) to its degree, we obtain an homomorphism of groups \( \text{Deg} : \text{Div}(X) \rightarrow \mathbb{Z} \).

Now let \( f \) be any non-zero rational function on \( X \). For all but finitely many points \( p \) on \( X \), the integer \( \text{ord}_p(f) \), defined in 2.2, is zero. So we may define the divisor of the function \( f \) on \( X \) by
\[
(f) = \sum_{p \in X} n_p \cdot p
\]
Divisors of rational functions form a subgroup of \( \text{Div}(X) \), called the subgroup of principal divisors on \( X \) and denoted \( \text{Princ}(X) \). The quotient group \( \text{Div}(X)/\text{Princ}(X) \) is called the divisor class group of \( X \) and is
denoted $\mathcal{C}(X)$. Two divisors $D$ and $D'$ on $X$ who belong to the same class in $\mathcal{C}(X)$ are said to be linearly equivalent. We then write $D \equiv D'$.

For any divisor $D$ on $X$, the complete linear system defined by $D$ is the set of all effective divisors $Q$ on $X$ with $Q \equiv D$. It is denoted as $|D|$.

The curve $X$ being complete and nonsingular, all principal divisors on $X$ are of degree zero (cf. [C. Chevalley, I, 8]). So the degree homomorphism $\text{Deg} : \text{Div}(X) \rightarrow \mathbb{Z}$ factors into a homomorphism $\mathcal{C}(X) \rightarrow \mathbb{Z}$, that we still denote $\text{Deg}$.

The curve $X$ being nonsingular, we can always find for any divisor $D$, an open covering $U = \{U_i\}_{i \in I}$ of $X$ and for each $i \in I$, an open set $U_i$ of $X$ and for each $i \in I$, an open covering $U = \{U_i\}_{i \in I}$ of $X$ and for each $i \in I$, a rational function $f_i$ on $X$ with the property that for every point $p \in U_i$ we have

$$\text{ord}_p (f_i) = n_p.$$

Such a collection $\{f_i\}_{i \in I}$ of rational functions is called a system of local equations for the divisor $D$ with respect to the covering $U$.

### 3.3 THE ISOMORPHISM: $\mathcal{C}(X) \cong \Delta(X)$

In order to construct such an isomorphism, we will start by associating to each divisor on $X$ an invertible sheaf on $X$.

So let $D$ be any divisor on $X$ and let $\{h_i\}_{i \in I}$ be a system of local equations for $D$ with respect to some open covering $U = \{U_i\}_{i \in I}$ of $X$. For each open subset $U$ of $X$, we have a $\Gamma(U, \mathcal{O}_X)$-module $M_U$ given as

$$M_U = \{f \in K(X) \mid \forall i \in I : f \cdot h_i \in \Gamma(U \cap U_i, \mathcal{O}_X)\}.$$

These modules are uniquely determined by the divisor $D$ and do not depend
upon the choice of local equations for \( D \). Suppose for example that
\[ \{g_j\}_{j \in J} \]
is another system of local equations for \( D \) with respect to an open covering \( U = \{V_j\}_{j \in J} \) of \( X \). Any element \( f \) of \( M_U \) satisfies
\[ \forall i \in I, \forall j \in J; f \cdot g_i / g_j \in \Gamma(U_i \cap V_j, \mathcal{O}_X). \]
By definition, the function \( h_i / g_j \) on \( U_i \cap V_j \) is regular and nowhere vanishing for each \( i \in I \) and each \( j \in J \). So \( f \) satisfies
\[ \forall j \in J; f \cdot g_j \in \Gamma(U \cap V_j, \mathcal{O}_X). \]

The associated sheaf of \( D \) is then defined to be the sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{O}_X(D) \), such that for each open subset \( U \) of \( X \), \( \Gamma(U, \mathcal{O}_X(D)) = M_U \).

The assignments \( f \longmapsto f \cdot h_i \) give local isomorphisms between \( \mathcal{O}_X(D) \) and \( \mathcal{O}_X \). So \( \mathcal{O}_X(D) \) is an invertible sheaf on \( X \). The corresponding line bundle will be denoted \( [D] \) and has \( \{h_i / g_j\}_{i,j \in I} \) for system of transition matrices over the covering \( U \) of \( X \).

Now suppose \( D \) and \( D' \) are divisors on \( X \), with respective systems of local equations \( \{h_i\}_{i \in I} \) and \( \{h_i'\}_{i \in I} \) over the same open covering \( U = \{U_i\}_{i \in I} \) of \( X \).

The collection \( \{h_i \cdot h_i'\}_{i \in I} \) is a system of local equations for the divisor \( D + D' \) with respect to the covering \( U \). In particular, the line bundle \( [D + D'] \) has transition matrices \( \{h_i / h_j \cdot h_i' / h_j'\}_{i,j \in I} \) over \( U \), and so is isomorphic to \( [D] \otimes [D'] \).

The line bundles \( [D] \) and \( [D'] \) will be isomorphic precisely when there exist on every open set \( U_i' \), a non-vanishing regular function \( f_i' \), with
\[ \forall i,j \in I; \ h_i' / h_j' = f_i' \cdot h_i / h_j \cdot f_j^{-1} \quad \text{on } U_i \cap U_j. \]
And these equalities hold if and only if the rational functions \( f_i \cdot h_i/h'_i \) are all equal. So \([D]\) and \([D']\) are isomorphic precisely when the rational functions \( h_i/h'_i \) are local equations with respect to \( U \) for a principal \( i^* \) divisor, and hence precisely when \( D = D' \). In particular, the assignment \( D \mapsto [D] \) induces an injective homomorphism of groups from \( C\mathcal{L}(X) \) to \( \Delta(X) \).

In order to show that this homomorphism is onto, let \( L \) be any line bundle over \( X \) with transition matrices \((a_{ij})_{i,j} \) over some open covering \( U = \{ U_i \}_{i \in I} \) of \( X \). Take any rational section \( s \) of \( L \), or, which is the same, a local section \( s \) of \( L \) over some open subset \( U \) of \( X \). Then there exist rational functions \( s_i \) on \( U_i \) such that \( s_i \) represent \( s \) over \( U_i \). Hence the \( s_i \) satisfy the equalities

\[
V_{i,j} \in I: \quad s_i/s_j = a_{ij} \quad \text{on} \quad U_i \cap U_j.
\]

So the collection \( \{ s_i \}_{i \in I} \) is a system of local equations for some divisor \( X \), with respect to the covering \( U \). This divisor is called the divisor of the rational section \( s \) and is denoted \((s)\). Clearly, the invertible sheaves \( L \) and \( \mathcal{O}_X((s)) \) are isomorphic, and we have proved

**Theorem 3.1:** When \( X \) is a complete nonsingular curve, the group \( \Delta(X) \) of isomorphism classes of line bundles over \( X \) is isomorphic to the divisor class group \( C\mathcal{L}(X) \) of \( X \).

Note that the ideal sheaf \( I_q \) of a point \( q \) on \( X \), introduced in paragraph 1.4.1, is the invertible sheaf \( \mathcal{O}_X(-q) \).

One remark ought to be made here. Given a divisor \( D \) on \( X \), we have the well known \( k \)-vector space \( L(D) \) of rational functions \( f \) on \( X \) satisfying \( (f) + D \geq 0 \). Now suppose that \( \{ h_i \}_{i \in I} \) is a system of local equations for \( D \) with respect to some open covering \( U = \{ U_i \}_{i \in I} \) of \( X \),
and let \( f \) be any rational function in \( L(D) \). For every \( i \in I \) the function \( f \cdot h_i \) is regular on \( U_i \). So the collection \( \{ f \cdot h_i \}_{i \in I} \) represents a section of \([D]\). We obtain in this way an isomorphism between the \( k \)-vector spaces \( L(D) \) and \( \Gamma(X, O_X(D)) \). The sheaf \( O_X(D) \) being locally free, and therefore coherent, the \( k \)-vector space \( \Gamma(X, O_X(D)) \) is finite dimensional (cf. [Shafarevich (2), V1, 4]), and so is \( L(D) \). So the complete linear system \([D]\) can be identified with the projective space associated to \( L(D) \), and has dimension \( \dim_k \Gamma(X, O_X(D)) - 1 \).

Our main tools for computing these dimensions will be Serre's duality theorem and the Riemann-Roch theorem for nonsingular curves. For this reason, we will need the canonical divisor class on the curve \( X \).

### 3.4 The Canonical Divisor Class on \( X \)

Let \( A \) be a commutative unitary ring and let \( B \) be an \( A \)-algebra (containing \( A \) as a subring). Recall that for any \( B \)-module \( M \), an \( A \)-derivation of \( B \) into \( M \) is an \( A \)-homomorphism \( d : B \to M \) which satisfies

\[
\forall b_1, b_2 \in B; \quad d(b_1 \cdot b_2) = b_1 \cdot d(b_2) + b_2 \cdot d(b_1).
\]

Then following [Hartshorne, II, 8], we define the module of relative differential forms of \( B \) over \( A \) to be a \( B \)-module \( \Omega^1_{B/A} \), together with an \( A \)-derivation \( d : B \to \Omega^1_{B/A} \), which has the following universal property:

For every \( B \)-module \( M \) and every \( A \)-derivation \( d' : B \to M \), there exists a unique \( B \)-homomorphism \( f : \Omega^1_{B/A} \to M \) making the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{d} & \Omega^1_{B/A} \\
\downarrow & & \downarrow \text{id} \\
M & \xrightarrow{\text{id}} & M
\end{array}
\]
This universal object \((\Omega_{B/A}, d)\) can be constructed as follows:

Let \(F\) be the free \(B\)-module generated by the symbols \(db\), with \(b \in B\). Then \(\Omega_{B/A}\) is the quotient \(B\)-module of \(F\) by the sub-\(B\)-module generated by the expressions of the form:

1. \(d(a \cdot b) - a \cdot db\)
2. \(d(b + c) - db - dc\)
3. \(d(b \cdot c) - b \cdot dc - c \cdot db\)

with \(a \in A\) and \(b, c \in B\). The derivation \(d : B \to \Omega_{B/A}\) is given by the assignment \(b \to db\).

The symbols \(db\) generate the \(B\)-module \(\Omega_{B/A}\). So in particular, when \(B\) is a finitely generated \(A\)-algebra, the \(B\)-module \(\Omega_{B/A}\) is finitely generated.

We now define

Definition 3.1: The sheaf of relative differentials \(\omega_X\) on the curve \(X\) is the sheaf of \(O_X\)-modules associated to the presheaf

\[ U \mapsto \Omega_{A_U/k}, \]

where \(A_U = \Gamma(U, O_X)\).

The fact that the assignment \(U \mapsto \Omega_{A_U/k}\) is a presheaf follows from the universal property of the modules \(\Omega_{A_U/k}\).
One important property of the sheaf $\omega_X$ is that it is locally free of rank one if and only if the curve $X$ is nonsingular (cf. [Hartshorne, II, Theorem 8.15]). We then have the following definition:

**Definition 3.2:** The canonical divisor class on $X$ is the divisor class on $X$ associated to the invertible sheaf $\omega_X$ and is denoted $K_X$. Accordingly, a canonical line bundle over $X$ is any line-bundle in the isomorphism class $[K_X]$.

Remark that the tangent bundle to $X$ is a member of the dual class $[-K_X]$. Also, remark that the global sections of a canonical line bundle over $X$ may be viewed as regular differentials on $X$ in the classical sense.

### 3.5 Serre's Duality and the Riemann-Roch Theorem for Complete Nonsingular Curves

The formulation of Serre's duality requires the use of cohomology, so we give here a brief review of cohomology of sheaves of abelian groups on $X$. For the defining axioms of an abelian category we refer to [A. Grothendieck (2), I, 4]. The category $\text{Ab}$ of abelian groups and the category $\text{Ab}(X)$ of sheaves of abelian groups on $X$ are examples of abelian categories. In particular, for all sheaves $F$ and $G$ in $\text{Ab}(X)$ the set $\text{Hom}_{\text{Ab}(X)}(F,G)$ of morphisms from $F$ to $G$ is an abelian group.

Recall that an object $I$ in an abelian category $\mathcal{C}$ is called injective when the functor $\text{Hom}_{\mathcal{C}}(-,I)$ is exact. Also, an injective resolution of an object $A$ in $\mathcal{C}$ is any exact sequence in $\mathcal{C}$ of the form
$0 \rightarrow A \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots$, 

where $I_0, I_1, \ldots$ are injective in $\mathcal{C}$. In the case of the abelian category $\text{Ab}(X)$, every sheaf of abelian groups on $X$ has an injective resolution (cf. [A. Grothendieck (2), III, 1]). The global sections functor, given by $\Gamma(F) = \Gamma(X,F)$ is a covariant left exact functor from $\text{Ab}(X)$ to $\text{Ab}$.

So we can define the cohomology functors $H^1(X, -)$ of $X$ to be the right derived functors $R^1(-)$ of $\Gamma(-)$. The idea goes as follows:

We choose for every sheaf of abelian groups $F$ on $X$ an injective resolution

$$0 \rightarrow F \xrightarrow{c} I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots.$$

The $i$th cohomology group of $F$ is then defined to be the $i$th cohomology group of the complex

$$0 \rightarrow \Gamma(I_0) \xrightarrow{\Gamma(d_0)} \Gamma(I_1) \xrightarrow{\Gamma(d_1)} \cdots,$$

i.e., $H^i(X, F) = \ker \Gamma(d_i)/\text{Im} \Gamma(d_{i-1})$ for $i > 0$ and $H^0(X, F) = \Gamma(X,F)$ since the functor $\Gamma(-)$ is left exact.

It would remain to show that the above construction is independent of the choice of an injective resolution of $F$, and that we have obtained in this way cohomology functors which satisfy the usual properties, in particular the long exact sequence property:

If $0 \rightarrow F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \rightarrow 0$ is an exact sequence in $\text{Ab}(X)$, then there exists an exact sequence in $\text{Ab}$:

$$0 \rightarrow H^0(X, F') \xrightarrow{H^0(\phi)} H^0(X, F) \xrightarrow{H^0(\psi)} H^0(X, F'') \xrightarrow{\delta} H^1(X, F') \xrightarrow{H^1(\phi)} \cdots$$

These results are well known and their proofs are long, so we refer to [A. Grothendieck (2), III, 2-6]. In the sequel we will need to compute explicitly some cohomology groups. As is known, the $i$th cohomology
group \( H^i(X, F) \) of any sheaf of abelian groups \( F \) on \( X \) vanishes for \( i > \dim X \) (cf. [A Grothendieck (2), Théorème 3.6.5]). We already know the group \( H^0(X, F) = \Gamma(X, F) \), and since \( X \) is a curve, we only need a method to compute the first cohomology group \( H^1(X, F) \). We will use the Čech cohomology whose complete construction can be found in [J.P. Serre (1), I, 3, p. 112 ff.]. That the Čech cohomology on \( X \) is equivalent to the one defined above is proved in [A. Grothendieck (2), III, 8]. In this section we will only give the construction of the first Čech cohomology group \( H^1(X, F) \) of \( F \) on \( X \).

So let \( U = \{U_1\}_{i \in I} \) be any open covering of \( X \). For any triple \( i, j, l \in I \), we denote \( \rho_{i,j,l} \) the restriction homomorphism of \( F \) from \( \Gamma(U_i \cap U_j, F) \) into \( \Gamma(U_i \cap U_j \cap U_l, F) \).

A 1-cocycle with coefficients in \( F \) over \( U \) is a collection \( \{c_{ij}\}_{i,j \in I} \) of local sections \( c_{ij} \in \Gamma(U_i \cap U_j, F) \) satisfying the 1-cycle condition over \( U \):

\[
(1-c): \quad \forall i, j, l \in I; \quad \rho_{i,j,l}(c_{ij}) = \rho_{i,l,j}(c_{ij}) + \rho_{j,l,i}(c_{jl}) = 0.
\]

These 1-cocycles form an abelian group in the obvious manner. We denote this group as \( Z^1(U, F) \). The elements of \( Z^1(U, F) \) which are of the form \( \{\rho_{i,j}(c_i) - \rho_{j,l}(c_j)\}_{i,j \in I} \) (with \( c_i \in \Gamma(U_i, F) \) and \( c_j \in \Gamma(U_j, F) \)) form a subgroup denoted \( B^1(U, F) \). We then define the first Čech cohomology group of \( F \) with respect to \( U \), to be the quotient group \( \check{H}^1(U, F) \) of \( Z^1(U, F) \) by \( B^1(U, F) \).

Since we will be exclusively dealing with locally free sheaves on \( X \), we may as well take the sheaf \( F \) to be a coherent sheaf of \( \mathcal{O}_X \)-modules. Then for any two affine open coverings \( U \) and \( V \) of \( X \), the groups
\( H^1(U,F) \) and \( H^1(U,F) \) are isomorphic (cf. [J.P. Serre (1), II, p. 239, Théorème 4]). So we take \( H^1(X,F) \) to be \( H^1(U,F) \) for any affine open covering \( U \) of \( X \).

Note that the 1-cocyle condition (1-c) given above can be restated as:

\[
\forall i, j, k \in I; \quad \rho_{ij,k}(c_{ij}) + \rho_{jk,k}(c_{jk}) + \rho_{ki,k}(c_{ki}) = 0 \quad \text{and} \quad \forall i \in I; \quad c_{ii} = 0.
\]

So condition (1-c) is simply the old condition (TM) written in additive form.

It appears from the construction of the Čech cohomology of \( X \) that the cohomology groups of any coherent sheaf on \( X \) are \( k \)-vector spaces. Since complete nonsingular curves are projective, these \( k \)-vector spaces are finite dimensional (cf. [J.P. Serre (1), III, p. 259, Théorème 1.]).

We may now state the Serre's duality theorem, whose proof can be found in [A. Grothendieck (3), Exp. XII].

Serre's Duality for the Complete Nonsingular Curve \( X \):

For any locally free sheaf \( E \) on \( X \), the \( k \)-vector spaces

\[
\begin{align*}
H^1(X,E) & \quad \text{and} \quad H^0(X,E^* \otimes_X \omega_X)
\end{align*}
\]

are dual to each other. In particular

\[
\dim_k H^1(X,\omega_X) = 1.
\]

We now state the Riemann-Roch theorem for divisors on a non-singular curve, the proof of which can be found in [J.P. Serre (4), II, 9].
Riemann-Roch Theorem for Divisors on the Curve $X$:

If $D$ is any divisor on $X$, then

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^1(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g,$$

where $g = \dim_k H^1(X, \mathcal{O}_X)$ is the genus of $X$.

The number $\dim_k H^1(X, \mathcal{O}_X)$ is usually referred to as the geometric genus of $X$, while the arithmetic genus of $X$ is defined to be $\dim_k H^1(X, \mathcal{O}_X) - \dim H^0(X, \mathcal{O}_X) + 1$. Thus in the special case of the zero divisor on $X$, the Riemann-Roch theorem states the equivalence of these notions of genus on a nonsingular curve.

By applying the Riemann-Roch theorem and Serre's duality to a canonical divisor $K$ (i.e. $K = K_X$), we find $\deg(K_X) = \deg(K) = 2g - 2$.

Also, suppose that $D$ is a divisor on $X$ with $\dim_k H^1(X, \mathcal{O}_X(D)) = 0$. The Riemann-Roch theorem gives the dimension of the $k$-vector space $L(D) \cong H^0(X, \mathcal{O}_X(D))$. Such a divisor $D$ on $X$ is said to be non-special, and accordingly a divisor $D$ on $X$ is said to be special when $H^1(X, \mathcal{O}_X(D))$ is non-zero.

It is clear that when $\deg(D)$ is negative, the $k$-vector space $L(D)$ must be zero. Thus, if $\deg(D) > 2g - 2$, the divisor $K - D$ has a negative degree and $H^0(X, \mathcal{O}_X(K-D))$ is zero. So by Serre's duality the divisor $D$ is non-special.

In order to extend the Riemann-Roch theorem to vector bundles over $X$ of arbitrary rank $r$, we make the following definition:

**Definition 3.3:** The degree of a line bundle over $X$ is the degree of its associated divisor class on $X$. Then, the degree of an arbitrary vector
bundle $E$ over $X$ is the degree of the line bundle $\text{det}(E)$ and is denoted $\deg(E)$.

The degree of a locally free sheaf on $X$ is defined accordingly.

There are two properties of the degree of vector bundles which will be needed in the sequel. We give them in a lemma:

**Lemma 3.1:** Let $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ be an exact sequence of vector bundles over $X$, then

$$\deg(E) = \deg(E') + \deg(E'').$$

In particular, if $E$ has rank $r$ and if $L$ is any line bundle, then

$$\deg(E \otimes L) = \deg(E) + r \deg(L).$$

**Proof:** The first equality is a direct consequence of Proposition 1.3, which says that $\text{det}(E) \cong \text{det}(E') \otimes \text{det}(E'')$.

The second equality is clearly true when $E$ is a line bundle. Let us suppose that it is true for all $r' < r$, with $r > 1$. By Proposition 2.3, $E$ can be written as an extension of a rank $r - 1$ vector bundle $E''$ by a line bundle $E'$. Tensoring the corresponding exact sequence with the given line bundle $L$ gives an exact sequence

$$0 \rightarrow E' \otimes L \rightarrow E \otimes L \rightarrow E'' \otimes L \rightarrow 0.$$ 

So, by the first equality and the induction hypothesis

$$\deg(E \otimes L) = \deg(E' \otimes L) + \deg(E'' \otimes L)$$

$$= \deg(E) + r \deg(L).$$

Knowing this, we give another version of the Riemann-Roch theorem for complete nonsingular curves.
Riemann–Roch Theorem for Vector Bundles over X:

Let the curve $X$ be of genus $g$ and let $K$ be any canonical line bundle over $X$. Then for any rank $r$ vector bundle $E$ over $X$

$$\dim_k \Gamma(E) - \dim_k \Gamma(E^* \otimes K) = \deg(E) + r(l - g),$$

where $\Gamma(E)$ is the $k$-vector space of global sections of $E$.

Proof. We have $\Gamma(E) \cong H^0(X, E)$, and by Serre’s duality,

$$\dim_k H^0(X, E^* \otimes K) = \dim_k H^1(X, E)$$

So there is nothing new here in the case $r = 1$. Let us suppose that the theorem is true for all $r' < r$, with $r > 1$. By Proposition 2.3 the rank $r$ vector bundle $E$ has a sub-line-bundle. Let

$$0 \longrightarrow L \longrightarrow E \longrightarrow F \longrightarrow 0$$

be the corresponding exact sequence of sheaves of sections. We have a long exact cohomology sequence

$$0 \longrightarrow H^0(X, L) \longrightarrow H^0(X, E) \longrightarrow H^0(X, F) \longrightarrow H^1(X, L) \longrightarrow \cdots,$$

which terminates at $H^1(X, F)$ since $X$ is a curve. By exactness of this sequence, we have:

$$\dim_k H^0(X, L) - \dim_k H^0(X, E) + \dim_k H^0(X, F) - \dim_k H^1(X, L) + \dim_k H^1(X, E) - \dim_k H^1(X, F) = 0.$$

Since $F$ is of rank $r - 1$, we can apply the induction hypothesis, and:

$$\dim_k \Gamma(E) - \dim_k \Gamma(E^* \otimes K) = \dim_k H^0(X, E) - \dim_k H^1(X, E)$$

$$= \deg(L) + \deg(F) + r(l - g)$$

$$= \deg(E) + r(l - g).$$
3.6 CLASSIFICATION OF DECOMPOSABLE BUNDLES

The Riemann-Roch theorem enables us to compute the divisor class group of \( X \) in the special case where \( X \) is of genus zero or one. So we may now return to decomposable rank 2 vector bundles and \( \mathbb{P}^1 \)-bundles over \( X \).

Suppose \( X \) has genus \( g = 0 \) and let \( D \) be any degree zero divisor on \( X \). The Riemann-Roch theorem tells in this case that

\[
\dim_k L(D) = \dim_k H^0(X, \mathcal{O}_X(D)) = 1.
\]

So \( D \) is linearly equivalent to an effective divisor of degree zero, which must therefore be the zero divisor on \( X \). Thus any zero degree divisor on \( X \) is principal and hence, the degree homomorphism gives an isomorphism of groups between \( \text{Cl}(X) \) and \( \mathbb{Z} \). So returning to the earlier results of paragraph 3.1, which said that the set of isomorphism classes of decomposable rank 2 vector bundles over \( X \) is bijective to the set of non-ordered pairs of elements of \( \Delta(X) \cong \text{Cl}(X) \), we see that isomorphism classes of decomposable rank 2 vector bundles over \( X \) are in one-to-one correspondence with the pairs of integers \((d_1, d_2)\) with \( d_1 \leq d_2 \).

We also found in paragraph 3.1 that for any decomposable \( \mathbb{P}^1 \)-bundle \( P \) over \( X \), there are exactly two isomorphism classes of line bundles \( L \) determined by the equation \( \text{Proj}(I \otimes L) \cong P \), namely \( L \) and \( L^* \).

Since \( \deg(L^*) = -\deg(L) \), the set of isomorphism classes of decomposable \( \mathbb{P}^1 \)-bundles over \( X \) is therefore in bijection with the set \( \mathbb{N} \) of non-negative integers.

Now suppose \( X \) has genus \( g = 1 \). The canonical divisor class \( K_X \)
on \( X \) has degree zero, so any degree one divisor \( D \) on \( X \) is non-special. By the Riemann-Roch theorem, we find that
\[
\dim |D| = \dim_k H^0(X, \mathcal{O}_X(D)) - 1 = 0.
\]
So there exists a unique point \( p \) on \( X \) which is linearly equivalent to the degree one divisor \( D \), when considered as a divisor.

Fix any point \( p_0 \) on the curve \( X \). Then for any integer \( d \) and for each divisor \( D \) on \( X \) of degree \( d \), there exists a unique point \( p_D \) on \( X \) with
\[
D \equiv (d - 1) \cdot p_0 + p_D.
\]
So the set of isomorphism classes of line bundles of degree \( d \) on the curve \( X \) of genus one, is in bijection with the set of points of \( X \).

Now, given any integer \( d \) and any point \( p \) on \( X \), the only point \( q \) on \( X \) for which we have an isomorphism:
\[
\mathcal{O}_X((d-1) \cdot p_0 + p) \cong \mathcal{O}_X((-d-1) \cdot p_0 + q)
\]
is the point given by the equation:
\[
p + q = 2 \cdot p_0.
\]

To sum up, we have

**Theorem 3.2:** On the complete nonsingular curve \( X \) of genus \( g = 0 \), or 1, let \( p_0 \) be a fixed point.

When \( g = 0 \), the set of isomorphism classes of decomposable rank 2 vector bundles is in bijection with the set of pairs \((d_1, d_2)\), with \( d_1 \leq d_2 \), and the set of isomorphism classes of decomposable \( \mathbb{P}^1 \)-bundles is in bijection with the set \( \mathbb{N} \) of non-negative integers.

When \( g = 1 \), isomorphism classes of decomposable rank 2 vector bundles correspond to the 4-tuples of the form \((d_1, p_1; d_2, p_2)\) with \( d_1, d_2 \in \mathbb{Z} \) and \( p_1, p_2 \in X \). And the isomorphism classes of decomposable
$l^1$-bundles are in one-to-one correspondence with the equivalence classes of pairs in $\mathbb{Z} \times X$ under the equivalence relation $\sim$ defined by $(d,p) \sim (d',p')$ if either $(d,p) = (d',p')$ or $d = -d'$ and $p + p' \equiv 2 \cdot p_0$. 
Chapter 4

INDECOMPOSABLE BUNDLES

In chapter 3 we derived a classification of decomposable \( F^1 \)-bundles over the curve \( X \) of genus zero or one from our knowledge of line bundles over \( X \). This was possible because \( F^1 \)-bundles can be viewed as equivalence classes of rank 2 vector bundles. It is reasonable to try the same approach in the case of indecomposable \( F^1 \)-bundles over \( X \). The first question which then arises is to know when there exists for given line bundles \( L \) and \( M \) over \( X \), an indecomposable rank 2 vector bundle over \( X \) which can be written as an extension of \( M \) by \( L \). We will show in section 4.1 that modulo a certain equivalence relation, the extensions of \( M \) by \( L \) are in one-to-one correspondence with the elements of the group \( h^1(X, \text{Hom}_X(M,L)) \). The group operation induced by this bijection on the set of equivalence classes of extensions of \( M \) by \( L \) is also known as the Baer sum. At the end of section 4.1, we will give the classical construction of the Baer sum.

The central step in classifying decomposable \( F^1 \)-bundles over \( X \), was to find for each such \( F^1 \)-bundle a representative rank 2 vector bundle of the form \( E \oplus L \), with \( L \) of non-negative degree. We will show in section 4.2 how this idea can be carried to indecomposable \( F^1 \)-bundles over \( X \) by defining normalized representatives. In general normalized representatives are not uniquely determined by their projectivizations, but when \( X \) has genus low enough they still yield a classification of indecomposable \( F^1 \)-bundles, and consequently of rank 2 vector bundles, over \( X \). This will be explained in section 4.3. At the end of the chapter we give a table...
summarizing the classification of $\mathbb{P}^1$-bundles and rank 2 vector bundles over the curve $X$ of genus zero or one.

### 4.1 Extensions of Line Bundles

**Definition 4.1:** Let $L$ and $M$ be line bundles. Two extensions of $M$ by $L$

(*) : $0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$

and

(*)' : $0 \rightarrow L \rightarrow E' \rightarrow M \rightarrow 0$

are said to be equivalent if there exists an isomorphism $\phi : E \rightarrow E'$ making the diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & L & \rightarrow & E & \rightarrow & M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \phi & & \downarrow & & \downarrow \\
0 & \rightarrow & L & \rightarrow & E' & \rightarrow & M & \rightarrow & 0
\end{array}
\]

commutative. We then write $\phi : (\ast) \sim (\ast)'$.

The set of equivalence classes of extensions of $M$ by $L$ is denoted $E(M,L)$.

Given any two line bundles $L$ and $M$ we want to establish a one-to-one correspondence between the equivalence classes of extensions of $M$ by $L$ and the elements of the group $H^1(X, \underline{\text{Hom}}_X^{\mathbb{O}}(M,L))$. We will start by constructing such a correspondence between a subset of $E(M,L)$ and $H^1(X, \underline{\text{Hom}}_X^{\mathbb{O}}(M,L))$. It will then suffice to show that this subset is the whole of $E(M,L)$.

In order to simplify the computations, we fix the transition matrices of $L$ and $M$ to be respectively $\{a_{ij}\}_{i,j \in I}$ and $\{b_{ij}\}_{i,j \in I}$ over some
affine open covering $U = \{U_i\}_{i \in I}$ of $X$. It should be noted, however, that the bijection we are going to construct is in fact natural and does not depend upon the choice of transition matrices for $L$ and $M$.

We say that an extension of $M$ by $L$

$$0 \longrightarrow L \xrightarrow{f} E \xrightarrow{g} M \longrightarrow 0$$

is reduced when the transition matrices of $E$ can be chosen to be of the form

$$g_{ij} = \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix}$$

in such a way that the morphisms $f$ and $g$ are respectively represented over every open set $U_i$ by matrices

$$f_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad g_i = (0, 1).$$

The condition $(TM)$ for the above transition matrices $g_{ij}$ is equivalent to

$$(TM)_R: \forall i, j, k \in I; \quad a_{ij} a_{jk} c_{kl} + a_{ij} c_{jk} b_{kl} + c_{ij} b_{jk} b_{kl} = 0$$

on $U_i \cap U_j \cap U_k$,

where $a_{ij}$ and $b_{ij}$ are the given transition matrices of $L$ and $M$.

For each pair $i, j \in I$ the regular function $c_{ij}$ on $U_i \cap U_j$ represents a unique morphism

$$\tau_{ij}: M|_{U_j \cap U_i} \longrightarrow L|_{U_i \cap U_j}.$$ 

Comparing $(TM)_R$ with the 1-cocycle condition (1-c) given in section 3.5, we see that the collection $\{\tau_{ij}\}_{i, j \in I}$ is a 1-cocycle with coefficients in $\underline{\text{Hom}}_X^0(M, L)$. Any two equivalent reduced extensions of $M$ by $L$ correspond in this way to the same element of $H^1(X, \underline{\text{Hom}}_X^0(M, L))$. To see this,
(\ast) : \quad 0 \xrightarrow{f} L \xrightarrow{g} E \xrightarrow{h} M \xrightarrow{i} 0 \quad \text{and}

(\ast)': \quad 0 \xrightarrow{f'} L \xrightarrow{g'} F \xrightarrow{h'} M \xrightarrow{i'} 0

be two reduced extensions of $M$ by $L$ and let the corresponding transition matrices of $E$ and $F$ over $U$ be respectively

$$g_{ij} = \begin{pmatrix} a_{ij} & c_{ij} \\ 0 & b_{ij} \end{pmatrix} \quad \text{and} \quad h_{ij} = \begin{pmatrix} a_{ij} & d_{ij} \\ 0 & b_{ij} \end{pmatrix}.$$  

Let $\phi : E \sim F$ define an equivalence between $(\ast)$ and $(\ast)'$, i.e.,

$\phi : (\ast) \sim (\ast)'$. Then over the covering $U$, the isomorphism $\phi$ is represented by a collection $\{\phi_i\}_{i \in I}$ of $2 \times 2$ invertible matrices satisfying over every open set $U_i$ the equalities:

$$\phi_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0,1 \end{pmatrix} = \begin{pmatrix} 0,1 \end{pmatrix} \cdot \phi_i$$

It follows from this that the matrices $\phi_i$ are of the form

$$\phi_i = \begin{pmatrix} 1 & u_i \\ 0 & 1 \end{pmatrix},$$

where $u_i$ is a regular function on $U_i$. But then for every pair $i,j \in I$, the equality

$$h_{ij} = \phi_i \cdot g_{ij} \cdot \phi_j^{-1} \quad \text{over} \quad U_i \cap U_j$$

reads as

$$d_{ij} = c_{ij} + u_i b_{ij} - a_{ij} u_j \quad \text{over} \quad U_i \cap U_j.$$  

Therefore the reduced extensions $(\ast)$ and $(\ast)'$ correspond to the same element of $\mathbb{H}^1(X, \text{Hom}_X(M,L))$, and we have constructed in this way the required
partial map which is clearly one-to-one. It is also onto because every element of $H^1(X, \text{Hom}_X(M,L))$ corresponds to a collection $\{c_{ij}\}_{i,j \in I}$ satisfying $(TM)_R$ from which we can build a reduced extension of $M$ by $L$, using transition matrices $g_{ij}$ as above.

Now, consider an arbitrary extension

$\quad (**): \quad 0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0.

It corresponds to an exact sequence of locally free sheaves

$\quad [**]: \quad 0 \longrightarrow L \longrightarrow E \longrightarrow M \longrightarrow 0.

Over any open set $U_i \subset U$, the restricted exact sequence

$\quad [**]_i: \quad 0 \longrightarrow L|_{U_i} \longrightarrow E|_{U_i} \longrightarrow M|_{U_i} \longrightarrow 0$

is split. Choose for each $i \in I$ a splitting $h_i^0: M|_{U_i} \longrightarrow E|_{U_i}$ of $[**]_i$ and a morphism $\xi^i: E|_{U_i} \longrightarrow L|_{U_i}$ making the sequence

$\quad 0 \leftarrow L|_{U_i} \leftarrow E|_{U_i} \leftarrow M|_{U_i} \leftarrow 0$

exact. Let also the morphisms $f_1, g_1, h_i^0$, and $\xi^i$ be respectively represented over $U_i$ by matrices $f_i^1, g_i^1, h_i$, and $\xi_i^1$.

For each $i \in I$, we define the isomorphism

$\quad \Theta_i: L|_{U_i} \longrightarrow L|_{U_i} \oplus M|_{U_i}$

to be the one represented over $U_i$ by the matrix $\{f_i^1, g_i^1\}$, with

$\quad V^{f_i^1, g_i^1} \in \Gamma(U_i, \mathcal{O}_X) \otimes \mathcal{O}_X \otimes \mathcal{O}_X; \quad (f_i^1, g_i^1)u = f_i^1u + g_i^1u.$

Then $\Theta_i^{-1}$ is represented over $U_i$ by the matrix $\langle f_i^1, h_i^0 \rangle$, with

$\quad V^{f_i^1, h_i^0} \in \Gamma(U_i, \mathcal{O}_X); \quad \langle f_i^1, h_i^0 \rangle(u \oplus v) = f_i^1u + h_i^0v,$

and for every pair $i, j \in I$, the isomorphism
\[ \Theta^i_1 \circ \Theta^{-1}_j : \mathcal{L}|_{U_i \cap U_j} \otimes \mathcal{M}|_{U_i \cap U_j} \overset{\sim}{\rightarrow} \mathcal{L}|_{U_i \cap U_j} \otimes \mathcal{M}|_{U_i \cap U_j} \]

is represented over \( U_i \cap U_j \) by the matrix

\[ \begin{pmatrix} a_{ij} & f^i_j & g_{ij} & h^i_j \\ 0 & b_{ij} \end{pmatrix} \]

By gluing together the sheaves \( \mathcal{L}|_{U_i} \otimes \mathcal{M}|_{U_i} \) with the isomorphisms \( \Theta^i_1 \)

we get a locally free sheaf \( \mathcal{E}' \), and the isomorphisms \( \Theta^i_1 \) induce an isomorphism \( \phi : \mathcal{E} \rightarrow \mathcal{E}' \). Let \( \mathcal{E}' \) be the vector bundle corresponding to \( \mathcal{E}' \),

and let \( \phi : \mathcal{E} \rightarrow \mathcal{E}' \) be the isomorphism corresponding to \( \phi \). Then the extension

\[ 0 \rightarrow \mathcal{L} \longrightarrow \mathcal{E} \rightarrow \mathcal{E}' \rightarrow \mathcal{M} \rightarrow 0 \]

is reduced and equivalent to the original extension (**) . We have proved

**Proposition 4.1** The set \( E(M, \mathcal{L}) \) of equivalence classes of extensions of \( \mathcal{M} \) by \( \mathcal{L} \) is in bijection with the underlying set of the group

\[ H^1(X, \mathcal{O}_X, (M, \mathcal{L})) \cong H^1(X, \mathcal{M}^* \otimes \mathcal{O}_X, \mathcal{L}). \]

Under this bijection the class of the trivial extension, \( \mathcal{L} \otimes \mathcal{M} \), corresponds to the zero element of \( H^1(X, \mathcal{M}^* \otimes \mathcal{O}_X, \mathcal{L}) \).

Recall that \( H^1(X, \mathcal{M}^* \otimes \mathcal{O}_X, \mathcal{L}) \) has a structure of \( k \)-vector space. Then, if any two non-zero elements \( \xi, \xi' \in H^1(X, \mathcal{M}^* \otimes \mathcal{O}_X, \mathcal{L}) \) are on the same line passing through the origin, i.e., \( \exists \lambda \in k - \{0\} \); \( \xi' = \lambda \xi \), the rank 2 vector bundles induced by \( \xi \) and \( \xi' \) are isomorphic. In particular, when the projective space associated to \( H^1(X, \mathcal{M}^* \otimes \mathcal{O}_X, \mathcal{L}) \) has dimension zero, there exists, up to isomorphism, a unique indecomposable rank 2 vector bundle over \( X \) which can be written as an extension of \( \mathcal{M} \) by \( \mathcal{L} \).
The group structure of $H^1(X, \mathbb{Z}/\mathcal{X})$ induces a group structure on the set $E(M, L)$, and we have a description of the corresponding addition in $E(M, L)$ in terms of transition matrices. This addition is known as the Baer sum and its classical construction is possible in any abelian category $\mathcal{C}$.

Recall that the pushout in $\mathcal{C}$ of a diagram

$$
\begin{array}{c}
B \\
\downarrow f \\
A \\
\downarrow g \\
C
\end{array}
$$

in $\mathcal{C}$, is a commutative square

$$
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow u \\
D
\end{array}
\quad \begin{array}{c}
g \\
\downarrow v \\
C
\end{array}
$$

such that for every diagram

$$
\begin{array}{c}
B \\
x \\
\downarrow y \\
J \\
\downarrow y \\
C
\end{array}
$$

in $\mathcal{C}$ with $x \circ f = y \circ g : A \to J$, there exists a unique arrow $h : D \to J$ with $x = h \circ u$ and $y = h \circ v$. The object $D$ is usually written as $B \amalg_A C$, and is called the pushout of $f$ and $g$.

The pullback in $\mathcal{C}$ of a diagram

$$
\begin{array}{c}
B \\
f \\
\downarrow f \\
A \\
\downarrow g \\
C
\end{array}
$$

is defined the same way as a pushout but with all arrows reversed. The pullback of $f$ and $g$ is denoted $B \times_A C$.

In an abelian category every morphism has a kernel and a cokernel, and any pair of objects has a direct sum. We then have the following characterizations of pushouts and pullbacks:
For any diagram $B \leftarrow f A \rightarrow g C$ in $\mathcal{C}$, an object $D$ together with arrows $u : B \rightarrow D$ and $v : C \rightarrow D$ forms a pushout of $f$ and $g$ if and only if the sequence

$$(p_0): \quad \begin{array}{ccc}
A & \rightarrow & B \oplus C \\
\{f,g\} & \rightarrow & D \\
<u,-v> & \rightarrow & 0
\end{array}$$

is exact in $\mathcal{C}$, where $\{f,g\}$ and $<u,-v>$ are the obvious arrows given by the universal properties of $B \oplus C$.

Reversing all the arrows, an object $D$ together with arrows $u : D \rightarrow B$ and $v : D \rightarrow C$, forms a pullback of the diagram $B \leftarrow f A \rightarrow g C$, if and only if the sequence

$$(pb): \quad \begin{array}{ccc}
0 & \rightarrow & D \\
\{u,-v\} & \rightarrow & B \oplus C \\
<f,g> & \rightarrow & A
\end{array}$$

is exact in $\mathcal{C}$.

From this it follows that abelian categories have all pullbacks and pushouts. Now let

$$(e_x): \quad \begin{array}{ccc}
0 & \rightarrow & A \\
f & \rightarrow & B \\
g & \rightarrow & C \\
0 & \rightarrow & 0
\end{array}$$

be an exact sequence in an abelian category $\mathcal{C}$, and let $\phi : A \rightarrow A'$ be an arrow in $\mathcal{C}$. We can take the pushout in $\mathcal{C}$ of $f$ and $\phi$. Let $u : B \rightarrow B \amalg_A A'$ and $v : A' \rightarrow B \amalg_A A'$ be the corresponding arrows. By the defining property of pushouts there exists a unique arrow $w : B \amalg_A A' \rightarrow C$ such that $w \circ u = g$ and $w \circ v = 0$. We obtain in this way an exact sequence

$$\phi_x(e_x): \quad \begin{array}{ccc}
0 & \rightarrow & A' \\
v & \rightarrow & B \amalg_A A' \\
w & \rightarrow & C \\
0 & \rightarrow & 0.
\end{array}$$
The Definition 4.1 of equivalence classes of extensions of vector bundles makes sense for extensions in any abelian category. Again, we write $E(C,A)$ for the set of equivalence classes of extensions of $C$ by $A$ in $\mathcal{C}$. The assignment $(e_x) \mapsto \phi_x(e_x)$ gives a mapping:

$$\phi_* : E(C,A) \rightarrow E(C,A').$$

Similarly, if $\psi : C' \rightarrow C$ is any arrow in $\mathcal{C}$ we obtain an exact sequence

$$\psi_*(e_x) : 0 \rightarrow A \rightarrow B \times C' \rightarrow C' \rightarrow 0$$

by taking the pullback of $\psi$ and $g$. We write

$$\psi_* : E(C,A) \rightarrow E(C',A)$$

for the resulting map.

Let $E$ be any object in $\mathcal{C}$. Let $\Delta_E : E \rightarrow E \oplus E$ be the arrow in the sequence (po) corresponding to the pushout $E \sqcup_E E$ of $\text{Id}_E$ and $\text{Id}_E'$. Let $\Sigma_E : E \oplus E \rightarrow E$ be the arrow in the sequence (pb) corresponding to the pullback $E \times_{E, E} E$ of $\text{Id}_E$ and $\text{Id}_E'$. Given any two equivalence classes of extensions $\xi$ and $\xi'$ of $C$ by $A$, the Baer sum of $\xi$ and $\xi'$, denoted $\xi \boxplus \xi'$, is constructed as follows:

Let $\xi$ and $\xi'$ be represented by exact sequences

$$(e_x) : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

and

$$(e'_x) : 0 \rightarrow A \rightarrow B' \rightarrow C \rightarrow 0.$$ These extensions determine a unique exact sequence

$$(e_x) \oplus (e'_x) : 0 \rightarrow A \oplus A \rightarrow B \oplus B' \rightarrow C \oplus C \rightarrow 0.$$ The Baer sum $\xi \boxplus \xi'$ is defined to be the equivalence class of the extension

$$\Delta^*_C \circ (\Sigma_A)^* ((e_x) \oplus (e'_x)).$$
In the case where \( A \) and \( C \) are the line bundles \( \mathcal{L} \) and \( \mathcal{M} \) one can check directly, with the help of the exact sequences (po) and (pb) above, that the Baer sum on \( E(\mathcal{M}, \mathcal{L}) \) is in fact the group operation on \( E(\mathcal{M}, \mathcal{L}) \) induced by the bijection onto \( H^1(X, \text{Hom}_X(\mathcal{M}, \mathcal{L})) \) given in Proposition 4.1.

4.2 NORMALIZED REPRESENTATIVES

Lemma 4.1: Any \( \mathbb{P}^1 \)-bundle over the curve \( X \) is isomorphic to the projectivization of a rank two vector bundle having a non-zero section.

Proof: If \( \mathcal{P} \) is any \( \mathbb{P}^1 \)-bundle over \( X \) we can find by Proposition 2.3 an exact sequence of the type

\[
0 \longrightarrow I \longrightarrow E \longrightarrow \mathcal{M} \longrightarrow 0,
\]

where \( I \) is the trivial line bundle and \( E \) is a rank 2 vector bundle whose projectivization is isomorphic to \( \mathcal{P} \). But a non-zero morphism \( I \longrightarrow E \) is represented by matrices satisfying the same conditions as matrices representing a non-zero section of \( E \). So \( E \) is as required and has in fact a non-vanishing section. \( \square \)

Let \( E \) be a rank 2 vector bundle. A non-zero section \( s \) of \( E \) induces a local section of the \( \mathbb{P}^1 \)-bundle \( \text{Proj}(E) \). By Propositions 2.1 and 2.2, \( s \) corresponds to an exact sequence

\[
0 \longrightarrow L \overset{f}{\longrightarrow} E \longrightarrow \mathcal{M} \longrightarrow 0.
\]

Let \( \mathcal{L} \) and \( E \) have respective systems of transition matrices \( \{a_{ij}\}_{i,j \in I} \) and \( \{g_{ij}\}_{i,j \in I} \) over some open covering \( \mathcal{U} = \{U_i\}_{i \in I} \) of \( X \), and let \( f \) and \( s \) be respectively represented by collections \( \{f_i\}_{i \in I} \) and \( \{s_i\}_{i \in I} \).

After possibly refining the open covering \( \mathcal{U} \), we can find a collection
\( \{h_i\}_{i \in I} \) of rational functions such that for every \( i \in I \) the \( 2 \times 1 \) matrix \( h_i^{-1}s_i \) is nowhere zero on \( U_i \) and has entries in \( \mathbb{C}(U_i, O_X) \). Then, following the proof of Proposition 2.2, we see that for every pair \( i, j \in I \) the following equalities hold

\[
a_{ij} = \frac{h_i}{h_j} \quad \text{over } U_i \cup U_j \quad \text{and} \\
s_i = f_i h_i \quad \text{over } U_i.
\]

Since for every \( i \in I \) the entries of \( f_i \) are nowhere both vanishing regular functions on \( U_i \), the restriction to \( U_i \) of the divisor \( (h_i) \) is effective. So the collection \( \{h_i\}_{i \in I} \) is a system of local equations for an effective divisor \( (s) \) on \( X \), called the divisor of \( s \). This divisor is independent of the choice of a trivialization of \( E \), and \( \mathcal{L} = \mathcal{O}_X((s)) \).

In particular, \( \deg(\mathcal{L}) \geq 0 \). So we have

**Lemma 4.2**: Let \( E \) be a rank 2 vector bundle having a non-zero section \( s \).

Then we can find an exact sequence

\[
0 \longrightarrow \mathcal{L} \longrightarrow E \longrightarrow \mathcal{M} \longrightarrow 0,
\]

where \( \mathcal{L} = \mathcal{O}_X((s)) \).

We may now prove

**Proposition 4.2**: Let \( E \) be an indecomposable rank 2 vector bundle over the curve \( X \). If \( E \) has a non-zero section, then

\[
\deg(E) \geq -2g + 2
\]

where \( g \) is the genus of \( X \).

**Proof**: By Lemma 4.2 there exists an exact sequence

\[
0 \longrightarrow \mathcal{L} \longrightarrow E \longrightarrow \mathcal{M} \longrightarrow 0,
\]

with \( \deg(\mathcal{L}) \geq 0 \). Since \( E \) is indecomposable, Proposition 4.1 tells us
that the group $H^1(X,M^* \otimes L)$ is non-zero. So the divisor class of $M^* \otimes L$ is special and in particular $\deg(L) - \deg(M) \leq 2g - 2$. Hence

$$\deg(E) = \deg(M) + \deg(L) \\ \geq \deg(M) - \deg(L) \\ \geq -2g + 2.$$

It now makes sense to give the following definition:

**Definition 4.2:** Let $P$ be an indecomposable $\mathbb{P}^1$-bundle. A normalized representative of $P$ is a rank 2 vector bundle which is of minimal degree among those vector bundles having a non-zero section and whose projectivizations are isomorphic to $P$.

From what we said above, the following is immediate:

**Proposition 4.3:** Every indecomposable $\mathbb{P}^1$-bundle has a normalized representative.

In order to find all normalized representatives of indecomposable $\mathbb{P}^1$-bundles we will need an upper bound on their possible degrees. We first prove

**Lemma 4.3:** Let $E$ be a normalized representative for some indecomposable $\mathbb{P}^1$-bundle. If $L$ is any sub-line-bundle of $E$, then

$$\deg(L) \leq 0.$$  

In particular, non-zero sections of $E$ are nowhere vanishing.

**Proof:** By tensoring the corresponding exact sequence

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$$

with $L^*$ we obtain by exactness of the tensor product an exact sequence
So \( E \otimes L^* \) has a non-zero section and by assumption

\[
\deg(E) \leq \deg(E \otimes L^*) \\
\leq \deg(E) - 2 \deg(L).
\]

Hence \( \deg(L) \leq 0 \).

Now, if \( s \) is a non-zero section of \( E \) there exists by Lemma 4.2, an exact sequence

\[
0 \rightarrow [(s)] \rightarrow E \rightarrow M \rightarrow 0,
\]

and by the above result \( \deg(s) \leq 0 \). Since \( (s) \) is an effective divisor we must have \( (s) = 0 \), and this means that \( s \) is nowhere vanishing. ∎

We may now find an upper bound for the degrees of normalized representatives. The following proposition is the key point of the classification; it is essentially the Lemma 10 of [M.F. Atiyah (2), II].

**Proposition 4.4:** Let \( E \) be a normalized representative for some indecomposable \( F^1 \)-bundle over the curve \( X \) of genus \( g \). Then

\[
\deg(E) \leq 2g - 1.
\]

**Proof:** Suppose \( \deg(E) \geq 2g \). Then according to the Riemann-Roch theorem, we have the inequality:

\[
\dim_k \Gamma(E) \leq \deg(E) + \chi(0) - \chi(1) + \chi(2) + \ldots + \chi(g)
\]

So let \( s_1 \) and \( s_2 \) be linearly independent sections of \( E \), and let

\[
f_1 : [(s_1)] \rightarrow E \quad \text{and} \quad f_2 : [(s_2)] \rightarrow E
\]

be the corresponding injections. By Lemma 4.3, for any pair \( \lambda, \mu \in k \),
the section \( s_1 + \mu s_2 \) is nowhere vanishing on \( X \) unless \( \lambda = \mu = 0 \). So the intersection of the images of \( f_1 \) and \( f_2 \) in \( E \) is the image of the zero section of \( E \). But then \( E \cong \langle (s_1) \rangle \oplus \langle (s_2) \rangle \), by Proposition 1.2.

This is a contradiction, so \( \deg(E) = 2g - 1 \). □

4.3 CLASSIFICATION OF INDECOMPOSABLE BUNDLES

In the case where the curve \( X \) is of genus zero, we see by Propositions 4.2 and 4.4 that there is no normalized representative on \( X \). We have:

Proposition 4.5: Over a complete nonsingular curve \( X \) of genus zero, there is no indecomposable \( \mathbb{P}^1 \)-bundle, and hence there is also no indecomposable rank 2 vector bundle over \( X \).

From now on the curve \( X \) will be of genus one. In this case Propositions 4.2 and 4.4 say that normalized representatives on \( X \) are of degree either zero or one.

According to Proposition 4.1 we see from the equalities

\[
\dim_k H^1(X, \mathcal{O}_X^* \otimes \mathcal{O}_X) = \dim_k H^1(X, \mathcal{O}_X) = g = 1,
\]

that up to isomorphism there is only one rank 2 vector bundle over \( X \) which can be written as a non-trivial extension of \( I \) by \( I \). We denote it as \( E_0 \). By Proposition 4.2, \( E_0 \) is a normalized representative of degree zero.

In fact, we have

Lemma 4.4: Any two normalized representatives of degree zero are isomorphic.
Proof: By what we said above it is sufficient to show that any normalized representative of degree zero can be written as a non-trivial extension of $I$ by $I$. So let $E$ be one such representative. By Lemma 4.3 there exists a non-trivial extension

$$0 \rightarrow I \rightarrow E \rightarrow M \rightarrow 0$$

corresponding to a non-zero section of $E$. By Proposition 4.1 the $k$-vector space $H^1(X, M^*)$ is non-zero. The curve $X$ being elliptic, the canonical divisor class on $X$ coincide with the zero divisor class on $X$, and Serre's duality gives

$$\dim_k H^1(X, M^*) = \dim_k H^0(X, M).$$

Since $\deg(E) = \deg(M) = 0$, the $k$-vector space $H^0(X, M)$ is non-zero if and only if $M \not\cong \mathcal{O}_X$. So $E$ can be written as a non-trivial extension of $I$ by $I$. 

For normalized representatives of degree one, we have the following

Lemma 4.5: All indecomposable rank 2 vector bundles of degree one are normalized representatives.

Proof: Let $E$ be an indecomposable rank 2 vector bundle of degree one. By the Riemann–Roch theorem we have

$$\dim_k \Gamma(E) \geq \deg(E) = 1.$$

So $E$ has a non-zero section. On the other hand, let $L$ be any line bundle such that $E \otimes L$ has a non-zero section. By Proposition 4.2

$$0 \leq \deg(E \otimes L) = \deg(E) + 2 \deg(L).$$

Hence $\deg(L) \geq 0$ and

$$\deg(E \otimes L) \geq \deg(E).$$

So $E$ is a normalized representative as required. 

\[ \square \]
We also have

Lemma 4.6: Let $E$ and $F$ be indecomposable rank 2 vector bundles of degree one. Then $E$ and $F$ are isomorphic if and only if $\det(E) \cong \det(F)$.

Proof: The condition is clearly necessary. To prove it is sufficient take two exact sequences

$$0 \to I \to E \to L \to 0 \quad \text{and} \quad 0 \to I \to F \to M \to 0.$$ 

By Proposition 1.3, we have

$$L \cong \det(E) \quad \text{and} \quad M \cong \det(F).$$

So by assumption, we have $L \cong M$. The Riemann-Roch theorem says that

$$\dim_k H^0(X, M^*) - \dim_k H^1(X, M^*) = \deg(M).$$

Since $\deg(M^*) = -1$, we have $\dim_k H^0(X, M^*) = 0$, and hence $\dim_k H^1(X, M^*) = 1$. So according to the remark following Proposition 4.1, $E$ and $F$ are isomorphic. \qed

When computing the group $\Delta(X)$, we fixed a point $p_o$ on $X$. We now define $L_o$ to be the line bundle $[p_o]$ and fix a normalized representative $E_1$ with $\det(E_1) = L_o$. Also, let the respective projectivizations of $E_0$ and $E_1$ be $P_0$ and $P_1$. Since $E_0$ and $E_1$ do not have the same degree, $P_0$ and $P_1$ cannot be isomorphic. So up to isomorphism there are at least two indecomposable $P^1$-bundles over $X$. In fact

Proposition 4.6: Over the curve $X$ of genus one, there are exactly two isomorphism classes of indecomposable $P^1$-bundles.

Proof: Let $P$ be any indecomposable $P^1$-bundle over $X$, and let $E$ be a normalized representative for $P$. We know that $E$ has degree either
zero or one.

If \( \deg(E) = 0 \), then by Lemma 4.4 we have \( E \cong E_0 \), and therefore \( P \cong P_0 \).
If \( \deg(E) = 1 \), then again \( P \) and \( P_0 \) are not isomorphic. Let \( p \) be the unique point on \( X \) such that \( \det(E) \cong [p] \), and take any point \( q \) on \( X \) such that \( p + p_0 = 2 \cdot q \). Such a point \( q \) is a branch point of the degree two morphism from \( X \) into \( \mathbb{P}^1 \) induced by the complete linear system \( \mathcal{O}(p + p_0) \), and always exists according to the Hurwitz formula (cf. [Hartshorne, IV, 2]). Now let \( N \) be the line bundle \( \mathcal{O}(q - p_0) \). Since

\[
[p_0] \otimes [2 \cdot q - 2 \cdot p_0] = [2 \cdot q - p_0] \cong [p],
\]

we have \( \det(E_1 \otimes N) \cong \det(E) \). Hence by Lemma 4.6, \( E_1 \otimes N \cong E \) and \( P \cong P_1 \). 

Given any integer \( d \), we denote by \( E_X(d) \) the set of isomorphism classes of indecomposable rank 2 vector bundles of degree \( d \) over \( X \).

The assignment \( E \mapsto \det(E) \) induces a one-to-one correspondence between \( E_X(d) \) and \( E_X(d + 2) \). So in order to classify all indecomposable rank 2 vector bundles over \( X \), it will suffice to enumerate \( E_X(0) \) and \( E_X(1) \).

On the elliptic curve \( X \) the divisor classes of degree one are in one-to-one correspondence with the points of \( X \). So, by Lemma 4.5 and Lemma 4.6 the assignment \( E \mapsto \det(E) \) induces a one-to-one correspondence between \( E_X(1) \) and the set of points of \( X \).

On the other hand the set \( E_X(0) \) is the orbit of the isomorphism class of \( E_0 \) under the action of \( \Delta(X) \). This is because all normalized representatives of even degree are isomorphic to \( E_0 \).

We now show
Lemma 4.7: Let $L$ be a line bundle.

If $E \otimes L \cong E$, then $L \cong I$.

Proof: Suppose we have $E \otimes L \cong E$. Then $\deg(L) = 0$ and there exists an exact sequence

$$0 \rightarrow I \rightarrow E \otimes L \rightarrow I \rightarrow 0,$$

which gives, when tensored by $L^*$, an exact sequence

$$0 \rightarrow L^* \rightarrow V \rightarrow L^* \rightarrow 0.$$

Consider the corresponding cohomology sequence

$$0 \rightarrow \Gamma(L^*) \rightarrow \Gamma(E) \rightarrow \Gamma(L) \rightarrow \cdots$$

and suppose that $L$ is not a trivial line bundle. The divisor class of $L^*$ is then non-zero and $\Gamma(L^*) = 0$. By exactness of the cohomology sequence this implies $\Gamma(E) = 0$ and contradicts the definition of $E$. So we must have $L \cong I$.

We have also proven

Proposition 4.7: There is a one-to-one correspondence between $E_x(0)$ and the set of points of $X$.

To sum up

Theorem 4.1: When the curve $X$ has genus zero, all $\mathbb{P}^1$-bundles and rank 2 vector bundles over $X$ are decomposable.

When the curve $X$ has genus one there are exactly two isomorphism classes of indecomposable $\mathbb{P}^1$-bundles over $X$, and the set $E_x(d)$ of isomorphism classes of indecomposable rank 2 vector bundles of given degree $d$ is in one-to-one correspondence with the set of points of $X$. 
### SUMMARY

<table>
<thead>
<tr>
<th>$\mathbb{P}^1$-BUNDLES</th>
<th>RANK 2 VECTOR BUNDLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decomposable</td>
<td>Indecomposable</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\mathbb{Z} \times \mathbb{X}/\pm 1$</td>
<td>${P_0, P_1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\text{LINE BUNDLES}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$</td>
</tr>
<tr>
<td>$\cup \mathbb{X}$ $d \in \mathbb{Z}$</td>
</tr>
</tbody>
</table>

$g = 0$ $g = 1$
BIBLIOGRAPHY

M. F. ATIYAH:


C. CHEVALLEY:


R. GANONG AND P. RUSSELL:


A. GROTHENDIECK:


R. HARTSHORNE:

Algebraic Geometry, Graduate Text in Mathematics #52, Springer-Verlag, 1977.
J.P. SERRE:


I.R. SHAFAREVICH:


A.N. TJURIN: