BEHAVIOUR OF ORTHOTROPIC
BRIDGE DECKS

by

René A. Tinawi, M.Sc.(Eng.), D.I.C.
To
my wife Liliane
and to
the memory of my father
COMPORTEMENT DES PLATELAGES DE PONTS ORTHOTROPIQUES

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RESUME

Le présent ouvrage comprend une étude du platelage des ponts orthotropiques par la méthode des éléments finis. Un élément rectangulaire compatible pour voiles minces et tenant compte des rotations coplanaires est formulé pour l'idéalisation de la dalle de patelage. Ce même élément est utilisé pour les raidisseurs de type fermé; pour les raidisseurs ouverts, un élément conforme du type "poutre excentrée" est présenté comme alternative. Les résultats obtenus se comparent favorablement aux valeurs expérimentales disponibles.

L'effet du changement de l'espacement des raidisseurs et des poutres transversales est également discuté. Des comparaisons sont établies avec les méthodes de calcul courantes et des suggestions sont faites pour l'augmentation de l'espacement des nervures en vue d'une plus grande économie dans la fabrication.

Les effets de la non-linéarité géométrique ont été étudiés au moyen d'éléments triangulaires pour le cas de raidisseurs trapézoïdaux à grand espacement.
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ABSTRACT

An analysis of orthotropic bridge decks is presented using the finite element technique. A compatible rectangular shell element with in-plane rotations is developed for simulation of the deck plate. For closed-type ribs, the same element is used; whereas, for open-type ribs, a compatible eccentric beam element is presented as an alternative. The results compare favourably with available experimental data.

An investigation into the effect of varying the stiffener and cross-beam spacings is also described. Comparisons are made with current design practice and suggestions, to increase the standard rib spacing, are made in order to achieve greater economy in the fabrication process.

Geometric nonlinearities are also studied, using triangular shell elements, for the case of trapezoidal stiffeners with large spacings.
ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

The symbols used in this thesis are defined in the text. However, for convenience of reference, these are summarized below for each chapter individually. Bold-type characters refer to matrix notation.

\[
\begin{align*}
\{ \} & \text{ Square or rectangular matrix} \\
\{ \} & \text{ Column matrix} \\
\Gamma & \text{ Diagonal matrix} \\
A^t & \text{ Transpose of matrix } A \\
I_n & \text{ Unit matrix of order } n \times n \\
0 & \text{ Null matrix}
\end{align*}
\]

CHAPTER 1

\[
\begin{align*}
D_x & \text{ Flexural rigidity in } x\text{-direction} \\
D_y & \text{ Flexural rigidity in } y\text{-direction} \\
H & \text{ Effective torsional rigidity} \\
w & \text{ Transverse displacement of plate} \\
p(x,y) & \text{ Loading on plate}
\end{align*}
\]
## CHAPTER 2

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<tr>
<td>$a,b,c$</td>
<td>Dimensions of a triangular subregion</td>
</tr>
<tr>
<td>$\bar{a},\bar{b},\bar{c}$</td>
<td>Dimensions of a triangular element</td>
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<tr>
<td>$E$</td>
<td>Elastic modulus</td>
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<td>$E_{ij}$</td>
<td>Component of elasticity matrix</td>
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<td>$F(m,n)$</td>
<td>Integral value of $\xi^m \eta^n$ over subregion area</td>
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<td>$m_i, n_i$</td>
<td>Exponent of polynomial expansion</td>
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<td>$q_0$</td>
<td>Pressure over subregion</td>
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<tr>
<td>$S_{ij}$</td>
<td>Length between points $i$ and $j$</td>
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<td>$t_i$</td>
<td>Thickness at point $i$</td>
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<td>$U_e$</td>
<td>Strain energy of a subregion</td>
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<td>$U_{eT}$</td>
<td>Strain energy of a triangle</td>
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<td>$V_e$</td>
<td>Work done for a subregion</td>
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<td>$V_{eT}$</td>
<td>Work done for a triangle</td>
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<td>$w$</td>
<td>Transverse displacement function</td>
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<td>$w_i$</td>
<td>Transverse deflection at point $i$</td>
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<td>( w_{\xi \xi} )</td>
<td>Partial derivatives of ( w )</td>
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<td>( X_i, Y_i, Z_i )</td>
<td>Global coordinates of node ( i )</td>
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<tr>
<td>( \bar{x}_i, \bar{y}_i )</td>
<td>Coordinates of node ( i ) in quadrilateral axes</td>
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<td>( x, y )</td>
<td>Cartesian coordinate axes of a triangle</td>
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<td>( z )</td>
<td>Distance from neutral plane of plate</td>
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<td>( A_i, A_j, A_k )</td>
<td>Matrices relating nodal degrees of freedom to polynomial constants for the subregions</td>
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<td>( A_n, A_p, A_q )</td>
<td>Matrices relating nodal degrees of freedom to polynomial constants for the subregions</td>
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<td>( A_{oo}, A_{or}, A_{rr} )</td>
<td>Partitioned submatrices</td>
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<td>( E_T )</td>
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<td>( \bar{E} )</td>
<td>Superdiagonal elasticity matrix</td>
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<td>( G )</td>
<td>Matrix relating constants to triangle degrees of freedom</td>
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<td>( H )</td>
<td>Superdiagonal matrix of ( \psi_0 )</td>
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<td>( k^l )</td>
<td>Stiffness matrix of subregion ( l )</td>
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<td>( \bar{k} )</td>
<td>Stiffness matrix of combined subregions</td>
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<td>Symbol</td>
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<td>$K_{11}K_{12}K_{22}$</td>
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<td>$M_Q$</td>
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<td>$\bar{P}$</td>
<td>Pressure vector for a general triangle</td>
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<tr>
<td>$P^\ell$</td>
<td>Pressure vector for triangle $\ell$</td>
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\( r \) Total triangle degrees of freedom in triangle axes

\( r_i \) Degrees of freedom at node \( i \)

\( r_T \) Total triangle degrees of freedom in quadrilateral axes

\( r_Q \) Quadrilateral degrees of freedom

\( r_Q^* \) Reduced quadrilateral degrees of freedom

\( r_{Q2} \) Interior quadrilateral degrees of freedom

\( R_c \) Subregion consistent load vector

\( R_{\sigma T} \) Triangle consistent load vector

\( R_{\sigma Q} \) Quadrilateral consistent load vector

\( R_{\sigma Q}^* \) Reduced quadrilateral consistent load vector

\( R_Q \) Force vector corresponding to \( r_Q^* \)

\( S_T \) Triangle stress matrix

\( T \) Diagonal matrix of thicknesses

\( \alpha_i \) Constant in polynomial expansion

\( \alpha, \beta, \gamma, \theta \) Angles between subregions, triangles and quadrilateral axes
\( \theta_x, \theta_y \) Rotation degrees of freedom about \( x \) and \( y \)

\( \theta_\xi, \theta_\eta \) Rotation degrees of freedom about \( \xi \) and \( \eta \)

\( \theta_n, \theta_p, \theta_q \) Normal slopes at \( n, p \) and \( q \)

\( \tau_i \) Thickness constant

\( \nu \) Poisson's ratio

\( a \) Vector of constants

\( a^l \) Vector of constants for subregion \( l \)

\( \bar{a} \) Total number of constants for the three subregions

\( a_r, a_o \) Subvectors of \( \bar{a} \)

\( \epsilon \) Strain vector in subregion axes

\( R \) Reduction matrix for quadrilateral degrees of freedom

\( \psi \) Transformation for elasticity matrix

\( \bar{\psi} \) Superdiagonal matrix of \( \psi \) inverse

\( \psi_\theta \) Bending moments transformation matrix

\( \phi \) Transformation matrix of triangle degrees of freedom to quadrilateral axes

\( \phi \) Transformation matrix of nodal degrees of freedom to quadrilateral axes
CHAPTER 3

\( \phi \)  
Matrix \( \phi \) for triangle \( \lambda \)

\( \rho \)  
Curvatures vector in triangle axes

\( \rho_S \)  
Curvatures vector in subregion axes

\( \rho_{S_i} \)  
Curvatures vector evaluated at node \( i \) of subregion

\( \sigma \)  
Stress vector

**A**  
Area of rectangle

**a, b**  
Sides of element

**E_{i,j}**  
Component of elasticity matrix

**F(m,n)**  
Integral value of \( \xi^m \eta^n \) over rectangle area

**p**  
Boundary load varying quadratically

**u**  
Polynomial expansion for in-plane displacement in \( \xi \) direction

**u_\xi, v_\eta, v_\xi**  
Displacements at node \( i \)

**u_\eta, v_\xi**  
Slope at node \( i \)

**v**  
Polynomial expansion for in-plane displacement in \( \eta \) direction

**U_e**  
Strain energy
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<td>Thickness at node $i$</td>
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<td>Strain matrix</td>
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<td>$C$</td>
<td>Matrix relating constants to degrees of freedom</td>
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<td>$\bar{C}$</td>
<td>&quot;Constraint&quot; matrix</td>
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<td>Elasticity matrix</td>
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<td>$\bar{E}$</td>
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<td>Stiffness matrix in terms of constants</td>
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<td>$K$</td>
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<td>$K^*$</td>
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<td>$N$</td>
<td>Polynomial vector for boundary loading</td>
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<td>$P$</td>
<td>Vector of work done by boundary loading</td>
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<td>&quot;Reduced&quot; consistent load vector</td>
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<td>$r$</td>
<td>Total rectangle degrees of freedom</td>
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<td>$r^*$</td>
<td>Degrees of freedom corresponding to $K^*$</td>
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\( r_i \)  
Degrees of freedom at node \( i \)

\( S_R \)  
Stress matrix

\( \alpha_i \)  
Typical constant in polynomial expansion

\( \beta_i \)  
Constant for boundary loading

\( \xi, \eta \)  
Local rectangle axes

\( \tau_i \)  
Thickness constant

\( a_u \)  
Column matrix of constants for \( u \) displacements

\( a_v \)  
Column matrix of constants for \( v \) displacements

\( a \)  
Column matrix of combined constants for \( u \) and \( v \)

\( \epsilon \)  
Strain vector

\( \bar{\epsilon} \)  
Strain vector at corners and centroid of rectangle

\( \bar{\sigma} \)  
Stress vector at corners and centroid of rectangle

\( \sigma_i \)  
Stress vector at typical point in rectangle

**CHAPTER 4**

\( A \)  
Cross-sectional area of beam element

\( a_{ij} \)  
Direction cosines
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<td>( e_y, e_z )</td>
<td>Eccentricities of nodes for beam element</td>
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<tr>
<td>G</td>
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<td>( I_y, I_z )</td>
<td>Inertia about y and z axes of beam element</td>
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<td>L</td>
<td>Length of beam element</td>
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<td>( S_{ij} )</td>
<td>Length of side ij of shell element</td>
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<td>u</td>
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<td>( u_i, v_i, w_i )</td>
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<tr>
<td>( u'_i, v'_i, w'_i )</td>
<td>Displacements at node i at eccentric axes</td>
</tr>
<tr>
<td>( \overline{u}_i, \overline{v}_i, \overline{w}_i )</td>
<td>Displacements at node i in global axes</td>
</tr>
<tr>
<td>( u_{12} )</td>
<td>Mid-side axial displacement of beam element</td>
</tr>
<tr>
<td>( u_{ij} )</td>
<td>Mid-side displacement of shell element</td>
</tr>
<tr>
<td>( U_e )</td>
<td>Strain energy</td>
</tr>
<tr>
<td>w</td>
<td>Transverse displacement function</td>
</tr>
<tr>
<td>x</td>
<td>Longitudinal axis of beam element</td>
</tr>
<tr>
<td>( x_L, y_L, z_L )</td>
<td>Local axes of shell element</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
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<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>X, Y, Z</td>
<td>Global axes</td>
</tr>
<tr>
<td>K</td>
<td>Stiffness matrix of a concentric beam element</td>
</tr>
<tr>
<td>K'</td>
<td>Stiffness matrix of an eccentric beam element</td>
</tr>
<tr>
<td>K_A</td>
<td>Axial stiffness contribution for beam</td>
</tr>
<tr>
<td>K_T</td>
<td>Torsion stiffness contribution for beam</td>
</tr>
<tr>
<td>K_{By}</td>
<td>Bending stiffness about y axis</td>
</tr>
<tr>
<td>K_{Bz}</td>
<td>Bending stiffness about z axis</td>
</tr>
<tr>
<td>K^*</td>
<td>Local membrane stiffness matrix for rectangle</td>
</tr>
<tr>
<td>K'_Q</td>
<td>Local bending stiffness matrix for quadrilateral</td>
</tr>
<tr>
<td>K_M</td>
<td>Global membrane contribution</td>
</tr>
<tr>
<td>K_B</td>
<td>Global bending contribution</td>
</tr>
<tr>
<td>K</td>
<td>Global total stiffness matrix</td>
</tr>
<tr>
<td>r</td>
<td>Beam degrees of freedom about centroidal axes</td>
</tr>
<tr>
<td>r'</td>
<td>Beam degrees of freedom about eccentric axes</td>
</tr>
<tr>
<td>r</td>
<td>Global degrees of freedom for beam or shell element</td>
</tr>
</tbody>
</table>
Local bending degrees of freedom for quadrilateral

Local membrane degrees of freedom for rectangle

Force vector corresponding to local centroidal degrees of freedom of beam

Force vector corresponding to local eccentric degrees of freedom of beam

Global force vector corresponding to \( \bar{F} \)

Eccentricity matrix for beam element

Constant in polynomial expansion

Non-dimensional longitudinal beam axis

Rotations about y and z local axes

Mid-side normal slope

Transformation matrix of direction cosines for bending contribution

Transformation matrix of direction cosines for membrane contribution

Transformation matrix of direction cosines for general beam element
CHAPTER 7

$V_e$ Potential energy due to in-plane stresses

$w$ Transverse displacement

$z$ Distance from neutral plane

$k'$ Geometric stiffness for a subregion

$\overline{K}'$ Geometric stiffness for combined subregions

$K_E$ Elastic local bending stiffness matrix for triangle

$K_G$ Local geometric stiffness matrix for triangle

$\overline{K}_E$ Combined elastic global stiffness matrix for triangle

$\overline{K}_G$ Global geometric stiffness matrix for triangle

$r$ Local triangle degrees of freedom

$\overline{r}$ Global triangle degrees of freedom

$\Delta \overline{r}$ Incremental displacement vector

$R$ Local force vector

$\overline{R}$ Global force vector

$\Delta \overline{R}$ Incremental force vector
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_\xi', \varepsilon_\eta', \varepsilon_\xi \eta$</td>
<td>Local bending strains in subregion axes</td>
</tr>
<tr>
<td>$\tau_\xi', \tau_\eta', \tau_\xi \eta$</td>
<td>Local in-plane stresses in subregion axes</td>
</tr>
<tr>
<td>$\beta_{kx}^x, \beta_{kx}^y, \beta_{xy}^k$</td>
<td>Constants for linear in-plane stress distribution</td>
</tr>
</tbody>
</table>
STATEMENT OF ORIGINAL CONTRIBUTION

1. The development of a rectangular shell element with 32 degrees of freedom making full use of automatic generation and including material anisotropy and taper in thickness.

2. The development of an eccentric general beam element compatible with the shell element for the simulation of torsionally weak ribs or cross-beams.

3. An algorithm to specify automatically the degrees of freedom pertaining to the mid-side nodes without reference to the data input.

4. Provision of a tool by which the behaviour of the class of structures considered can be accurately investigated. This has been used to indicate that changes from current practice are possible such that the economics of fabrication are improved and the effects of which cannot be predicted by previously available methods of analysis.

5. The investigation of a three-dimensional geometrically non-linear analysis for bridge decks with closed ribs at large intervals. A new triangular shell element is proposed for this analysis.
CHAPTER 1

INTRODUCTION

1.1 THE ORTHOTROPIC DECK: DEFINITION AND GEOMETRY

The basic system of an orthotropic steel bridge deck, Fig. 1.1, consists of a steel plate supported by and welded to stiffeners placed in two mutually perpendicular directions. The longitudinal stiffeners are referred to as ribs, or stiffeners, and the transverse stiffeners are called cross-beams or cross-girders. The whole deck is supported by the main girders or may form part of the top flange of a box bridge.

Two types of ribs are usually considered in the design of the deck: the open type (torsionally weak) and the closed type (torsionally stiff). The open ribs are usually manufactured as inverted T-sections, flats, angles or bulb sections, etc., that is, sections which possess negligible torsional rigidity. The closed stiffeners can have rectangular, trapezoidal or semi-circular cross-sections, all of which offer considerable resistance to torsion due to the closed attachment to the deck.

The cross-beams are usually made of heavy inverted T-sections, and these are welded to both the deck plate and the longitudinal ribs. The whole deck, which consists of the top
plate, the ribs and the cross-beams, forms one complete monolithic unit, which is the key to efficient utilization of steel and maximum reduction of the dead weight of the structure.

Since the behaviour of a cross-stiffened deck may be likened to that of a plate having dissimilar elastic properties in two mutually perpendicular directions, known as orthogonal-anisotropic plate (or orthotropic plate), the steel plate deck for bridges of this type are often referred to as "orthotropic steel decks".

1.2 THE PROBLEM

The design of orthotropic decks is governed by individual truck wheel loads. When the stiffened plate is loaded by such wheel loads applied directly to a rib, some of the load is shed from the loaded stringer to the adjacent ones. The torsional rigidity plays an important role in this transverse dispersion of the load. From the stiffeners, the load is then carried by the cross-beams to the main girders then, finally, to the foundations.

The stresses in any member of a loaded steel plate bridge deck, and especially in the deck plate, are due to the combined effects of the various functions performed by the deck in the bridge structure. In the design stage of the deck, the following structural component systems are treated separately:
System I - The steel plate deck and the longitudinal stiffeners acting as part of the main carrying members of the bridge. That is, acting as top flange to the main girders.

System II - The stiffened plate deck, consisting of the longitudinal ribs and the transverse floor beams and the deck plate as their common upper flange, acting as a bridge floor.

System III - The deck plate acting locally as an isotropic plate directly supporting the concentrated wheel loads placed between the ribs and transmitting the reactions to the ribs.

It is seen that the function of the steel deck plate, participating in all three structural systems, is especially significant. Furthermore, the resulting systems are interrelated contrary to the design assumptions.

The purpose of this work is to analyse and understand the behaviour of stiffened decks, not as an equivalent continuum, but as an actual monolithic unit where the interactions of all the systems are considered. To date, the finite element method is the most appropriate tool for analysis of this kind. Special emphasis, in this work, will be placed on the closed stiffeners of trapezoidal shape. The case of open stiffeners will be restricted to inverted T-sections.
1.3 REVIEW OF EXISTING LITERATURE

Many investigators have attempted to simplify the analysis of a stiffened plate by idealizing it into another structure for which the solution can be obtained.

One of the early attempts was to visualize the stiffened plate as an equivalent grillage. An orthotropic deck is not a true grillage since the system formed by intersecting ribs and floor beams is rigidly connected to the deck plate. However, by selecting suitable section properties, an appropriate equivalent grillage system can be obtained providing the stiffeners are closely spaced. Many methods of analysing regular grids have been proposed such as moment distribution, the force method involving calculation of redundancies or a harmonic analysis as proposed by Hendry and Jaeger.

Another approach to predict the behaviour of stiffened plates is to replace such a system by an equivalent orthotropic plate of constant thickness having the same stiffness characteristics. The differential equation of equilibrium for materially orthotropic plates in flexure was first developed by Huber:

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p(x,y) \quad (1.1)
\]

where \( D_x \) and \( D_y \) are the flexural rigidities of the plate in the \( x \) and \( y \) directions respectively, and \( H \) the effective torsional...
A great number of publications have appeared since the development by Huber, and these can be classified into two categories: work dealing with the evaluation of the flexural and torsional rigidities, and work on the various methods of solution as given by Timoshenko and Lekhnitskii. An interesting historical review can be found in the AISC Design Manual and the book by Troitsky.

In the more recent literature, Chu and Krishnamoorthy, Cusens and Pama presented solutions to Huber's equation where the original work of Cornelius, Guyon and Massonet is generalized. However, no account was taken of the eccentricity effects which will be described later. It should be noted that the procedure recommended by the AISC Design Manual, largely based on the work of Pelikan and Esslinger also ignores the effects of eccentricity.

The rigorous concept of treating orthogonally stiffened plates by an equivalent orthotropic plate of constant thickness assumes that the ribs are disposed symmetrically with respect to the middle surface. However, a typical bridge deck consists of a plate reinforced by ribs located on only one side of the plate, thus disposed asymmetrically with respect to the middle surface. This eccentricity introduces deformations and shear stresses into the plate. Therefore, an "exact" analysis of the
problem should include the effect of in-plane strains which are disregarded in Huber's equation. One of the most widely accepted approximate methods for torsionally soft ribs makes use of the concept of effective stiffness introduced by Giencke. This concept is based on the assumption that the longitudinal strain is zero at the adjusted centroids of the cross-sections.

A further improvement was developed by Clifton, Chang and Au treating orthotropic decks with eccentrically closed or open stiffeners. A set of three coupled fourth-order partial differential equations were obtained for $u, v$ and $w$ displacements. However, by using single Fourier series functions for the displacements, an eighth-order partial differential equation, in terms of $w$ only, was obtained. The same results were also shown by Vitols, Clifton and Au for the case of a slab with longitudinal girders attached to it. Since the effect of stiffeners is always "smeared out" over the plate width, the interpretation of the plate and stiffener moments is difficult. In any case, the computation procedures presented in these studies are far too complicated to be considered for engineering applications.

Numerical solutions, such as finite difference, based on Huber's equation have been introduced recently by Heins and Looney and Dowling, taking into account the girders and cross-beam flexibilities. Adotte, also using finite differences, introduced a second order theory for orthotropic plates in flexure.
Another approach, based on a Fourier series representation of the bending moments that satisfies the equilibrium conditions of the plate, has been proposed by Coull\textsuperscript{21}. The drawback of this method is the difficulty in evaluating the deflections of the plate.

All the approaches discussed above are restricted to geometrically simple boundary conditions and uniformity in both the shape and spacing of the stiffeners. The finite element method, however, possesses the versatility and accuracy desired by the engineer and does not require the "smearing out" of the stiffeners. Furthermore, it is not restricted to special geometries or boundary conditions. Orthotropic plates analysed by finite elements are reviewed in the next section.

1.4 THE FINITE ELEMENT METHOD

The finite element method has been used extensively during the past decade. An important factor that speeded the development of the method is the advent of the faster, larger and more reliable digital computers.

A number of books, dealing with matrix methods, have appeared such as those by Livesley\textsuperscript{22}, Gere & Weaver\textsuperscript{23}, Martin\textsuperscript{24} and Meek\textsuperscript{25}. More recently, books treating solely the subject of finite elements, have been published by Zienkiewicz\textsuperscript{26 27}, Przemieniecki\textsuperscript{28}, Holland & Bell\textsuperscript{29}. Conferences entirely devoted
to the subject were held at Wright-Patterson Air Force Base, Ohio, Nashville, Tennessee, Japan and Germany.

In this vast amount of literature, the number of papers dealing with stiffened plates analysed by finite elements is very limited. Zienkiewicz & Cheung used rectangular two-dimensional flexural elements and modified the elasticity matrix to account for the effects of orthotropy. Fam used the same approach with triangular elements. Powell & Ogden used also two-dimensional flexural orthotropic elements but the displacement variation within the element was represented by a Levy-type series with a cubic polynomial assumption for each harmonic. This approach is called finite segments. Similarly, Willam & Scordelis used both finite segments and finite elements for open-type stiffeners but included the effects of eccentricity by introducing suitable coupling between the membrane and bending actions of the deck. Nevertheless, all the approaches discussed above, required the "smearing out" of the stiffeners.

Among researchers who treated the stiffeners of the deck as discrete elements, the work of Gustafson & Wright is a typical example where the rib stiffness matrix is evaluated independently about the plate nodal points rather than the centroid of the stiffener. Coupling is automatically introduced, in this case, between the membrane and bending actions. Furthermore, the internal forces in the deck plate and the ribs can be
evaluated separately without reference to the concept of effective width. Mehrain, Eka and McBean presented a number of different idealizations for both the deck and the attached stiffeners. In the latter work, conforming elements were used throughout. Harris, also used conforming elements but his mathematical model treated concentric ribs only. Olson & Lindberg, in using higher order triangular elements for the dynamics of stiffened panels, neglected the effects of in-plane deformations.

All the finite element models developed above for concentric and eccentric stiffeners treated only the case of torsionally weak stiffeners (open-type). The reason is that an open-type rib joins the plate nodes at just one point and, if all the stiffness relations are with respect to the plate nodes, then a two-dimensional grid of elements is obtained. This has the added advantage of being able to choose different polynomial functions for both the in-plane and bending elements. Similarly, the rib stiffness matrix would have corresponding functions for the coupled axial and flexural behaviour.

In the case of closed stiffeners, the deck structure has to be idealized by a three-dimensional model. A recent independent study by Dowling, using non-conforming elements, has demonstrated the capability of simulating closed stiffeners using rectangular shell elements. In this work, conforming elements are used throughout for the case of open or closed stiffeners.
1.5 OBJECT AND SCOPE

The principal objectives of this investigation are as follows:

(a) The formulation of conforming finite elements suitable for idealizing open or closed-type stiffeners.

(b) The development of a computer program for the analysis of any type of stiffened plates.

(c) The verification of the finite element predictions with available experimental tests or other possible theoretical solutions.

(d) To study the effect of some of the principal parameters entering into the problem, such as stiffener spacing, for torsionally weak or torsionally stiff ribs.

(e) Comparison of the theoretical predictions with the design method proposed by AISC.

(f) To conduct an exploratory study on the problem of geometric non-linearities in the case of closed stiffeners.

The material is assumed elastic throughout. Material non-linearities due to plasticity effects are not considered. It is not the objective of this investigation to propose a new design code for orthotropic decks. However, it is intended to shed some light into the complex three-dimensional behaviour of such structures.
1.6 OUTLINE OF THE THESIS

The fundamentals of the finite element method will not be reported here. It will be assumed that the reader is familiar with the basic concepts. However, details will be given as to the formulation of the stiffness matrices since the approach adopted here makes full use of automatic generation techniques to avoid lengthy, if not impossible, algebraic derivations.

Chapter 2 contains the derivation of a conforming quadrilateral element in bending. This element is identical to the development by Fraeijs de Veubeke but the derivation is based on ideas from Clough & Felippa. Automatic generation of the stiffness matrices as developed by Cowper et al., is used. Anisotropy in material properties, as well as a linear variation in the element thickness, have been included as shown by Tinawi.

Chapter 3 contains the development of a new rectangular plane stress element with in-plane rotations as generalized freedoms acting at the corners of the element. Care was taken to insure that the element is compatible with its plate bending partner when the combination is used to form a rectangular shell. A linear variation of the stress was permitted by the introduction of a mid-side displacement.

Chapter 4 describes the development of a general eccentric beam element which would serve to simulate open stiffeners or
cross beams. The axial, bending and torsional behaviour of the element is again compatible with the elements mentioned above. Also, the combination of the membrane and bending contributions to form a shell element in a three-dimensional frame of reference is discussed.

Chapter 5 describes the computer program, in general terms, and the algorithm used in assembling the stiffness matrices corresponding to the structural elements without reference to the mid-side nodes in the data input. Comparison is made between the finite element results and the folded plate theory by Scordelis \(^{39}\) for two examples with open stiffeners. Two experimental tests by Erzurumlu \(^52\) and Toprac \(^52\) on single cell closed stiffeners are used to check the validity and accuracy of the computer results. Finally, the results of a half-scale model of a deck with six closed stiffeners, as tested by Dowling \(^19\), are used to compare the behaviour of the mathematical model with a realistic situation.

Chapter 6 describes the behaviour of torsionally stiff (trapezoidal sections) and torsionally weak (inverted T-sections) orthotropic decks when the stiffener spacings are varied but keeping the total cross-section area constant. The increase in rib spacing is compensated by either thicker stiffeners or an increase in the deck plate thickness. Comparisons are made with the proposed AISC design formulae.

Chapter 7 contains studies into the effect of geometric
non-linearities in the elastic range. In this case, a triangular, rather than a rectangular shell element, is used. The flexural part of the element is described and is similar to the concepts introduced in Chapter 2. For the membrane part, a new element with in-plane rotations is proposed. A step-by-step linear incremental approach is used to investigate the existence of geometric non-linearities for one particular case of closed stiffeners where the stiffener spacing is rather large.

Finally, in Chapter 8, the work is briefly summarized and conclusions are drawn. Recommendations are also made for further research in this relatively unexplored territory.
Types of ribs

(a) Deck with open ribs

(b) Deck with closed ribs

FIGURE 1.1
BASIC TYPES OF STEEL PLATE BRIDGE DECKS
(taken from ref. 6)
CHAPTER 2

QUADRILATERAL PLATE BENDING ELEMENT

2.1 INTRODUCTION

This chapter is concerned with the formulation of a plate bending finite element. The importance of developing an "accurate" element is necessary since the element is intended to represent the local bending behaviour in the deck plate under wheel loads.

The formulation of plate bending elements has been covered extensively in the literature. These can be classified into three main categories:

(a) Displacement models based on minimum potential energy

(b) Equilibrium models based on minimum complementary energy

(c) Mixed models based on Reissner's principle.

This classification has been reported by Pian and Tong.

The displacement models for flat plate bending elements have been the most widely used. A quadrilateral element is presented in this chapter based on a displacement model as originally presented by Fraeijs de Veubeke. The reason for
the choice of this element in particular, will be apparent from the discussion in the next section. Linear variation in thickness and material anisotropy have been added to the original formulation by treating the quadrilateral as an assemblage of four triangles as proposed by Clough & Felippa.

2.2 ELEMENT CHOICE

The importance of selecting proper displacement functions to describe the element behaviour has been reviewed in a survey paper by Gallagher. The chosen function, to yield monotonic convergence of the strain energy, must satisfy the following conditions:

(a) Inter-element continuity of displacements and normal slopes

(b) Proper representation of rigid body motion states

(c) Inclusion of all pertinent constant strains.

One of the earlier developments for a rectangular element is based on a 12-term polynomial where a full cubic expansion, in x and y, for the w displacement plus two quartic terms are chosen. Twelve freedoms (a translation plus two rotations at each corner node) describe the element displacement. This element, as developed by Melosh and described by Zienkiewicz, converges, but violates the normal slope continuity. Other rec-
tangular elements based on beam-functions lacked a constant twist term and, therefore, did not converge to the correct solution.

In the case of triangular elements, the problem is more complicated since the full cubic expansion results in ten terms while only nine freedoms are required. A pair of terms, such as \(x^2y\) and \(xy^2\), could be combined to produce a 9-term representation, but as noted by Tocher, this is prone to singularities for certain geometries. On the other hand, discarding any term in the polynomial expansion will either violate the constant strain criteria or the "geometric isotropy" of the element.

The first conforming triangle was introduced by Clough \& Tocher where the element is subdivided into three subregions with a linear variation of normal slope. Although this element converges monotonically, a great number of them is required to achieve such convergence.

The same year, Bogner et al. introduced a rectangular element based on a Hermitian polynomial expansion. To insure conformity, a twist term \(\frac{\partial^2 w}{\partial x \partial y}\) was introduced as generalized freedom at the corners. Excellent results were obtained; however, difficulties are encountered if this element is to be transformed to a global three-dimensional set of axes.
More work on conforming elements was introduced by Cowper et al., Argyris and Bell using quintic polynomials and allowing, at each node, a displacement, its first and also second derivatives. Although these elements exhibit a very high degree of precision, their application has been mainly directed towards two-dimensional problems since a transformation of the second derivatives to a global system of axes does present difficulties.

To compromise between the "simple" non-conforming elements mentioned earlier and the high-precision ones available now, the quadrilateral elements of Fraeijs de Veubeke seems to be the most appropriate choice. The element is conforming and a quadratic variation of the normal slopes is allowed. This means the introduction of a mid-side degree of freedom. At the corners, however, only a translation and two rotations are described. More details on the very complicated formulation of the element was given by McBean. The work of Clough and Felippa is virtually identical in principle but the approach differs. Furthermore, they imposed a linear variation of the normal slopes, thus eliminating the mid-side normal freedom. This approach, based on dividing the quadrilateral into four triangles, will be developed here but retaining the mid-side normal freedoms in order to achieve higher accuracy.

The important properties of the element are:

(a) Conformity, hence, monotonic convergence

(b) Linear variation of internal forces between the nodes

* Meaning generalized degrees of freedom.
(c) Can be easily transformed to a global system of axes.

This last property is extremely important, in this work, since the element is intended to simulate closed stiffened decks and, therefore, the arbitrary orientation of the element in space should be possible.

2.3 TRIANGULAR ELEMENT IN BENDING

2.3.1 Element Geometry and Displacement Assumption

Figure 2.1 illustrates the element and related coordinate systems. The centroid of the element is point 0 and the lines drawn from the centroid to the vertices form the three sub-regions 1, 2 and 3. The element axes (x, y) are shown in the figure and the subregion axes (ξ, η) are such that the η axis for each subregion passes through 0. Table 2-1 shows all the formulae required to calculate the various geometric quantities needed in the development. These quantities are evaluated in terms of the global (X, Y, Z) coordinates which define the element position in cartesian space.

A full cubic polynomial is chosen for each subregion to represent the normal displacement w. Therefore, for a typical subregion:

\[ w = a_1 + a_2 \xi + a_3 \eta + a_4 \xi^2 + a_5 \xi \eta + a_6 \eta^2 + a_7 \xi^3 + a_8 \xi^2 \eta + a_9 \xi \eta^2 + a_{10} \eta^3 \]  \hspace{1cm} (2.1)
or
\[ w = \sum_{i=1}^{10} \alpha_i \xi_i \eta_i \]  
(2.2)

where \( m_i \) and \( n_i \) are obviously given by:

\[ m_i : 0,1,0,2,1,0,3,2,1,0 \quad \text{for } i=1,10 \]  
(2.3)

\[ n_i : 0,0,1,0,1,2,0,1,2,3 \quad \text{for } i=1,10 \]

The 10 constants
\[ \alpha = \{\alpha_1 \alpha_2 \ldots \alpha_{10}\} \]  
(2.4)

will be evaluated in terms of ten degrees of freedom for each subregion as shown below. It must be noted that, by simple inspection of (2.1), the normal slope variation along any side of any subregion is quadratic.

Defining the vectors \( r_i, r_j \) and \( r_k \) for a subregion by

\[ r_i = \{w_i \theta^x_i \theta^y_i\} \]  
(2.5)

\[ r_j = \{w_j \theta^x_j \theta^y_j\} \]  
(2.6)

\[ r_k = \{w_k \theta^x_k \theta^y_k\} \]  
(2.7)

where \( i, j \) and \( k \) define the three vertices of a typical subregion as shown in Fig.2.2 and \( \theta^x \) and \( \theta^y \) are the rotations about the \( x \) and \( y \) axes respectively. From Fig.2.3, it is clear that these rotations are given by:
\[ \theta_x = \theta_\xi \cos \gamma - \theta_\eta \sin \gamma \quad (2.8) \]
\[ \theta_y = \theta_\xi \sin \gamma + \theta_\eta \cos \gamma \quad (2.9) \]

Noting that \( \theta_\xi = \frac{\partial w}{\partial \eta} \) and \( \theta_\eta = -\frac{\partial w}{\partial \xi} \), then from (2.2), the following expressions for the rotations are obtained:

\[ \theta_x = \sum_{i=1}^{10} a_i (m_i \xi - l \eta \sin \gamma + n_i \xi \eta \cos \gamma) \quad (2.10) \]
\[ \theta_y = \sum_{i=1}^{10} a_i (-m_i \xi - l \eta \cos \gamma + n_i \xi \eta \sin \gamma) \quad (2.11) \]

If the coordinates of nodes \( i,j \) and \( k \) as shown in Fig. 2.2 are substituted into (2.2), (2.10) and (2.11), then a relationship of the kind

\[ r_i = A_i a \quad (2.12) \]
\[ r_j = A_j a \quad (2.13) \]
\[ r_k = A_k a \quad (2.14) \]

can be automatically generated within the computer once the values of \( m_i \) and \( n_i \) are furnished. The matrices \( A_i, A_j, \) and \( A_k \) are each of order \((3\times10)\).

Relations (2.12), (2.13) and (2.14) constitute a set of nine equations and obviously these are insufficient to calculate the ten constants \( a \) of (2.4). Therefore, extra mid-side degrees
of freedom are introduced on each side of the subregion. The normal slope \( \theta_n \) will constitute the last equation desired for the solution of the ten constants. The normal slopes \( \theta_p \) and \( \theta_q \) will serve later as compatibility conditions between the subregions. Hence, from the subregions (Figs. 2.2 and 2.4) and previously established relations, the normal slopes are:

\[
\theta_n = \sum_{i=1}^{10} \alpha_i n_i \xi_i \eta_i \xi_i \eta_i -\lambda 
\]  

(2.15)

\[
\theta_p = \sum_{i=1}^{10} \alpha_i (-m_i \xi_i \eta_i \cos \alpha + n_i \xi_i \eta_i \sin \alpha) 
\]  

(2.16)

\[
\theta_q = \sum_{i=1}^{10} \alpha_i (-m_i \xi_i \eta_i \cos \beta + n_i \xi_i \eta_i \sin \beta) 
\]  

(2.17)

Substituting the coordinates of points \( n, p \) and \( q \) into the above relations, the following expressions are directly obtained within the computer.

\[
\theta_n = A_n \alpha 
\]  

(2.18)

\[
\theta_p = A_p \alpha 
\]  

(2.19)

\[
\theta_q = A_q \alpha 
\]  

(2.20)

where the matrices \( A_n \), \( A_p \) and \( A_q \) are each of order (1x10).
If the full cubic expansion of (2.1) yielded ten constants for each subregion, a total of 30 constants needs to be evaluated for the triangle. Twelve of these constants will be in terms of the triangle degrees of freedom which are the displacement and two rotations at each vertex plus the three normal slopes and the mid-sides of the triangle as shown in Fig.2.5. The remaining 18 equations are obtained using various compatibility relations between the subregions.

Denoting first the triangle 12 degrees of freedom by the vector \( r \) where

\[
r = \{r_1, r_2, r_3, \theta_{12}, \theta_{23}, \theta_{31}\}
\]  \hspace{1cm} (2.21)

and

\[
r_1 = \{w_1, \theta_{x_1}, \theta_{y_1}\} \text{ etc...}
\]  \hspace{1cm} (2.22)

also using superscripts to define individual subregions, then the 30 equations needed to evaluate the 30 constants of (2.4) are obtained by reference to Fig.2.6. Hence,

\[
r_1 = r_1
\]

\[
r_2 = r_2 \hspace{1cm} 9 \text{ equations}
\]

\[
r_3 = r_3
\]

\[
\theta_{12} = \theta_{1n}
\]

\[
\theta_{23} = \theta_{2n} \hspace{1cm} 3 \text{ equations}
\]

\[
\theta_{31} = \theta_{3n}
\]
These equations can be written in matrix form. Using (2.12), (2.13), (2.14), (2.18), (2.19) and (2.20) the result is:

\[
\begin{bmatrix}
  r_j \\
  r_k \\
  \theta \\
  \theta \\
  \theta \\
\end{bmatrix} =
\begin{bmatrix}
  A_i^1 & 0 & 0 \\
  0 & A_i^2 & 0 \\
  0 & 0 & A_i^3 \\
  A_n^1 & 0 & 0 \\
  0 & A_n^2 & 0 \\
  0 & 0 & A_n^3 \\
  A_j^1 & -A_j^2 & 0 \\
  0 & A_j^2 & -A_j^3 \\
  -A_j^1 & 0 & A_j^3 \\
  A_k^1 & -A_k^2 & 0 \\
  0 & A_k^2 & -A_k^3 \\
  A_k^1 & 0 & A_k^3 \\
  A_q^1 & -A_q^2 & 0 \\
  0 & A_q^2 & -A_q^3 \\
  -A_q^1 & 0 & A_q^3 \\
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
\end{bmatrix}
\]

\[= A \begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
\end{bmatrix}
\]

(2.24)
Partitioning the above matrix into

\[
\begin{bmatrix}
  r \\
  0 \\
\end{bmatrix} = 
\begin{bmatrix}
  A_{rr} & A_{ro} \\
  A_{or} & A_{oo} \\
\end{bmatrix}
\begin{bmatrix}
  a_r \\
  a_o \\
\end{bmatrix}
\]  \hspace{1cm} (2.25)

and solving simultaneously, the following relations are obtained:

\[
a_o = -A_{oo}^{-1}A_{or}a_r
\]  \hspace{1cm} (2.26)

\[
a_r = \bar{A}^{-1}r
\]  \hspace{1cm} (2.27)

where

\[
\bar{A} = [A_{rr} - A_{ro}A_{oo}^{-1}A_{or}]
\]  \hspace{1cm} (2.28)

Hence

\[
\begin{bmatrix}
  a_r \\
  a_o \\
\end{bmatrix} =
\begin{bmatrix}
  \bar{A}^{-1} \\
  -A_{oo}^{-1}A_{or}\bar{A}^{-1} \\
\end{bmatrix}
\begin{bmatrix}
  r \\
\end{bmatrix}
\]  \hspace{1cm} (2.29)

or, more concisely, (2.29) is written as

\[
\bar{a} = \bar{G}r
\]  \hspace{1cm} (2.30)

Relation (2.30) simply determines the 30 constants \( \bar{a} \) in terms of the chosen degrees of freedom of the triangle.
2.3.2 **Stiffness Matrix**

The stiffness matrix of the element is obtained by calculating the strain energy in the individual subregions from the relation

\[
U_e = \frac{1}{2} \int \epsilon^T E \epsilon \, dV \tag{2.31}
\]

The superscripts corresponding to subregions are dropped temporarily for simplicity. \( E \) is the stress/strain relation for anisotropic materials defined in the local \( \xi, \eta \) axes of the subregion. Hence, if the value of \( E_T \) corresponds to the elasticity matrix in the triangle \((x,y)\) axes, transformation to the subregion axes is obtained, according to Lekhnitskii \(^6\), from the relation

\[
E = \begin{bmatrix}
E_{11} & E_{12} & E_{13} \\
E_{12} & E_{22} & E_{23} \\
E_{13} & E_{23} & E_{33}
\end{bmatrix} = \psi^T E_T \psi \tag{2.32}
\]

where

\[
\psi = \begin{bmatrix}
\cos^2\gamma & \sin^2\gamma & -\sin\gamma\cos\gamma \\
\sin^2\gamma & \cos^2\gamma & \sin\gamma\cos\gamma \\
2\sin\gamma\cos\gamma & -2\sin\gamma\cos\gamma & \cos^2\gamma - \sin^2\gamma
\end{bmatrix} \tag{2.33}
\]

The values of \( \gamma \) have been defined in Table 2-1. The vector \( \epsilon \) defines the strains obtained from classical plate bending theory such that

\[
\epsilon = -z\{w_{\xi\xi} \ w_{\eta\eta} \ 2w_{\xi\eta}\} \tag{2.34}
\]
The subscripts denote partial differentiation and \( z \) is the distance from the neutral plane. Substituting (2.34) and (2.32) into (2.31) and integrating over the element thickness yields

\[
U_e = \frac{1}{2} \int_A \frac{\partial^3}{\partial z^3} \left( E_{11} w_1^2 + E_{12} w_1 w_2 + 2E_{13} w_1 w_3 + E_{22} w_2^2 + 2E_{23} w_2 w_3 + E_{33} w_3^2 \right) \, dA
\]

where \( A \) is the area of the subregion.

For the case of isotropic materials, (2.35) reduces considerably due to the fact that

\[
E_{11} = E_{22} = E/(1-\nu^2) \\
E_{12} = \nu E_{11} \\
E_{13} = E_{23} = 0 \\
E_{33} = E/2(1+\nu)
\]

where \( \nu \) is Poisson's ratio and \( E \) the elastic modulus.

To proceed further with the evaluation of the strain energy, a close examination of a typical second derivative of \( w \) in (2.35) yields, according to the format of (2.2), the following expression:
Integrating the typical term in (2.38) over the subregion area gives

\[ \int w_{\xi}^2 \, dA = \sum_{i=1}^{10} \sum_{j=1}^{10} \alpha_{i} \alpha_{j} \xi^{m_{i}-1} \eta^{n_{i}} + \sum_{i=1}^{10} \sum_{j=1}^{10} \alpha_{i} \alpha_{j} \xi^{m_{i}-1} \eta^{n_{i}} \]  

where, in general,

\[ F(m,n) = \int \xi^{m} \eta^{n} \, dA \]  

If the thickness is a linear function of \( \xi \) and \( \eta \), the term \( t^3 \) inside the integral of (2.35) must be evaluated before performing the integration. Let \( t_1 \), \( t_2 \) and \( t_3 \) denote the thickness at nodes \( i, j \) and \( k \) of a typical subregion. Hence

\[ t = t_1 + t_2 \xi + t_3 \eta = \sum_{i=1}^{3} t_i \xi^{m_{i}} \eta^{n_{i}} \]  

where

\[ t_1 = (at_1 + bt_2)/(a+b) \]
\[ t_2 = (t_2 - t_1)/(a+b) \]  

\[ t_3 = [a(t_3 - t_1) + b(t_3 - t_2)]/(a+b) \]
hence

\[ t^3 = \sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} \tau_{k_1} \tau_{k_2} \tau_{k_3} \xi \]

If (2.43) is now included in the integration of the strain energy, the typical term shown in (2.39) becomes

\[ \int \frac{t^3}{12} W^2 dA = \frac{1}{12} \sum_{i=1}^{10} \sum_{j=1}^{10} \sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} \alpha_i \alpha_j \tau_{k_1} \tau_{k_2} \tau_{k_3} [m_i m_j (m_i - 1)(m_j - 1)F(II - 4, JJ)] \]

(2.44)

where

\[ II = m_i + m_j + 1 \]

(2.45)

\[ JJ = n_i + n_j + 1 \]

(2.46)

Evaluating all the other components in the strain energy expression in a similar fashion to (2.44) and writing the answer in quadratic form in \( a \) gives

\[ U_e = \frac{1}{2} a^T k a \]

(2.47)

where \( k \) is a square symmetric matrix of order 10x10 given by

\[ k_{i,j} = \frac{1}{12} \sum_{k_1=1}^{3} \sum_{k_2=1}^{3} \sum_{k_3=1}^{3} \tau_{k_1} \tau_{k_2} \tau_{k_3} \{ \]

\[ E_1 m_i m_j (m_i - 1)(m_j - 1)F(II - 4, JJ) + E_2 n_i n_j (n_i - 1)(n_j - 1)F(II, JJ - 4) + \]
All the ingredients necessary to evaluate (2.48) automatically, within the computer, are now available except for the value of F(m,n) as defined by (2.40). The general expression for the integration over the subregion turns out to be extremely simple, and is given by:

\[ F(m,n) = c^{m+n} \eta^m \left[ a^{m+1} - b^{m+1} \right] \frac{m! n!}{(m+n+2)!} \]  

(2.49)

The proof of the above expression will not be repeated here as details were given by Cowper et al.\(^6\).

It is important to note that equation (2.48) is valid not only for the triangular element in question but for any other polynomial function as described by the author in ref.51.

Once the value of \( k \) has been obtained for a typical subregion, the strain energy of the complete triangular element can be written in the form

\[ U_{T} = \frac{1}{2} \overline{\alpha}^T K \overline{\alpha} \]  

(2.50)
where

\[
\bar{\mathbf{a}} = \{a^1, a^2, a^3\} \quad (2.51)
\]

\[
\bar{\mathbf{k}} = \begin{bmatrix} k^1 & k^2 & k^3 \end{bmatrix} \quad (2.52)
\]

and superscripts define the subregions, as usual. Substituting (2.30) into (2.50) yields

\[
U_{er} = \frac{1}{2} \mathbf{r}^T G^T \bar{\mathbf{k}} G \mathbf{r} \quad (2.53)
\]

Hence, the stiffness matrix of the triangle is simply

\[
\mathbf{K} = G^T \bar{\mathbf{k}} G \quad (2.54)
\]

which is of order 12x12. It can be appreciated, now, that the explicit algebraic evaluation of the matrix \( \bar{\mathbf{k}} \) is extremely lengthy, if not impossible, to derive. More confidence can, therefore, be placed in the programming phase using this technique.

2.3.3 Consistent Load Matrix

The consistent load matrix is established by calculating the work done by the applied loads over the individual sub-regions. If \( q_o \) is the load intensity, then

\[
V_e = q_o \int_{A} w dA \quad (2.55)
\]

Substituting for \( w \) from (2.2) yields
\[ V_e = q_o \sum_{i=1}^{10} \alpha_i \int_0^1 \int_0^1 \xi^m \eta^n dA \]

\[ = q_o \sum_{i=1}^{10} \alpha_i F(m_i, n_i) \]  

\[ = \mathbf{a}^t \mathbf{p} \]  

where the entries in the vector \{p\} are given by

\[ p_i = q_o F(m_i, n_i) \]

For the whole triangle, the work done is

\[ V_{et} = \overline{\mathbf{a}}^t \overline{\mathbf{p}} \]

where

\[ \overline{\mathbf{p}} = \{p^1, p^2, p^3\} \]

and for each subregion, the corresponding value of \( F \) is used as shown in (2.59).

Let \{R_e\} be the vector of forces corresponding to the element freedoms. The work done by these forces is, therefore

\[ V_{et} = \mathbf{r}^t \mathbf{R}_e \]  

Equating (2.62) and (2.60) and substituting for \( \overline{\mathbf{a}} \) from (2.30) yields

\[ \mathbf{R}_e = G^t \overline{\mathbf{p}} \]  

(2.63)
which is the consistent load vector of order 12 and, therefore, labeled with the suffix $c$.

2.3.4 The Stress Matrix

The stress matrix for the triangular element is obtained by calculating the stresses for the individual subregions and averaging the values at the vertices. For any particular subregion, the strains are evaluated at nodes $i$ and $j$ only as shown in Fig.2.2. Adopting Timoshenko's sign convention for the bending moments at any particular point, then

\[
M = \frac{t^3}{12} E_T \rho
\]  

(2.64)

where

\[
M = \begin{bmatrix}
M_x & M_y & M_{xy}
\end{bmatrix}
\]  

(2.65)

\[
\rho = \begin{bmatrix}
-w_{xx} & -w_{yy} & 2w_{xy}
\end{bmatrix}
\]  

(2.66)

refer to the triangle $(x,y)$ axes. Transforming the curvatures to the subregion axes and substituting back into (2.64), yields

\[
M = \frac{t^3}{12} E_T \psi^{-1} \rho_S
\]  

(2.67)

where

\[
\rho_S = \begin{bmatrix}
-w_{\xi\xi} & -w_{\eta\eta} & 2w_{\xi\eta}
\end{bmatrix}
\]  

(2.68)

and $\psi^{-1}$ is the inverse of (2.33) which is obtained by simply replacing $\gamma$ by $-\gamma$. Hence
\[
\psi^{-1} = \begin{bmatrix}
\cos^2 \gamma & \sin^2 \gamma & \sin \gamma \cos \gamma \\
\sin^2 \gamma & \cos^2 \gamma & -\sin \gamma \cos \gamma \\
-2\sin \gamma \cos \gamma & 2\sin \gamma \cos \gamma & \cos^2 \gamma - \sin^2 \gamma
\end{bmatrix}
\] (2.69)

Evaluating the vector \( \mathbf{p}_s \) in terms of the polynomial constants and substituting the coordinates \( i \) and \( j \) of the subregion as shown in Fig. 2.2 gives

\[
\mathbf{p}_{s_i} = B_i a
\] (2.70)

\[
\mathbf{p}_{s_j} = B_j a
\] (2.71)

where

\[
B_i = \begin{bmatrix}
\ldots & -2 & \ldots & 6b & \ldots \\
\ldots & \ldots & -2 & \ldots & 2b \\
\ldots & \ldots & 2 & \ldots & -4b \\
\end{bmatrix}
\] (2.72)

and

\[
B_j = \begin{bmatrix}
\ldots & -2 & \ldots & -6a & \ldots \\
\ldots & \ldots & -2 & \ldots & -2a \\
\ldots & \ldots & 2 & \ldots & 4a \\
\end{bmatrix}
\] (2.73)

Hence, by introducing superscripts again to define the particular subregions, the bending moments at the nodes are defined by the vector

\[
\mathbf{M} = \{M^1_i, M^1_j, M^2_i, M^2_j, M^3_i, M^3_j\}
\] (2.74)

Substituting (2.70), (2.71) into (2.67) and putting the result
in the form introduced by (2.74) yields

\[ \mathbf{\bar{M}} = \mathbf{T} \mathbf{E} \mathbf{\bar{y}} \mathbf{\bar{B}} \mathbf{\bar{a}} \]  

(2.75)

where the supermatrices \( \mathbf{T}, \mathbf{E}, \mathbf{\bar{y}} \) and \( \mathbf{\bar{B}} \) are shown in Table 2-2. The vector of constants \( \mathbf{\bar{a}} \) can be substituted for from equation (2.30). Averaging the moments between the subregions and introducing now the vector

\[ \mathbf{M}_T = \{M_1, M_2, M_3\} \]  

(2.76)

where \( M_1, M_2 \) and \( M_3 \) define the bending and twisting moments at the triangle nodes 1, 2 and 3 respectively, then

\[ \mathbf{M}_T = \mathbf{L} \mathbf{T} \mathbf{E} \mathbf{\bar{y}} \mathbf{\bar{B}} \mathbf{G} \mathbf{r} \]  

(2.77)

where \( \mathbf{L} \) is a boolean matrix shown in Table 2-2. Hence

\[ \mathbf{M}_T = \mathbf{S}_T \mathbf{r} \]  

(2.78)

where

\[ \mathbf{S}_T = \mathbf{L} \mathbf{T} \mathbf{E} \mathbf{\bar{y}} \mathbf{\bar{B}} \mathbf{G} \]  

(2.79)

is the stress matrix for the triangular element.

2.4 QUADRILATERAL ELEMENT AS AN ASSEMBLAGE OF TRIANGLES

Four triangular elements may be combined to form a quadrilateral as shown in Fig. 2.7. The local axes \((\overline{x}, \overline{y})\) of the quadrilateral are shown and the four x-axes of the triangles form angles \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \). The intersection of the diagonals is
given as node number 5. All the pertinent geometry is evaluated in Table 2-3, with reference to the global (X,Y,Z) axes. For each triangle, the stiffness, stress and consistent load matrices are shown in Section 2.3.

2.4.1 **Stiffness Matrix of the Quadrilateral**

Before assembling the stiffness contributions of the individual triangles, a transformation to the quadrilateral \((x,y)\) axes is required. If \(r\) was defined in (2.21) as the vector of generalized displacements for the triangle in the corresponding \((x,y)\) axes, then \(r_T\) will refer to the same quantities but in the local quadrilateral axes. Hence, if

\[
\mathbf{r} = \phi \mathbf{r}_T
\]

where

\[
\mathbf{r}_T = \{w_1 \, \theta_x \, \theta_y \, w_2 \, \theta_x \, \theta_y \, w_3 \, \theta_x \, \theta_y \, \theta_{12} \, \theta_{23} \, \theta_{31}\}
\]

then, matrix \(\phi\) is given by

\[
\phi = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{bmatrix}
\]
The stiffness matrices of the triangles in the \((\bar{x}, \bar{y})\) axes are therefore

\[ K^i_T = \Phi^i_t K^i \Phi^i \quad i = I, II, III, IV \]  

(2.84)

where \(K^i\) is derived from (2.54) and superscripts refer to the corresponding triangles shown in Fig.2.7.

The total degrees of freedom in the quadrilateral are shown in Fig.2.8 and when expressed as a vector, take the following form:

\[ r_Q = \{w_1 \theta_{x_1} \theta_{y_1} w_2 \theta_{x_2} \theta_{y_2} w_3 \theta_{x_3} \theta_{y_3} w_4 \theta_{x_4} \theta_{y_4} \theta_{12} \theta_{23} \theta_{34} \theta_{41} w_5 \theta_{x_5} \theta_{y_5} \theta_{15} \theta_{25} \theta_{35} \theta_{45}\} \]  

(2.85)

Hence, the assembled stiffness matrix is therefore

\[ K_Q = \sum_{i=I}^{IV} p^i_t K^i_T p^i \]  

(2.86)

where \(p^I, p^{II}, p^{III}\) and \(p^{IV}\) are boolean matrices, shown in Table 2-4, relating the degrees of freedom in the triangle to the corresponding ones in the quadrilateral, that is

\[ r_T^i = p^i_T r_Q^i \quad i = I, II, III, IV \]  

(2.87)

The matrix \(K_Q\) is now partitioned according to the exterior and interior nodes of the element respectively.
where $R_Q$ corresponds to the exterior nodal forces and the interior nodal forces being zero. Expansion of (2.88) shows that

$$r_Q = \Gamma r_Q^*$$  \hspace{1cm} (2.89)

where

$$r_Q^* = \{w_1 \theta_{\hat{x}_1} \theta_{\hat{y}_1} w_2 \theta_{\hat{x}_2} \theta_{\hat{y}_2} w_3 \theta_{\hat{x}_3} \theta_{\hat{y}_3} \theta_{\hat{x}_4} \theta_{\hat{y}_4} \theta_{12} \theta_{23} \theta_{34} \theta_{41}\}$$  \hspace{1cm} (2.90)

and

$$\Gamma = \begin{bmatrix} I_{16} \\ -K_{22}^{-1}K_{21} \end{bmatrix} \hspace{1cm} (2.91)$$

Therefore, the reduced stiffness matrix is given by the relation

$$R_Q = K_Q^* r_Q^*$$  \hspace{1cm} (2.92)

where

$$K_Q^* = \Gamma^t K_Q \Gamma$$  \hspace{1cm} (2.93)

is of order $16 \times 16$. 
2.4.2 Consistent Loading for Quadrilateral Element

The consistent load vector for the quadrilateral is made up of contributions from the individual triangles. Transforming, first, the vector $R_a$ of (2.63) into the quadrilateral axes,

$$R_{aT} = \Phi^t R_a$$  \hspace{1cm} (2.94)

the required vector for the quadrilateral is then

$$R_{aQ} = \sum_{i=1}^{IV} p_i^t R_{aT}^i$$  \hspace{1cm} (2.95)

Hence, the reduced consistent load vector $R_{aQ}^*$ corresponding to $r_Q^*$ is simply obtained using (2.89) and equating the work done. Therefore,

$$R_{aQ}^* = R^t R_{aQ}$$  \hspace{1cm} (2.96)

2.4.3 Stress Matrix for Quadrilateral Element

It is desired to calculate the bending and twisting moments at the four corners of the element, also at the intersection of the diagonals. The corner values from the individual triangles, as shown by (2.76) and obtained using (2.78), are averaged. However, transformation of the bending moments to a common set of axes $(\bar{x}, \bar{y})$ is necessary before such averaging can take place. The transformation for a
A typical triangle is of the form

$$\overrightarrow{M}_T^i = H \overrightarrow{M}_T^i \quad i = I, II, III, IV \quad (2.97)$$

where

$$H = \begin{bmatrix} \psi_\theta & \psi_\theta & \psi_\theta \\ \psi_\theta & \psi_\theta & \psi_\theta \end{bmatrix} \quad (2.98)$$

and

$$\psi_\theta = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -\sin \theta \cos \theta \\ -2 \sin \theta \cos \theta & 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \quad (2.99)$$

Replacing, now, the triangle node numbers by the corresponding quadrilateral nodes for the vectors $\overrightarrow{M}_T^i$ yields

$$\overrightarrow{M}_T^I = \{M_1, M_2, M_5\}$$

$$\overrightarrow{M}_T^{II} = \{M_2, M_3, M_5\} \quad (2.100)$$

$$\overrightarrow{M}_T^{III} = \{M_3, M_4, M_5\}$$

$$\overrightarrow{M}_T^{IV} = \{M_4, M_1, M_5\}$$

where

$$\overrightarrow{M}_i = \{\overrightarrow{M}_x, \overrightarrow{M}_y, \overrightarrow{M}_{xy}\} \quad (2.101)$$
Hence, the bending moments for the quadrilateral nodes are simply obtained by averaging the desired values in (2.100). If

\[ M_Q = \{M_1, M_2, M_3, M_4, M_5\} \]  

for the quadrilateral element, then

\[
\begin{bmatrix}
M_1^I \\
M_2^I \\
M_3^I \\
M_4^I \\
M_5^I
\end{bmatrix} = L
\begin{bmatrix}
M_1^I \\
M_2^I \\
M_3^I \\
M_4^I \\
M_5^I
\end{bmatrix}
\]  

(2.103)

where

\[
L = \frac{1}{2}
\begin{bmatrix}
I_3 & \ldots & \ldots & \ldots & \ldots & I_3 \\
\ldots & I_3 & I_3 & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & I_3 & I_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & I_3 \\
\ldots & \frac{1}{2}I_3 & \frac{1}{2}I_3 & \frac{1}{2}I_3 & \frac{1}{2}I_3 & \frac{1}{2}I_3
\end{bmatrix}
\]

(2.104)

From the bending moments, the stresses are simply

\[
\sigma_i = \frac{6}{t_i^2} M_i
\]

(2.105)

where

\[
\sigma_i = \{\sigma_x, \sigma_y, \sigma_{xy}\} \]

(2.106)

refer to the local quadrilateral axes.
2.5 **EVALUATION OF THE ELEMENT ACCURACY**

The accuracy of the element is examined, in this section, by comparison with exact solutions or experimental studies. The purpose of determining the element accuracy is two-fold: First, to verify the coding of the program. Second, and more important, to confirm that convergence of deflections and stresses is obtained with grid refinements.

### 2.5.1 Simply Supported Square Plate under Uniform Load

Figure 2.9 shows the idealization of an isotropic simply supported square plate and loaded uniformly over its surface. An exact solution to this problem is given by Timoshenko. A plot of error for central deflection and bending moments, as the grid is refined, is shown in Figs.2.10 and 2.11 respectively. The central deflection, using one element idealization only, is less than 0.5% in error. However, due to the linear variation in the bending moments between the element nodes, convergence of the bending stresses is slower in comparison to deflections. Nevertheless, even a 2×2 grid results in less than 3% error for the maximum bending stress at the centre. Other classical problems related to square plates have been studied by McBean and in all cases, the element behaviour is commendable.
2.5.2 Rhombic Cantilever Plate

This example has been chosen for three reasons: First, experimental data is available as reported by Adini. Secondly, a comparative study using other finite element models for the same problem has been reported by Clough & Tocher. Lastly, to examine the element behaviour for the case of nonrectangular structures.

Figure 2.12 shows the geometry and the idealization of the plate. A 4x4 grid was chosen. Figure 2.13 shows the deflection of the plate together with the experimental values at six points. The theoretical results are consistently, slightly higher. Comparison with other finite element models using the results reported in ref.57 shows the excellent behaviour of the element despite the fact that a coarser grid was chosen here.

2.6 PARTIAL LOADING ON THE ELEMENT

In section 2.3, the development of the triangular element was based on the idea of subregions where the point 0 was chosen at the centroid as shown in Fig.2.1. Later on, in the formulation of the quadrilateral as an assemblage of four triangles, a common point, 5, was defined at the intersection of the diagonals as shown in Fig.2.7. The choice of location of points 0 and 5, just mentioned above, was more a matter of
geometric convenience rather than a restriction to the development of the element itself.

Advantage is, therefore, taken of the arbitrary choice of position of these points to simulate uniform loading over a portion of the element. This formulation is restricted to nine different types of loading as shown in Fig. 2.14. For these types of loading the node number 5, where the four triangles meet, and their respective centroids \( O_I, O_{II}, O_{III}, \) and \( O_{IV} \) are chosen to coincide with the boundaries of the partial loading in question. Furthermore, each subregion is divided by its local \( \eta \) axis into a left and right part. This is shown in Fig. 2.15 where superscripts refer to triangle numbers and subscripts to their corresponding subregion. The loading case described in Fig. 2.14 is evaluated by integration over the appropriate areas using relation (2.59). For the left region, equation (2.49) reduces to

\[
F(m,n)_L = -c^{n+1}(-b)^m+1 \frac{m!n!}{(m+n+2)!} 
\]

(2.106)

For the right region, the corresponding relation is

\[
F(m,n)_R = c^{n+1} a^{m+1} \frac{m!n!}{(m+n+2)!} 
\]

(2.107)

Figure 2.16 shows the participation of each area, denoted by an asterisk, for the nine loading cases in question.

To demonstrate the effectiveness of the procedure described above, a study is conducted, first of all, on a 20 in. simply
supported square plate loaded uniformly by a total load of 400 lbs. The flexural rigidity of the plate is assumed, for convenience, $10^7/12$ lb.in$^2$/in. Using one element only to idealize a quarter of the plate, the element is constructed such that node 5 is chosen at other locations than the conventional diagonal intersection point. Figure 2.17 shows the effect of various geometries on the central deflections. The dots, on the graph, relate to a load formulation consistent with the stiffness and the geometry of the element. The crosses, however, refer to a consistent load formulation but applied the same stiffness developed by the standard idealization No.5 of the element. In other words, in the first case, both the stiffness and loading formulations changed with the geometry, while in the second case, only the loading was simulated by the varying geometry and the stiffness as conventionally derived, was kept constant. The results of the second case are closer to the correct answer. This is due to better "conditioning" of the stiffness matrix.

Figure 2.18 shows the same square plate analysed using partial loading varying from uniformly distributed over the whole area to a concentrated point load. If $P$ is the total load applied over an area $u \times v$, then the central deflection $w_C$, as given by Timoshenko, is simply:

$$w_C = \frac{16PL^4}{\pi^6Duv} \sum_{m=1}^{\text{odd}} \sum_{n=1}^{\text{odd}} \frac{(-1)^{m+n-2}}{mn(m^2+n^2)} \sin \frac{mn\pi u}{2L} \sin \frac{mn\pi v}{2L}$$ (2.108)

where $D$ is the plate flexural rigidity and $L$, the side length.
A plot of the central deflection for various values of $u$ and $v$ is shown in Fig. 2.19. The largest error of 5.5% is obtained when the area reduces to zero producing, therefore, a concentrated point load. It must be pointed out that only one element was used in the idealization.

Another case of loading is shown in Fig. 2.20 where the loaded area extends over the whole width of the plate but varies crosswise from a uniform load down to a line load. The agreement between the classical solution and the finite element is again excellent. Lastly, Fig. 2.21 shows a different kind of partial loading not exceeding one quarter of the plate and reducing down to a point load.

It can, therefore, be concluded that by using the approach discussed above to correctly simulate the partial loading on a plate, a much smaller error is involved. For instance, from Fig. 2.21, if for the case of a plate where $\frac{u}{L} = \frac{1}{2}$ and $\frac{v}{L} = \frac{1}{4}$, the loading is taken as a concentrated load rather than a uniform load over that portion of the plate, an error of about 20% is obtained for the deflection. There is always, undeniably, the other obvious alternative of choosing a grid which would coincide with the boundaries of the loaded region. However, this will involve more unknowns and more elements. Due to the relatively high performance of the element, even when choosing very few elements, a finer grid is not really necessary when the plate is partially loaded over a particular region. Examples on the application of this idea will be demonstrated in Chapters 5 and 6.
\[ X_i = \frac{(X + Y + Z)}{3} \]
\[ Y_i = \frac{(X + Y + Z)}{3} \]
\[ Z_i = \frac{(X + Y + Z)}{3} \]

\[ S_{11} = \sqrt{(X - X_1)^2 + (Y - Y_1)^2 + (Z - Z_1)^2} \]
\[ S_{21} = \sqrt{(X - X_2)^2 + (Y - Y_2)^2 + (Z - Z_2)^2} \]
\[ S_{31} = \sqrt{(X - X_3)^2 + (Y - Y_3)^2 + (Z - Z_3)^2} \]

\[ \mathbf{A} = \frac{(X - X_1)(X - X_2) + (Y - Y_1)(Y - Y_2) + (Z - Z_1)(Z - Z_2)}{S_{12}} \]
\[ \mathbf{B} = S_{12} \mathbf{A} \]
\[ \mathbf{C} = \sqrt{S_{12}^2 - \mathbf{A}^2} \]

<table>
<thead>
<tr>
<th>SUBREGION 1</th>
<th>SUBREGION 2</th>
<th>SUBREGION 3</th>
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</thead>
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<td>( A_i )</td>
<td>( (X_i - X_1)(X_i - X_2) + (Y_i - Y_1)(Y_i - Y_2) )</td>
<td>( (X_i - X_2)(X_i - X_3) + (Y_i - Y_2)(Y_i - Y_3) )</td>
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<td>( + (Z_i - Z_2)(Z_i - Z_3)/S_{12} )</td>
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<td>( + (Z_i - Z_2)(Z_i - Z_3)/S_{12} )</td>
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<tr>
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<td>( S_{12} - A_i )</td>
</tr>
<tr>
<td>( C_i )</td>
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<td>( \sqrt{S_{12}^2 - A_i^2} )</td>
</tr>
<tr>
<td>( \cos \theta_i )</td>
<td>( b_1/S_{12} )</td>
<td>( b_2/S_{12} )</td>
</tr>
<tr>
<td>( \sin \theta_i )</td>
<td>( c_1/S_{12} )</td>
<td>( c_2/S_{12} )</td>
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<td>( -a_2/S_{12} )</td>
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<td>( \sin \phi_i )</td>
<td>( c_1/S_{12} )</td>
<td>( c_2/S_{12} )</td>
</tr>
<tr>
<td>( \cos \gamma_i )</td>
<td>( 1 )</td>
<td>( -\bar{A}/S_{12} )</td>
</tr>
<tr>
<td>( \sin \gamma_i )</td>
<td>( 0 )</td>
<td>( \bar{C}/S_{12} )</td>
</tr>
</tbody>
</table>

**Table 2-1**

GEOMETRY OF TRIANGULAR ELEMENT
\[ T = \frac{1}{12} \begin{bmatrix} t_1^3 & t_1^3 & t_2^3 & t_2^3 & t_3^3 & t_3^3 \end{bmatrix} \]

\[ \bar{E} = \begin{bmatrix} E_T & E_T & E_T & E_T & E_T \end{bmatrix} \]

\[ \bar{\psi} = \begin{bmatrix} \psi^1 & \psi^1 & \psi^2 & \psi^2 & \psi^3 & \psi^3 \end{bmatrix}^{-1} \]

\[ B = \begin{bmatrix} B_1^1 & 0 & 0 \\ B_j^1 & 0 & 0 \\ 0 & B_1^2 & 0 \\ 0 & B_j^2 & 0 \\ 0 & 0 & B_1^3 \\ 0 & 0 & B_j^3 \end{bmatrix} \]

\[ L = \frac{1}{2} \begin{bmatrix} 1 & . & . & . & . & . \\ . & 1 & 1 & . & . & . \\ . & . & 1 & 1 & . & . \end{bmatrix} \]

**TABLE 2-2**

TRANSFORMATION MATRICES
\[ S_{12} = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \]
\[ S_{23} = \sqrt{(X_3 - X_2)^2 + (Y_3 - Y_2)^2 + (Z_3 - Z_2)^2} \]
\[ S_{34} = \sqrt{(X_4 - X_3)^2 + (Y_4 - Y_3)^2 + (Z_4 - Z_3)^2} \]
\[ S_{41} = \sqrt{(X_1 - X_4)^2 + (Y_1 - Y_4)^2 + (Z_1 - Z_4)^2} \]
\[ S_{13} = \sqrt{(X_3 - X_1)^2 + (Y_3 - Y_1)^2 + (Z_3 - Z_1)^2} \]
\[ S_{15} = \sqrt{(\bar{X}_5 + \bar{X}_1)^2 + \bar{Y}_5^2} \]

\[
\bar{x}_1 = \frac{(X_4 - X_1)(X_2 - X_1) + (Y_4 - Y_1)(Y_2 - Y_1) + (Z_4 - Z_1)(Z_2 - Z_1)}{S_{12}}
\]
\[
\bar{x}_2 = \frac{(X_2 - X_4)(X_2 - X_1) + (Y_2 - Y_4)(Y_2 - Y_1) + (Z_2 - Z_4)(Z_2 - Z_1)}{S_{12}}
\]
\[
\bar{x}_3 = \frac{(X_3 - X_1)(X_2 - X_1) + (Y_3 - Y_1)(Y_2 - Y_1) + (Z_3 - Z_1)(Z_2 - Z_1)}{S_{12} - \bar{x}_1}
\]
\[
\bar{y}_3 = \sqrt{S_{13}^2 - (\bar{x}_3 + \bar{x}_1)^2}
\]
\[
\bar{y}_4 = \sqrt{S_{41}^2 - \bar{x}_1^2}
\]
\[
\bar{x}_5 = \frac{\bar{x}_2(\bar{x}_1 \bar{y}_4 + \bar{x}_3 \bar{y}_4 - \bar{x}_1 \bar{y}_3)}{\bar{x}_1 \bar{y}_4 + \bar{x}_3 \bar{y}_3 + \bar{x}_2 \bar{y}_3}
\]
\[
\bar{y}_5 = \frac{\bar{y}_4 \bar{y}_3(\bar{x}_1 + \bar{x}_2)}{\bar{x}_1 \bar{y}_4 + \bar{x}_3 \bar{y}_3 + \bar{x}_2 \bar{y}_3}
\]

\[
X_5 = X_1 + (X_3 - X_1)S_{15}/S_{13}
\]
\[
Y_5 = Y_1 + (Y_3 - Y_1)S_{15}/S_{13}
\]
\[
Z_5 = Z_1 + (Z_3 - Z_1)S_{15}/S_{13}
\]

<table>
<thead>
<tr>
<th>Triangle I</th>
<th>Triangle II</th>
<th>Triangle III</th>
<th>Triangle IV</th>
</tr>
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<tr>
<td>(\cos \theta)</td>
<td>1</td>
<td>((\bar{x}_3 - \bar{x}<em>2)/S</em>{23})</td>
<td>(-\bar{x}<em>3/S</em>{34})</td>
</tr>
<tr>
<td>(\sin \theta)</td>
<td>0</td>
<td>(\bar{y}<em>3/S</em>{23})</td>
<td>((\bar{y}_4 - \bar{y}<em>3)/S</em>{34})</td>
</tr>
</tbody>
</table>

**TABLE 2-3**

GEOMETRY OF QUADRILATERAL ELEMENT
TABLE 2-4

TRANSFORMATION MATRICES $p^I$ AND $p^{II}$
### TABLE 2.4 (CONT'D.)

**Transformation Matrices $p^{III}$ and $p^{IV}$**

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<th>$\Theta_{23}$</th>
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</tr>
</tbody>
</table>

**$p^{III}$**

|   | 1   |     |     |     |               |               |               |               |     |               |               |               |               |
|---|-----|-----|-----|-----|---------------|---------------|---------------|---------------|-----|---------------|---------------|               |               |
| 1 | 1   |     |     |     |               |               |               |               |     |               |               |               |               |
| 2 |     | 1   |     |     |               |               |               |               |     |               |               |               |               |
| 3 |     |     | 1   |     |               |               |               |               |     |               |               |               |               |
| 4 |     |     |     | 1   |               |               |               |               |     |               |               |               |               |
| 5 |     |     |     |     |               |               |               |               |     |               |               |               |               |

**$p^{IV}$**

|   | 1   |     |     |     |               |               |               |               |     |               |               |               |               |
|---|-----|-----|-----|-----|---------------|---------------|---------------|---------------|-----|---------------|---------------|               |               |
| 1 | 1   |     |     |     |               |               |               |               |     |               |               |               |               |
| 2 |     | 1   |     |     |               |               |               |               |     |               |               |               |               |
| 3 |     |     | 1   |     |               |               |               |               |     |               |               |               |               |
| 4 |     |     |     | 1   |               |               |               |               |     |               |               |               |               |
| 5 |     |     |     |     |               |               |               |               |     |               |               |               |               |
FIGURE 2.1

TRIANGULAR PLATE BENDING: ELEMENT AND SUBREGION
DIMENSIONS AND AXES
FIGURE 2.2

TYPICAL SUBREGION

FIGURE 2.3

ROTATION DEGREES OF FREEDOM AT THE VERTICES OF A SUBREGION

FIGURE 2.4

NORMAL SLOPES AT POINTS P AND Q
FIGURE 2.5
DEGREES OF FREEDOM IN THE TRIANGULAR ELEMENT

FIGURE 2.6
COMPATIBILITY CONDITIONS BETWEEN SUBREGIONS
FIGURE 2.7

QUADRILATERAL ELEMENT AS AN ASSEMBLY OF FOUR TRIANGLES
FIGURE 2.8

TOTAL DEGREES OF FREEDOM IN QUADRILATERAL ELEMENT
PRIOR TO STATIC CONDENSATION
FIGURE 2.9
SIMPLY SUPPORTED SQUARE PLATE WITH UNIFORM PRESSURE

% ERROR
Central Deflection

FIGURE 2.10
CONVERGENCE OF CENTRAL DEFLECTION WITH GRID ELEMENT

% Error for $M_x$ at centre of plate

FIGURE 2.11
CONVERGENCE OF CENTRAL BENDING MOMENT WITH GRID REFINEMENT
pressure = 0.26066 ksi. 

\[ E = 10.5 \times 10^6 \text{ psi.} \]

\[ v = 0.3 \]

\[ t = 0.125'' \]

**FIGURE 2.12**
RHOMBIC CANTILEVERED PLATE

**FIGURE 2.13**
DEFLECTION PROFILE OF RHOMBIC CANTILEVERED PLATE
FIGURE 2.14

STANDARD CASES OF PARTIAL LOADING ON AN ELEMENT
FIGURE 2.15
TYPICAL ARRANGEMENT OF TRIANGLES AND SUBREGIONS
FOR PARTIAL LOADING ON AN ELEMENT

FIGURE 2.16
CONTRIBUTION OF VARIOUS SUBREGIONS TO THE PARTIAL
LOADING ON AN ELEMENT
Figure 2.17 Comparison of central deflection for uniformly loaded square plate using different triangular arrangements within the element.
FIGURE 2.18 SIMPLY SUPPORTED SQUARE PLATE WITH PARTIAL CENTRAL LOADING
FIGURE 2.19 VARIATION OF CENTRAL DEFLECTION UNDER VARYING PRESSURE. TOTAL LOAD KEPT CONSTANT.

- Timoshenko
- Finite element (1 element idealization for \(1/4\) plate)
Figure 2.20 Variation of central deflection under various loading conditions. Total load constant and $\xi = 1$. 

- Timoshenko

- Finite element (1 element idealization for $\frac{1}{4}$ plate)
Figure 2.21 Variation of central deflection under various loading conditions. Total load constant and $\eta = 1/4$. 

- Timoshenko
- Finite element (1 element idealization for $1/4$ plate)
3.1 INTRODUCTION

In this Chapter, a plane stress element compatible with the plate bending element described in Chapter 2, is developed. The purpose of the plane stress formulation is to create a flat shell element when combined with its plate bending partner. Since the shell element is intended to simulate stiffened decks, it must, therefore, adequately describe such behaviour when used to simulate either the web of the stiffeners or the deck plate itself. Certain requirements on the element performance are imposed. These can be summarized by the following points:

a) Must represent the direct bending stresses in the web of stiffeners.

b) Compatibility of deformation with the plate bending element.

c) Must simulate the membrane action in the deck plate due to eccentricity of stiffeners.

d) In-plane rotations at the nodes so that they can join to the out-of-plane rotations from the bending action.
e) Transformable to a global system of axes.

f) Accurate enough so that one element is sufficient to idealize the stiffener depth.

g) Gives reasonable results with high aspect ratio.

h) A linear variation in stresses between the nodes.

### 3.2 ELEMENT CHOICE

The abundant literature on the development of plane stress elements is narrowed down due to the requirements imposed on the element performance mentioned in the previous section.

In the case of triangular elements, the constant strain element of Turner et al. \(^6^5\) does not exhibit any of the desired features listed above. The linear strain triangle of Argyris \(^6^6\) behaves well but does not have in-plane rotations. Furthermore, the quadratic function is not compatible with the cubic one of the plate bending element. A quadratic strain element as proposed by Felippa \(^6^7\) exhibits very high accuracy but the higher derivatives used as generalized freedoms present a problem when transforming them to a global system of axes. Using the same principle, Tocher and Hartz \(^6^8\) combined the derivatives to yield generalized strains and an in-plane rotation at the nodes. These generalized strains are not easily transformable to a global system of axes.
Various rectangular elements, based on the same functions as the triangles mentioned above, have been summarized by Holand. The same difficulties are encountered in the choice of an adequate element.

Elements with in-plane rotations at the nodes have been proposed by a number of authors: Shieh used a non-conforming triangle based on a quadratic function. A rectangle by Scor-delis lacked certain conformity requirements such as the constant stress criterion. Willam studied a number of quadrilateral elements using various shape functions but introduced the shear strains as generalized displacements at the node to insure conformity between the elements. MacLeod developed a rectangular element with nodal rotations defined by either \( \frac{\partial v}{\partial x} \) or \( \frac{\partial u}{\partial y} \) at alternate nodes. Sisodiya, using quadrilateral elements, introduced \( \frac{\partial v}{\partial x} \) only as an in-plane rotation at the nodes. Furthermore, the constant stress criterion was only satisfied for the special case of a parallelogram or a rectangle. Oakberg and Weaver, using rectangular elements for shear wall analysis, adequately described the in-plane rotations but the stress variation along the length of the element was constant.

For the problem in hand, three rectangular elements are examined in this Chapter, but only the most "accurate" one will have details of the derivation presented. This element has two translations and an in-plane rotation at the nodes. However, a mid-side freedom is introduced for higher accuracy
and a linear variation in stresses. This extra freedom is parallel to the element sides and does not undergo any transformation to the global system of axes.

3.3 PLANE STRESS RECTANGULAR ELEMENT

3.3.1 Displacement Function

The displacement function for the element must contain cubic terms if compatibility with the plate bending element is to be maintained. The first function attempted is based on a cubic-linear assumption for \( u \) and \( v \) displacements. Thus

\[
 u = \left(1 \xi \eta \xi \eta \eta^2 \eta^2 \xi \eta^3 \xi^3 \eta \right) \{a_u\} \tag{3.1}
\]

\[
 v = \left(1 \xi \eta \xi \eta \xi^2 \xi^2 \eta \eta^3 \eta \right) \{a_v\} \tag{3.2}
\]

The 16 unknowns \( a_u \) and \( a_v \) are evaluated in terms of \( u_i \), \( v_i \), \( u_{\xi_i} \) and \( v_{\xi_i} \) at each node \( i=1,2,3 \) and 4 as shown in Fig.3.1. In fact, the choice of terms in (3.1) and (3.2) yields the same shape functions given by Oakberg and Weaver. Inspection of the polynomials above shows that the strain \( \epsilon_{\xi\xi} \) is constant, along the sides 1-2 and 3-4. Similarly, for the strain \( \epsilon_{\eta\eta} \). It can, therefore, be appreciated that if the element is used to simulate the direct bending stresses in the web of a stiffener, a rather large discontinuity in stresses will occur at the nodes. For any confidence in the
results, a large number of elements would have to be used.

To improve on the situation, two extra terms are added to the \( u \) function only, such that

\[
u = [1 \xi \eta \xi^2 \eta^2 \xi^2 \eta \xi^3 \eta^3] \{a_u\}
\]

(3.3)

\[
v = [1 \xi \eta \xi^2 \eta^2 \xi^3 \eta^3] \{a_v\}
\]

(3.4)

The same degrees of freedom are used at the corner nodes but two displacements \( u_{12} \) and \( u_{34} \) are introduced as shown in Fig.3.2. For this case, only the stresses along the \( \xi \) direction would exhibit a linear variation. However, the element would have a preferential direction.

To overcome this last difficulty, the following polynomials are chosen for the displacements:

\[
u = [1 \xi \eta \xi^2 \eta^2 \xi^2 \eta \xi^3 \eta^3] \{a_u\}
\]

(3.5)

\[
v = [1 \xi \eta \xi^2 \eta^2 \xi^3 \eta^3] \{a_v\}
\]

(3.6)

The corner freedoms are given by \( r_i \) where

\[
r_i = \{u \ u_\eta \ v \ v_\xi\}
\]

(3.7)

and the remaining mid-side displacements are \( u_{12} \), \( v_{23} \), \( u_{34} \) and \( v_{41} \) yielding a total of 20 freedoms. Expressed in vector
form, this gives, as shown in Fig. 3.3,

\[ r \{ r_1, r_2, r_3, r_4, u_{12}, v_{23}, u_{34}, v_{41} \} \]  \hspace{1cm} (3.8)

The constants \( a_u \) and \( a_v \) are also combined in one vector, thus

\[ \mathbf{a} = \{ a_u, a_v \} \]  \hspace{1cm} (3.9)

The functions in (3.5) and (3.6) are now written as

\[ u = \sum_{i=1}^{10} a_{\xi i} \xi_i^{m_{\xi i}} \eta_i^{n_{\xi i}} \]  \hspace{1cm} (3.10)

\[ v = \sum_{i=11}^{20} a_{\xi i} \xi_i^{r_{\xi i}} \eta_i^{s_{\xi i}} \]  \hspace{1cm} (3.11)

where

\[ m_{\xi i} = 0,1,0,1,2,0,2,1,0,1,0,0,0,0,0,0,0,0,0 \]

\[ n_{\xi i} = 0,0,1,1,0,2,1,2,3,3,0,0,0,0,0,0,0,0,0 \]  \hspace{1cm} (3.12)

\[ r_{\xi i} = 0,0,0,0,0,0,0,0,0,0,0,0,1,0,1,2,0,2,1,3,3 \]

\[ s_{\xi i} = 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1 \]

Hence

\[ u_i = \sum_{i=1}^{10} a_{\xi i} n_{\xi i}^{m_{\xi i}} \eta_i^{n_{\xi i}-1} \]  \hspace{1cm} (3.13)
Substituting the coordinates of the nodes into (3.10), (3.11), (3.13) and (3.14) and writing the result in matrix form yields

\[ \mathbf{r} = C \mathbf{a} \]  

(3.15)

where \( C \) is a square matrix of order 20\times20 which is automatically generated within the computer once the values of \( m_i \), \( n_i \), \( r_i \) and \( s_i \) are furnished. Therefore

\[ \mathbf{a} = C^{-1} \mathbf{r} \]  

(3.16)

### 3.3.2 Stiffness Matrix of the Element

As described earlier in Chapter 2, the stiffness matrix of the element is obtained by evaluation of the strain energy

\[ U_e = \frac{1}{2} \int_{V} \mathbf{\epsilon}^T \mathbf{E} \mathbf{\epsilon} \, dV \]  

(3.17)

where \( \mathbf{E} \) has already been defined in (2.32) as the elasticity matrix for anisotropic material and \( \mathbf{\epsilon} \) is the strain vector defined as

\[ \mathbf{\epsilon} = \{ u_\xi \, u_\eta \, (u_\eta + v_\xi) \} \]  

(3.18)
Substituting (3.18) into (3.17) and integrating over the thickness yields

\[
U_e = \frac{1}{2} \int \left[ E_{12}u_{\xi}^2 + E_{12}v_{\eta}u_{\xi} + E_{13}(u_{\eta}u_{\xi} + v_{\xi}u_{\eta}) + E_{12}u_{\xi}v_{\eta} + E_{22}v_{\eta}^2 + E_{23}(u_{\eta}v_{\eta} + v_{\xi}v_{\eta}) + E_{13}(u_{\xi}u_{\eta} + u_{\xi}v_{\eta}) + E_{23}(v_{\eta}u_{\eta} + v_{\xi}v_{\eta}) + E_{33}(u_{\eta}^2 + v_{\eta}^2 + u_{\xi}v_{\xi} + v_{\xi}u_{\eta}) \right] dA
\]  

(3.19)

where \( A \) is the area of the element.

Examining a typical term in (3.19),

\[
u_{\xi}^2 = \sum_{i=1}^{20} \sum_{j=1}^{20} \alpha_i \alpha_j m_i m_j \xi^{m_i + n_j - 2} \eta^{n_i + n_j}
\]

(3.20)

hence

\[
\int u_{\xi}^2 dA = \sum_{i=1}^{20} \sum_{j=1}^{20} t \alpha_i \alpha_j F(m_i + m_j - 2, n_i + n_j)
\]

(3.21)

where, in general,

\[
F(m,n) = \int \xi^m \eta^n dA = \frac{a^{m+1} b^{n+1}}{(m+1)(n+1)}
\]

(3.22)

and \( a, b \) are the sides of the rectangle.
If the thickness is allowed a linear variation such that

\[ t = \tau_1 + \tau_2 \xi + \tau_3 \eta + \tau_4 \xi \eta \]

\[ = \sum_{\xi} \tau_{\xi} \xi \eta \]

(3.23)

where

\[ \tau_1 = t_1 \]
\[ \tau_2 = (t_2 - t_1)/a \]
\[ \tau_3 = (t_4 - t_1)/b \]
\[ \tau_4 = (t_1 - t_2 - t_4 + t_3)/ab \]

and \( t_1 \ldots t_4 \) are the thicknesses at the four nodes. Equation (3.21) reduces, therefore, to

\[ \int t u_{\xi}^2 dA = \sum_{k=1}^{4} \sum_{i=1}^{20} \sum_{j=1}^{20} \tau_k \alpha_{i} \alpha_{j} F(m_{i} + m_{j} + m_{k} - 2, n_{i} + n_{j} + n_{k}) \]  

(3.25)

All the other components of the strain energy are evaluated in a similar fashion. Writing the result in quadratic form in \( a \) yields

\[ U_e = a^T k a \]

(3.26)

where \( k \) is a square matrix given by
\[ k_{i,j} = \sum_{k=1}^{4} \alpha_k \left\{ E_{11} m_i m_j F(II-2, JJ) + E_{22} s_i s_j F(RR, SS-2) + \\
E_{33} [n_i n_j F(II, JJ-2) + r_i r_j F(RR-2, SS)] + \\
[E_{12} m_i s_j + E_{33} n_i r_j] F(MRI-1, NSI-1) + [E_{12} m_j s_i + E_{33} n_j r_i] F(MRJ-1, NSJ-1) \\
E_{13} [m_i n_j + m_j n_i] F(II-1, JJ-1) + E_{23} [r_i s_j + r_j s_i] F(RR-1, SS-1) + \\
E_{13} [m_i r_j F(MRI-2, NSI) + m_j r_i F(MRJ-2, NSJ)] + \\
E_{23} [n_i s_j F(MRI, NSI-2) + n_j s_i F(MRJ, NSJ-2)] \} \] (3.27)

where

\begin{align*}
II & = m_i + m_j + m_k \\
JJ & = n_i + n_j + n_k \\
RR & = r_i + r_j + m_k \\
SS & = s_i + s_j + n_k \\
MRJ & = m_j + r_i + m_k \\
NSJ & = n_j + s_i + n_k \\
MRI & = m_i + r_j + m_k \\
NSI & = n_i + s_j + n_k
\end{align*}

(3.28)

and is automatically generated in the machine.
Substituting (3.16) into (3.26) yields the stiffness matrix of the element, that is

\[ K = C^{-1} t \kappa C^{-1} \]  \hspace{1cm} (3.29)

To introduce in-plane rotations \( \theta \) as degrees of freedom in the element, it is assumed that the two slopes \( v_\xi \) and \(-u_\eta\) are equal to \( \theta \). Therefore, if \( r^* \) defines the following displacement vector:

\[ r^* = \{ u_1 v_1 \theta, u_2 v_2 \theta, u_3 v_3 \theta, u_4 v_4 \theta, u_{12} v_2 \theta, u_{13} v_3 \theta, u_{14} v_4 \theta \} \]  \hspace{1cm} (3.30)

the relation between \( r \) and \( r^* \) is given by matrix \( \overline{C} \) shown in Table 3-1, where

\[ r = \overline{C} r^* \]  \hspace{1cm} (3.31)

The "reduced" or "constrained" matrix is, therefore

\[ K^* = \overline{C}^t \kappa \overline{C} \]

\[ = \overline{C}^t C^{-1} t \kappa C^{-1} \overline{C} \]  \hspace{1cm} (3.32)

which is of order 16x16 and the vector \( r^* \) is shown in Fig.3.4.

For this element in particular, the effect of using the "constraint" matrix \( \overline{C} \) on the stiffness matrix as shown by (3.32) has been studied in comparison to relation (3.29). As
expected, this results in a slightly stiffer matrix. However, for the two examples discussed later in the Chapter, the difference between the two formulations (equation (3.29) and (3.32) is small. The advantage of using a displacement vector such as the one shown in (3.30) is that the degree of freedom $\theta \zeta$ has a corresponding physical meaning in the force vector. This is in fact an in-plane couple.

### 3.3.3 Consistent Load Matrix

Any boundary loading along the edges of the element is obtained by equating the work done, by the equivalent nodal forces, to the work done by the distributed boundary load. If $R_e$ corresponds to the equivalent load, then

$$V_e = r^t R_e \quad (3.33)$$

If the boundary load is allowed a quadratic variation in the $\xi$ direction along a line of constant $\eta$, then

$$p = \beta_1 + \beta_2 \xi + \beta_3 \xi^2 \quad (3.34)$$

where

$$\beta_1 = p_1$$

$$\beta_2 = (4p_2 - p_3 - 3p_1)/a \quad (3.35)$$

$$\beta_3 = 2(p_1 - 2p_2 + p_3)/a^2$$
and $p_1$, $p_2$ and $p_3$ are the actual values along the side. The work done by this loading is simply

\[ V_e = \int u^t p t d\xi \tag{3.36} \]

Substituting for $u$ from (3.10) and using (3.16) yields

\[ V_e = r^t C^{-1t} \int N^t p t d\xi \tag{3.37} \]

Hence, from (3.33) and (3.37) $R_e$ is given as

\[ R_e = C^{-1t} \int N^t p t d\xi \tag{3.38} \]

where $N^t$ is a vector such that

\[ N^t = \xi^t n^t \tag{3.39} \]

and $n_0$ is the ordinate where the boundary load is applied.

If the integral in (3.38) is given by a vector \{P\} such that

\[ \{P\} = \int N^t p t d\xi \tag{3.40} \]

then, the corresponding consistent load vector to the displacements $r^*$ is given by

\[ \{R_e^*\} = \bar{C}^t C^{-1t} P \tag{3.41} \]
The same procedure is repeated for any other kind of boundary load acting along different sides. The only information required is the value of \( P_1, P_2, P_3 \) and the values of \( \xi_0, \eta_0 \) where the load is acting. Here again, the whole process is automated.

### 3.3.4 Stress Matrix

The formulation of the stress matrix is straightforward once the strain vector is obtained at the four corners of the element and the centroid. The choice of location is the same as the case of plate bending so that the total stress could be found. The strain vector is given by

\[
\bar{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\} \quad (3.42)
\]

where \( \varepsilon_i \) has already been defined by (3.18). Substituting the coordinates of the four corner nodes and the centroid into the strain vector yields

\[
\bar{\varepsilon} = \bar{B} \quad \varepsilon \quad (3.43)
\]

where \( \bar{B} \) is a 15x20 strain matrix.

Substituting for \( a \) from (3.16) and using (3.31) yields

\[
\bar{\varepsilon} = \bar{B} C^{-1} \bar{C} \quad \mathbf{r}^* \quad (3.44)
\]
Hence, the stress matrix $S_R$ is given by

$$\sigma = S_R \mathbf{r}^*$$  \hspace{1cm} (3.45)

where

$$\sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$$  \hspace{1cm} (3.46)

with

$$\sigma_z = \{\tau_{\xi \xi}, \tau_{\eta \eta}, \tau_{\xi \eta}\}$$  \hspace{1cm} (3.47)

and

$$S = \mathbf{E} \mathbf{B} \mathbf{C}^{-1} \mathbf{C}$$  \hspace{1cm} (3.48)

The Elasticity matrix $\mathbf{E}$ is a super-diagonal matrix such that

$$\mathbf{E} = \begin{bmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & E \end{bmatrix}$$  \hspace{1cm} (3.49)

It is important to point out at this stage that the shear strains at the corners of the element are zero. This stems from the assumption that the slopes $v_{\xi_i}$ and $-u_{\eta_i}$ were set equal to the in-plane rotation $\theta_{\xi_i}$. To overcome this difficulty, the shear strains can still be evaluated approximately at the nodes by calculating the difference in nodal displacements. For example, for node 1

$$\gamma_{\xi \eta_1} = (u_4-u_1)/b + (v_2-v_1)/a$$  \hspace{1cm} (3.50)

Alternatively, the shear strains can be obtained at the mid-side nodes by simply referring to the corresponding local coordinates in the strain matrix $\mathbf{E}$.  

3.4 EVALUATION OF THE ELEMENT ACCURACY

In this section, the element accuracy is tested. Two examples are shown such that they can individually describe the behaviour of the element when performing as a web for the stiffeners or when simulating the membrane action in the deck plate due to eccentricity of stiffeners or other effects.

3.4.1 Cantilever Beam

The cantilevered beam shown in Fig. 3.5 is idealized into one, two and four elements. The choice of this example, in particular, is to investigate whether the bending effects in the stiffeners are correctly represented by the element. This is particularly important since the overall bending of the deck is governed by the correct behaviour of the stiffeners. Table 3-2 shows the results for both deflections and stresses.

Even for a single element idealization where the aspect ratio is 4:1, the error in the deflection of the free end, is less than 5%. The stresses are very well-behaved due to the inherent linear variation between the nodes of the element. It must be pointed out that the exact solution is based on the assumption that the built-in end is allowed to warp. In the present idealization, warping was prevented which accounts for part of the error involved in the finite element solution. The other elements discussed earlier in the Chapter and shown in
Figs. 3.1 and 3.2 were also tested using the same example. The error involved was slightly higher for the deflection, but the stresses at the nodes exhibited large discontinuities between the elements using the very first idealization.

3.4.2 Square Plate with Parabolic In-Plane Load

Figure 3.6 shows a square plate under parabolic in-plane tension along two parallel edges. The choice of this example is to investigate the behaviour of the element under in-plane stresses. The reason is that when a wheel load is placed over the deck, it produces, at the bottom of a trapezoidal stiffener, a state of tension while the deck plate itself is under compression. Hence, the capability of reproducing such in-plane states of stress, using the rectangular element, is important.

An exact solution to the problem shown in Fig. 3.6 is given by Timoshenko. A 4x4 grid idealization for the quarter plate yielded excellent agreement with the classical solution for both deflections and stresses. The results are shown in Table 3.3. In conclusion, it can be seen from this example and the previous one that the element behaviour is quite satisfactory.
TABLE 3-1

TRANSFORMATION MATRIX $\bar{C}$
TABLE 3-2

CONVERGENCE OF TIP DEFLECTION FOR CANTILEVER BEAM
(see Fig. 3.5)

<table>
<thead>
<tr>
<th></th>
<th>TIP DEFLECTION</th>
<th>STRESS AT A</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in.</td>
<td>ksi</td>
</tr>
<tr>
<td>1 Element</td>
<td>.3393</td>
<td>40.22</td>
</tr>
<tr>
<td>2 Elements</td>
<td>.3462</td>
<td>40.15</td>
</tr>
<tr>
<td>4 Elements</td>
<td>.3513</td>
<td>39.92</td>
</tr>
<tr>
<td>Exact</td>
<td>.3558</td>
<td>40.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>POINT</th>
<th>EXACT $\times 10^{-3}$</th>
<th>F.E. $\times 10^{-3}$</th>
<th>STRESS $\sigma_x$ psi</th>
<th>STRESS $\sigma_y$ psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4764</td>
<td>1.4724</td>
<td>975.0</td>
<td>976.4</td>
</tr>
<tr>
<td>2</td>
<td>1.3895</td>
<td>1.3863</td>
<td>852.6</td>
<td>852.4</td>
</tr>
<tr>
<td>3</td>
<td>1.1428</td>
<td>1.1413</td>
<td>609.4</td>
<td>606.9</td>
</tr>
<tr>
<td>4</td>
<td>.7786</td>
<td>.7807</td>
<td>249.2</td>
<td>247.9</td>
</tr>
<tr>
<td>5</td>
<td>.3674</td>
<td>.3777</td>
<td>456.2</td>
<td>455.6</td>
</tr>
</tbody>
</table>

TABLE 3-3

DEFLECTION AND STRESSES FOR SQUARE PLATE UNDER PARABOLIC LOADING
(see Fig. 3.6)
FIGURE 3.1 POLYNOMIAL FUNCTION FOR 16 D.O.F. ELEMENT
FIGURE 3.2 POLYNOMIAL FUNCTIONS FOR 18 D.O.F. ELEMENT
FIGURE 3.3 POLYNOMIAL FUNCTION FOR 20 D.O.F. ELEMENT
FIGURE 3.4 DEGREES OF FREEDOMS OF A PLANE STRESS RECTANGULAR ELEMENT
\[ E = 30000 \text{ ksi} \]
\[ v = 0.25 \]

**Figure 3.5 Cantilever Beam**

**Figure 3.6 Square Plate Under Parabolic Inplane Loading**

\[ N_x = 100 \left( 1 - \left( \frac{y}{a} \right)^2 \right) \]
\[ t = 0.1 \text{ in} \]
\[ E = 10^7 \text{ psi} \]
\[ v = 0.3 \]
CHAPTER 4

ORTHOTROPIC DECK ELEMENTS

4.1 INTRODUCTION

The plate bending element developed in Chapter 2 and the plane stress one of Chapter 3 were tested individually, and in both cases their behaviour was shown to be very satisfactory.

In this Chapter, the bending and membrane actions are combined and transformed to a global system of axes. Since the plane stress element was developed for the special case of a rectangle, then only a rectangular shell can be formed. This shell can idealize both the deck plate and the stiffeners. However, in the case of open stiffeners, such as flats or inverted T-sections, the stiffener could be simulated by means of an eccentric beam, if it is so desired. The eccentric beam, developed in this Chapter, is similar to the work by McBean. However, a more general case is presented here. The beam element has a further application, which is to simulate the cross-girders in the deck. Figures 4.1 and 4.2 summarize the different elements used in different idealizations.
4.2 FLAT SHELL ELEMENT

The degrees of freedom of the plate bending and the plane stress elements are combined into a single vector. This vector will have six components (3 translations and 3 rotations) at each node, plus two freedoms (a translation and a normal slope) at the mid-sides as shown in Fig.4.3. Referring to the global displacements by the vector \( \vec{r} \) such that

\[
\vec{r} = \{ \bar{U}_1 \bar{V}_1 \bar{W}_1 \bar{\theta}_x \bar{\theta}_y \bar{\theta}_z \ u_{12} \ \theta_{12} \ \bar{U}_2 \ \bar{V}_2 \ \bar{W}_2 \ \bar{\theta}_x \ \bar{\theta}_y \ \bar{\theta}_z \ u_{23} \ \theta_{23} \ \\
\bar{U}_3 \ \bar{V}_3 \ \bar{W}_3 \ \bar{\theta}_x \ \bar{\theta}_y \ \bar{\theta}_z \ u_{34} \ \theta_{34} \ \bar{U}_4 \ \bar{V}_4 \ \bar{W}_4 \ \bar{\theta}_x \ \bar{\theta}_y \ \bar{\theta}_z \ u_{41} \ \theta_{41} \}
\]

(4.1)

then, the vector \( \vec{r}_Q^* \) corresponding to the local bending freedoms, given by (2.90), is transformed such that

\[
\vec{r}_Q^* = \Lambda_B \vec{r}
\]

(4.2)

Similarly, for the local in-plane freedoms \( \vec{r}^* \), given by (3.30), is transformed such that

\[
\vec{r}^* = \Lambda_M \vec{r}
\]

(4.3)

where \( \Lambda_B \) and \( \Lambda_M \) are matrices of direction cosines shown in Tables 4-1 and 4-2 respectively and described later in this section. The bending stiffness matrix corresponding to the global displacements is obtained using (2.92) and (4.2), thus
Similarly, the membrane stiffness corresponding to the global system is obtained using (3.32) and (4.3). Hence,

\[ \bar{K}_M = \Lambda^t M K^* \Lambda_M \]  

The total stiffness \( \bar{K} \) which is the sum of the bending and membrane components is, therefore

\[ \bar{K} = \bar{K}_B + \bar{K}_M \]  

Matrix \( \bar{K} \) is of order 32x32. The global forces corresponding to the displacement \( \bar{F} \) is given by \( \bar{K} \). Hence,

\[ \bar{K} = \bar{K} \bar{F} \]  

Referring to Table 4-1 and 4-2, the direction cosines are now calculated from the relation

\[
\begin{bmatrix}
    x_L \\
    y_L \\
    z_L
\end{bmatrix} =
\begin{bmatrix}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    X \\
    Y \\
    Z
\end{bmatrix}
\]  

where \( x_L, y_L \) and \( z_L \) refer to the local system of axes for the
shell and X, Y, Z the global axes, as shown in Fig.4.3. Furthermore, in both the cases of the plate bending element (Fig.2.8) and the plane stress (Fig.3.4), the $x_L$ axis runs along the side 1-2. The $y_L$ axis is perpendicular to it and passes through node 4. Hence,

$$a_{11} = \frac{(X_2 - X_1)}{S_{12}}$$
$$a_{21} = \frac{(Y_2 - Y_1)}{S_{12}}$$
$$a_{31} = \frac{(Z_2 - Z_1)}{S_{12}}$$

(4.9)

Similarly, for a rectangular element,

$$a_{21} = \frac{(X_4 - X_1)}{S_{41}}$$
$$a_{22} = \frac{(Y_4 - Y_1)}{S_{41}}$$
$$a_{23} = \frac{(Z_4 - Z_1)}{S_{41}}$$

(4.10)

where $X_i$, $Y_i$, and $Z_i$ are the global coordinates of node $i$ and $S_{ij}$ the length of the side $ij$ as defined in Table 2-3.

The last set of direction cosines is obtained using the orthonormality condition. That is

$$a_{11}a_{11} + a_{22}a_{22} + a_{33}a_{33} = 1$$

(4.11)

Using also the fact that the determinant of the matrix of direction cosines is +1 for right-handed systems of axes (see Jaeger [1]), then expansion of the determinant of (4.8) yields:
Equating coefficients of (4.11) and (4.12) yields

\[ a_{31}(a_{12}a_{23}-a_{2}a_{13}) + a_{32}(a_{21}a_{13}-a_{11}a_{23}) + a_{33}(a_{11}a_{22}-a_{12}a_{21}) = 1 \]  (4.12)

Equating coefficients of (4.11) and (4.12) yields

\[ a_{31} = a_{12}a_{23} - a_{2}a_{13} \]
\[ a_{32} = a_{21}a_{13} - a_{11}a_{23} \]
\[ a_{33} = a_{11}a_{22} - a_{12}a_{21} \]  (4.13)

Expressions (4.9), (4.10) and (4.13) are very general and do not have any restrictions on the spatial position of the element providing both the local and global axes are right-handed.

4.3 ECCENTRIC BEAM ELEMENT

As mentioned in the introduction of this Chapter, the beam element is used to simulate the cross beams in the orthotropic deck, since these are usually inverted T-sections. It could also be used as an alternative to the flat shell when idealizing open stiffeners. However, effects such as warping in the stiffeners cannot be simulated using the beam element.

The stiffness matrix is derived on the basis of a space frame, that is, with six degrees of freedom at each node. However, the element here is different from the conventional one given by Gere and Weaver, since two extra freedoms must be introduced at the mid-point nodes for compatibility between the
beam and shell element. This has the effect of producing a linear variation in the axial forces and torques when the beam is attached to a shell. The stiffness is derived, first of all, about the centroid of the element, then a transformation is performed to account for the eccentricity effects.

4.3.1 Stiffness Matrix of a Concentric Beam

The stiffness matrix of the general beam, shown in Fig. 4.4, is obtained by adding the axial, torsion and the bending (about two axes) contributions, since they are uncoupled.

The origin is chosen at node 1 and the local x-axis along the line 1-2. A third node is used to determine the orientation of the element in space. The local z-axis is parallel to the plane formed by nodes 1, 2 and 3. Then, the y-axis is simply perpendicular to that plane and forming a right-hand system of axes.

Compatibility of deformations and strains, with the plane stress rectangle, requires the axial displacement to vary quadratically. This variation is defined in terms of $u_1$, $u_2$ and $u_{12}$ as shown in Fig. 4.5. Therefore, if

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad (4.14)$$

and the constants $\alpha_1$, $\alpha_2$ and $\alpha_3$ are determined in terms of $u_1$, $u_2$ and $u_{12}$, then

$$u = (1-3\xi+2\xi^2)u_1 + (-\xi+2\xi^2)u_2 + (4\xi-4\xi^2)u_{12} \quad (4.15)$$
where
\[ \xi = \frac{x}{L} \]  
\hspace{1cm} (4.16)

Evaluation of the strain energy for the axial contribution yields

\[ U_e = \frac{1}{2} \frac{EA}{L} \int_0^1 u_1^2 d\xi \]  
\hspace{1cm} (4.17)

where E is the elastic modulus, A the cross-sectional area of the element and L the length. Substituting the derivative of (4.15) into (4.17) and writing the result in matrix form yields the stiffness matrix of the axial component. Hence

\[
\begin{bmatrix}
    7 & 1 & -8 \\
    1 & 7 & -8 \\
    -8 & -8 & 16
\end{bmatrix}
\]
\hspace{1cm} (4.18)

The case of the torsion stiffness matrix is derived in a similar manner. Here again, a quadratic polynomial in x is used since the normal slope variation in the plate bending element assumes such a variation. Hence, replacing the axial rigidity EA/L by its torsional equivalent, GJ/L yields

\[
\begin{bmatrix}
    7 & 1 & -8 \\
    1 & 7 & -8 \\
    -8 & -8 & 16
\end{bmatrix}
\]
\hspace{1cm} (4.19)
where $G$ is the shear modulus and $J$ the St. Venant torsion constant.

For the case of bending about the $y$-axis, the polynomial for the transverse displacement is cubic and is defined in terms of the two translations and rotations at the nodes. Hence,

$$w = (1-3\xi^2+2\xi^3)w_1 + (-\xi+2\xi^2-\xi^3)\theta_y + (3\xi^2-2\xi^3)w_2 + (\xi^2-\xi^3)\theta_y$$  \hspace{1cm} (4.20)

The stiffness matrix, which can be found in any standard text on matrix structural analysis, takes the form:

$$
K_{By} = \frac{EI_y}{L} \begin{bmatrix}
w_1 & \theta_y & w_2 & \theta_y \\
12/L^2 & -6/L & -12/L^2 & -6/L \\
-6/L & 4 & 6/L & 2 \\
-12/L^2 & 6/L & 12/L^2 & 6/L \\
-6/L & 2 & 6/L & 4
\end{bmatrix}
$$  \hspace{1cm} (4.21)

Similarly, for bending about the $z$-axis

$$
K_{Bz} = \frac{EI_z}{L} \begin{bmatrix}
u_1 & \theta_z & u_2 & \theta_z \\
12/L^2 & 6/L & -12/L^2 & 6/L \\
6/L & 4 & -6/L & 2 \\
-12/L^2 & -6/L & 12/L^2 & -6/L \\
6/L & 2 & -6/L & 4
\end{bmatrix}
$$  \hspace{1cm} (4.22)
where \( I_y \) and \( I_z \) are the moments of inertia about the \( y \) and \( z \)-axes respectively.

Combining now the individual components, yields a stiffness matrix for the general beam element with

\[
R = K r
\]

where

\[
\mathbf{r} = \{u_1 \ v_1 \ w_1 \ \theta_{x_1} \ \theta_{y_1} \ \theta_{z_1} \ u_{12} \ \theta_{x_{12}} \ u_2 \ v_2 \ w_2 \ \theta_{x_2} \ \theta_{y_2} \ \theta_{z_2}\}
\]

and \( \mathbf{R} \) the corresponding force vector. The matrix \( K \) is of order \( 14 \times 14 \) and is shown in Table 4-3.

### 4.3.2 Beam with Eccentric Nodes

The stiffness matrix derived in (4.23) corresponds to the displacement vector (4.24) where the degrees of freedom act at the centroid of the element. However, if the nodes of the beam are to coincide with those of the shell to which it is attached, a transformation to the plate nodal position is thus required. For the general case, shown in Fig. 4.6, the \( y \) and \( z \)-axes are shifted to \( y' \) and \( z' \) by a distance \( e_y \) and \( e_z \). The signs of \( e_y \) or \( e_z \) are considered positive when the movement takes place along the positive directions of the \( y \) and \( z \)-axes respectively. In fact, \( e_y \) and \( e_z \) are simply the distances between the deck plate nodal points and the centroid of the stiffener. Thus, if
r' defines the displacements and rotations about the dashed axes,

\[
r' = \{u'_1, v'_1, w'_1, \theta'_1, x'_1, y'_1, z'_1, u'_2, v'_2, w'_2, \theta'_2, x'_2, y'_2, z'_2\}\tag{4.25}
\]

then

\[
r = T r'
\]

where T is shown in Table 4-4. It must be noted that since no rotation about the y-axis is defined at the mid-point node, then transformation of \(u_{12}\) to \(u'_{12}\) is done by calculating the slope \(\partial w/\partial \xi\) from (4.20) and substituting \(\xi = 0.5\). The result is given by the seventh row of T in Table 4-4. The corresponding force vector about the dashed axes is defined by \(\{R'\}\). Hence, from (4.26) and (4.23), the required stiffness matrix is given by

\[
R' = K' r'
\]

where

\[
K' = T^T K T
\]

Finally, transformation of the matrix K' to a global system of axes is achieved through the relation

\[
r' = \Lambda_8 \bar{r}
\]

where \(\Lambda_8\) is shown in Table 4-5, and the direction cosines have already been defined by (4.9), (4.10) and (4.13). The vector \(\bar{r}\) defines the global displacements
The stiffness matrix in the global axes is, therefore
\[ R = K \] (4.31)

where \( \{R\} \) is the force vector in the global system. Using (4.29) and (4.27) yields
\[ K = A^t K'A \] (4.32)

Substituting from (4.28) for \( K' \) gives
\[ \bar{K} = A^t \bar{K}' A \] (4.33)

which is the stiffness matrix for an eccentric beam element.

4.4 DIRECTION OF MID-SIDE DISPLACEMENTS

The transformation of the shell element to a global system of axes was obtained in Section 4.2. The translations and rotations at the nodes were transformed by means of direction cosines. However, the mid-side freedoms, in the case of the shell or beam element, remained in the local axes. When the assembly of the elements is performed at the various nodes of the structure, it must be insured that, along any edge, be-
between two or more adjacent elements, the translation $u_{ij}$ and the normal slope $\theta_{ij}$ between nodes $i$ and $j$, have the same direction. A consistent method of insuring this, is to assume the positive direction of the mid-side freedoms from node $i$ to $j$ providing $i < j$. Figure 4.7 shows a typical positive direction for the mid-side freedoms.
TRANSFORMATION MATRIX OF DIRECTION COSINES FOR PLATE BENDING PART

<table>
<thead>
<tr>
<th></th>
<th>(a_{11})</th>
<th>(a_{12})</th>
<th>(a_{13})</th>
<th>(u_{12})</th>
<th>(u_{13})</th>
<th>(u_{14})</th>
<th>(\theta_{x_1})</th>
<th>(\theta_{y_1})</th>
<th>(w_1)</th>
</tr>
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<tbody>
<tr>
<td>(A)</td>
<td>(a_{21})</td>
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<td>(u_{22})</td>
<td>(u_{23})</td>
<td>(u_{24})</td>
<td>(\theta_{x_2})</td>
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<tr>
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<td>(a_{33})</td>
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<td>(\theta_{x_4})</td>
<td>(\theta_{y_4})</td>
<td>(w_4)</td>
</tr>
</tbody>
</table>

**TABLE 4-1**
TABLE 4-2

TRANSFORMATION MATRIX OF DIRECTION COSINES FOR PLANE STRESS PART

\[
\begin{pmatrix}
\begin{array}{ccc|ccc|ccc|ccc|ccc}
1 & 0 & 0 & u_1 & v_1 & w_1 & x_1 & y_1 & z_1 & u_{12} & \theta_{12} & u_{13} & \theta_{13} & u_{14} & \theta_{14} \\
0 & 1 & 0 & u_2 & v_2 & w_2 & x_2 & y_2 & z_2 & u_{23} & \theta_{23} & u_{24} & \theta_{24} \\
0 & 0 & 1 & u_3 & v_3 & w_3 & x_3 & y_3 & z_3 & u_{34} & \theta_{34} \\
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{ccc}
\begin{array}{ccc}
1 & 0 & 0 & a_{11} & a_{12} & a_{13} \\
0 & 1 & 0 & a_{21} & a_{22} & a_{23} \\
0 & 0 & 1 & a_{31} & a_{32} & a_{33}
\end{array}
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{ccc}
\begin{array}{ccc}
1 & 0 & 0 & u_1 & v_1 & \theta_{1} \\
u_2 & v_2 & \theta_{2} \\
u_3 & v_3 & \theta_{3}
\end{array}
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{ccc}
\begin{array}{ccc}
1 & 0 & 0 & u_{12} & \theta_{12} & u_{13} & \theta_{13} & u_{14} & \theta_{14} \\
u_{23} & \theta_{23} & u_{24} & \theta_{24} \\
u_{34}
\end{array}
\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{ccc}
\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
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a_{31} & a_{32} & a_{33}
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\end{array}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\begin{array}{ccc}
\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}
\end{array}
\end{pmatrix}
\]
TABLE 4-3
STIFFNESS MATRIX FOR CONCENTRIC BEAM ELEMENT

\[ K = \begin{bmatrix}
\frac{7EA}{L} & -\frac{8EA}{L} & \frac{EA}{L} & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
\frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} & -\frac{6EI_y}{L^2} & \frac{7GJ}{3L} & -\frac{8GJ}{3L} & \frac{GJ}{3L} & \theta_{x_1} \\
\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & \frac{6EI_y}{L^2} & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & \frac{2EI_y}{L} & \theta_{y_1} \\
\frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} & \frac{16EA}{L} & -\frac{8EA}{L} & \frac{16GJ}{3L} & \theta_{z_1} \\
\frac{16GJ}{3L} & -\frac{8GJ}{3L} & \frac{7EA}{3L} & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & \frac{12EI_y}{L^3} & \frac{6EI_y}{L^2} \\
\frac{12EI_y}{L^3} & -\frac{6EI_y}{L^2} & \frac{6EI_y}{L^2} & \frac{7GJ}{3L} & -\frac{8GJ}{3L} & \frac{GJ}{3L} & \theta_{x_2} \\
\frac{4EI_y}{L} & \theta_{y_2} & \frac{4EI_z}{L} & \theta_{z_2}
\end{bmatrix} \]
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</tbody>
</table>

**TABLE 4-4**

ECCENTRICITY MATRIX $T$
$\Lambda_g =$

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33} \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

**TABLE 4-5**

TRANSFORMATION MATRIX OF DIRECTION COSINES
FOR ECCENTRIC BEAM ELEMENT
FIGURE 4.1 TYPICAL IDEALIZATION FOR TORSIONALLY WEAK RIBS

FIGURE 4.2 TYPICAL IDEALIZATION FOR TORSIONALLY STIFF RIBS
**FIGURE 4.3** RECTANGULAR SHELL ELEMENT WITH 32 DEGREES OF FREEDOM
FIGURE 4.4 BEAM ELEMENT GEOMETRY

FIGURE 4.5 BEAM ELEMENT WITH 14 DEGREES OF FREEDOM

FIGURE 4.6 ECCENTRICITIES IN THE BEAM ELEMENT
FIGURE 4.7 SHELL ELEMENTS IN GLOBAL SYSTEM OF AXES
5.1 INTRODUCTION

In this Chapter, the computer program developed for the analysis of stiffened orthotropic decks is described very briefly. The elements used in the program as described in Chapters 2, 3 and 4 have been tested separately. It is necessary now to establish that when the element combinations are used to simulate a stiffened deck, the behaviour is still satisfactory.

Two examples dealing with open stiffeners and three others dealing with closed stiffeners are discussed. The open stiffeners are compared with folded plate theory, which is a harmonic analysis based on plate bending and plane stress classical theory. The closed stiffened decks are compared with experimental results. But first, a little summary of the computer program itself is presented.

5.2 COMPUTER PROGRAM

Table 5-1 shows a flow-chart of the whole program which contains five overlays. Each overlay uses a maximum of 300K bytes on the 360/75 IBM computer. Information is passed between
the various overlays through disks with Direct Access Files to minimize I/O time. Double precision is used throughout which, in the author's opinion, is compulsory on the 360 series. The work of Jofriet also confirms the deterioration of the results with single precision arithmetic. Fortran H, which optimizes the sequence of operations, is used.

The first overlay generates the stiffness, stress and consistent load matrices for the minimum number of elements that will be used throughout the structural idealization. For example, in Fig.2.13, the rhombic cantilevered plate had 16 elements but only one stiffness matrix was generated since all the elements were identical. Each type of element is stored on a Direct Access File for quick recovery during the assembly.

In the second overlay, the topology of the structure and the numbering system determine the size and bandwidth of the problem. To minimize on the input data, only the corner nodes of the elements are fed as input together with a code to identify its type as determined in the first overlay. The addresses of each degree of freedom are determined using the following algorithm:

a) Form an integer matrix with its first column having the corner node numbers throughout the structure.

b) Subsequent columns contain integers which represent the mid-side nodes between the original node (in column one)
and the actual numbers in the other columns. For example, a 6 in the first column with 12 and 14 in the other columns means there is a mid-side node 6-12 and 6-14. The sequence determines the direction of the mid-side freedoms as established in Section 4.4. A mid-side node at the junction of many elements is referred to just once with the smaller node in the first column.

c) From the layout of the integer matrix, the "addresses" of each degree of freedom, which correspond to a row number in the assembled stiffness matrix, is determined. Note that six degrees of freedom correspond to a corner node while only two to the mid-side nodes.

d) Test for maximum address, which determines the size of the assembled matrix, while the maximum difference between any two addresses in any element determines the bandwidth.

In the third overlay, using the established location for each freedom, the assembly of the total stiffness matrix is performed employing the results obtained from the first overlay. A lower banded matrix is stored row-wise on a Direct Access File.

The fourth overlay solves the system of equations for any given number of loading cases, as explained in the next section.

Finally, the fifth overlay calculates the membrane and bending stresses for each element.
5.2.1 A Note on the Solution Procedure

The solution of the simultaneous equations of equilibrium is the most time-consuming effort in a finite element analysis. For small size problems, where a banded matrix fits in the core of the machine, any suitable algorithm presents no difficulties, as shown by Weaver\cite{79}. For problems where either the size or the bandwidth or both prohibit an in-core solution, other methods have to be devised. Some of these are described below.

5.2.1.1 Cholesky's square root algorithm

The algorithm for upper and lower decomposition described by Weaver\cite{79} has been generalized to handle large matrices. During the decomposition stage, portions of the matrix are brought into core from an external file and are operated upon in-core. The restriction is that an array of size \((\text{NBW})(\text{NBW}+1)/2\) is required where NBW is the bandwidth.

5.2.1.2 Modified Cholesky approach

A method proposed by Rubenstein and Rosen\cite{80} avoids the square root operations by decomposing the matrix into a triple product. The upper and lower matrices are the transpose of each other with unity along their diagonals, while the middle matrix is a diagonal one. Melosh and Bamford\cite{81} argue that
significant loss of accuracy occurs during the square root operation and improved answers are obtained using the triple product decomposition. However, this was not found to be the case in this work, probably due to the Double Precision arithmetic. This method was programmed using Direct Access Files and operating virtually on one row at a time and hence, in theory, any size problem with any bandwidth up to 900 could be solved. However, the computing time involved was rather high. To overcome this problem, the method was reformulated operating on a lower diagonal matrix with a number of rows stored in a single array and using the maximum core capability of the computer.

5.3 ORTHOTROPIC DECKS WITH OPEN STIFFENERS

Two examples of an orthotropic deck with open-type stiffeners are analysed. In the first case, the stiffeners are treated as shell elements while in the second case, the stiffeners are idealized by eccentric beams. The finite element results are compared with folded plate theory in both cases.

5.3.1 Deck with Four Open Stiffeners

The choice of this example is to evaluate the accuracy of the shell element as both the membrane and bending capabilities are added. It also provides a check on the computer program
since an "exact" solution has been given for the structure shown in Fig. 5.1. Due to the symmetry of the structure and the loading, only half the model was idealized. The idealization is shown in Fig. 5.2 where 48 nodes and 35 elements are used. Only two different types of elements are generated: one for the deck and one for the stiffeners. Although the dimensions of the structure and the loading conditions differ completely from a steel orthotropic deck, the availability of a harmonic analysis by Willam and Scordelis makes this example ideal for testing the accuracy of the program.

The first loading case is a uniformly distributed load over the entire deck. The second case is a concentrated vertical load at mid-span over the outer stiffener. The third load case is a horizontal point load acting in the transverse direction at mid-span.

The results are compared with the "exact" solution for the displacements $u$, $v$ and $w$ corresponding to the $X$, $Y$ and $Z$ directions. The membrane stresses $\sigma_x$ in the deck plate and at the bottom of the stiffeners are also compared.

For the first loading case, all the deflections and stresses agree extremely well with folded plate theory. The results are shown in Table 5-2. For the second case, the results are shown in Table 5-3. Note the gradual decrease in rib stresses at stations B, C and D despite the fact that only one element was used transversely between the rib stations. The high performance
of the plate bending element correctly transferred some of the load to the adjacent stiffeners. Finally, for the third loading, the results are shown in Table 5-4. Agreement with folded plate theory is, in general, very good except for the stresses at the point of load application where a finer grid should have been used.

The analysis of this problem by orthotropic plate theory, based on Huber's equation as given by (1.1), would not have been very easy for the third loading case. Furthermore, the in-plane displacements u and v are not considered. This demonstrates the advantage of the finite element method to handle any loading case and any boundary conditions, not just simple supports, as required by the folded plate theory.

5.3.2 Deck with Three Open Stiffeners

The plan view and cross-section of the structure is shown in Fig.5.3. A concentrated load of 400 lbs is placed over the centre of the deck. Four grids were chosen to idealize a quarter of the structure as shown in Fig.5.4. The stiffeners are treated as eccentric beams in this case, with their centroids 1.5 in. below the deck plate nodal points. The finite element solution is compared with folded plate theory as reported by Mehrain.

A plot of the central deflection versus grid refinement is shown in Fig.5.3. The convergence is very rapid and a lower
bound as expected when using conforming elements. However, the beam elements did not include shear deformations.

The in-plane forces in the deck are shown in Fig.5.6. Note the discontinuities at the nodal points in the x-direction, which will be discussed below.

The bending moments for the outer and inner beams are shown in Fig.5.7. Excellent agreement is obtained with the exact solution if the moment at the mid-length of the element is considered. The discontinuities in the moments at the nodes arise from the fact that gross equilibrium at the nodes must be satisfied. In other words, the difference between the nodal bending moments (ΔM) is taken by the deck plate itself. Since the nodal points of the beam are eccentric with respect to the centroid, then ΔM gives rise to a discontinuity ΔP in the axial loads such that

\[(ΔM) = (ΔP) \text{ eccentricity} \quad \text{(5.1)}\]

It is this axial load ΔP that gives rise to the discontinuities in the direct stresses of the deck plate as observed in Fig.5.6. It can, therefore, be concluded that averaging the nodal stresses is not strictly desirable in this case. A more reliable procedure would be to consider the stress values at the mid-sides of the elements. The linear variation in axial stresses in both the beam and shell element, through the introduction of mid-side nodes, can therefore be appreciated. If that variation were constant, then very large discontinuities would have arisen at the
nodes which would have obscured the stress pattern.

Finally, a plot of the moments in the deck plate is shown in Fig.5.8. The exact solution is calculated by Mehrain at a distance \( x = 14 \frac{2}{3} \) in. from the origin, while the finite element results were obtained for \( x = 14 \) inch. This could explain the small discrepancies near the regions of high-stress gradients.

## 5.4 DECKS WITH CLOSED STIFFENERS

In this section, the developed program is used to investigate closed stiffened decks of trapezoidal shape. The results are compared with experimental work performed elsewhere.

### 5.4.1 Single Cell Closed Deck

Tests performed on two single cell closed stiffeners have been reported by Erzurumlu and Toprac. The geometry of both specimens is shown in Fig.5.9. The loading applied on specimen A was 5.32 kips while on specimen B, this was doubled. The load was applied, at mid-span, over a pad with dimension 20x10 in. For each specimen, three idealizations were performed.

The first two idealizations considered half the deck as shown in Fig.5.10. Idealization 1 considered the loading uniformly distributed over one element of area 9x12 in. (for half the model) while in idealization 2, the correct loading area was
simulated as explained in Section 2.6. The third idealization is shown in Fig. 5.11 where only a quarter of the structures were considered due to the double symmetry. As a matter of fact, the first two idealizations showed symmetry in both deflections and stresses for symmetric node positions, but the effects of local bending stresses or shear lag could not be seen. Hence, idealization 3 was adopted since the nodal points along the span centreline coincide with the strain gauge locations at the top of the deck plate and the bottom of the stiffener.

The deflections at the mid-span are shown in Fig. 5.12 for both specimens. Note that for a given load, the ratio of deflection of the specimens is roughly inversely proportional to the ratio of their cross-sectional inertia. Agreement with the measured deflections needs no elaboration.

The stress at the external surface of the webs are shown in Fig. 5.13 at three stations along the span. The agreement with the measured values at a specific distance from the deck plate is excellent. This is due to the linear variation in direct stresses which permitted the simulation of the whole depth of the web by one single element. As expected, the stresses at the bottom of the web are positive and increasing in magnitude towards the centreline of the specimens. The stresses at the top of the webs are compressive but decrease slightly near the centrelines of the models. This fact is explained by examining the bending stresses in the webs which, at the outer surface
near the deck plate, are positive. These positive bending stresses tend to decrease the total stress at the top of the webs. This is further confirmed by examination of the local bending stresses, in specimen B, which are higher due to higher load intensity.

Stresses at the top of the deck plate and the bottom of the stiffener, along the line symmetry in the span direction, are shown in Fig.5.14. The stresses vary linearly up to the region of load application where bending stresses are developed. The agreement with the experimental results is, in general, quite good especially at the bottom of the stiffeners where high tensile stresses are developed. It is interesting to note that, for specimen B, the stresses at the top of the deck, due to local bending effects, are of the same order of magnitude as the stresses at the bottom of the stiffeners. These effects are usually neglected in the design procedure of orthotropic decks.

5.4.2 Deck Panel with Six Closed Stiffeners

A test on a full scale closed stringer deck panel has been reported by Dowling. In this section, the experimental tests are compared with the finite element program for both deflections and stresses.
5.4.2.1 Brief description of the model

The panel, shown in Fig.5.15, was constructed from a 3/8 in. deck plate, stiffened in one direction by six closed stiffeners of trapezoidal shape with 3/16 in. thickness. Four L-shaped open stiffeners acted as edge beams to the panel. High tensile weldable steel was used throughout. The whole panel was supported on twelve rubber pads as shown in Fig.5.16. Four loading cases were studied and these are shown in Fig.5.17.

5.4.2.2 Idealization of the panel

Due to the symmetry of the structure and the loadings, only half the panel, spanwise, was considered. Four different finite element idealizations were tried to reach the desired accuracy. The first idealization considered only one element between and over the stiffeners and three elements spanwise. The edge beams were considered infinitely stiff. The second idealization was similar to the first, except that four elements were considered spanwise. In the third trial, an extra node was introduced between, and over, the stiffeners as shown in Fig.5.18. The last and final idealization adopted, considered the effects of both the edge beams and the rubber pads. The same grid as shown in Fig.5.18 was used. The edge beams were treated as eccentric stiffeners around the panel and the rubber pads were idealized as elastic springs of stiffness 8 tons/in.
5.4.2.3 Comparison of deflections

The theoretical deflections across the deck, at mid-span, for the four loading conditions, are shown in Fig.5.19 to 5.22. Unfortunately, no measured experimental deflections for load case 3 were reported. Comparison between the measured and calculated results show excellent agreement as far as the deflected shape is concerned. However, the computed results appear consistently stiffer. This is probably attributed to the rubber pad stiffness which restricted the whole panel to undergo a nearly rigid body movement.

It is interesting to note the inflection points in the deflection curves due to the torsional rigidity of the stiffeners. Such effects are disguised if Huber's equation is used, no matter how accurately the flexural or torsional rigidities are calculated. In fact, the results of orthotropic plate theory have been reported by Dowling\textsuperscript{19} using finite differences and the "smooth" deflection curves obtained bear little resemblance to the actual deflected profile.

Another drawback of orthotropic plate theory is the fact that it is not sensitive to load position or area of contact of loads with the deck plate. It is seen from Fig.5.19 to 5.22 that loads of the same magnitude, but different in contact area or position, exhibit different profiles which are correctly predicted using finite elements.
5.4.2.4 **Comparison of direct stresses**

One of the main advantages of the finite element method, in the analysis of stiffened structures, is the capability of calculating the stresses in the individual members without reference to the effective width or any other approximate methods. This is clearly the case here where direct stresses, at the bottom of the stiffeners or the deck plate itself, can be directly compared with experimental values.

The stresses for the four loading cases are shown in Fig. 5.23 to 5.26. When the load is placed between two stiffeners, considerable rotation and bending of the adjacent sections occur simultaneously which give rise to a rather large variation in the stresses at the bottom of these stiffeners. Since only one element was used to simulate the bottom of the troughs, then only a linear variation of stresses can be obtained crosswise. The agreement with the experimental values, measured in the middle at the bottom of stiffeners, is excellent.

For cases where the load is placed symmetrically over the stiffeners, no rotation of the stiffener occurs and only membrane stresses due to direct bending are developed. Comparison with the measured values show an error of the order of 8% for the highest stresses.

It is instructive to compare the roles played by the torsional rigidity of the stringers in distributing the load sideways.
For a load placed symmetrically between two stiffeners, the majority of the load is carried between these two stiffeners and the torsional rigidity of the sections is the effective agent that comes into play. However, for a load placed symmetrically over a stiffener, no twisting of the section occurs and it is the plate bending rigidity that is solely responsible for transferring the load sideways. It can, therefore, be appreciated that the influence of the plate rigidity in rendering the torsional rigidity of the stringers less effective is of greater importance when the load is in position 1 and 3 rather than in 2 or 4.

5.4.2.5 Comparison of transverse bending stresses

A plot of the transverse bending stresses at mid-span of the panel is shown in Fig. 5.27. For the stiffener walls only, the membrane stresses have been added since the strain gauges were placed on the outside of the stiffeners. By doing so, a better agreement with experiment was obtained especially for the webs that are closest to the load. The other transverse membrane stresses in the troughs were not significant.

For the points directly under the load, erratic readings were obtained for the transverse bending stresses in the deck plate as reported by Dowling 19. However, in general, a fairly
good agreement between theory and experiment is obtained for the stresses away from the load.

Points of contraflexure may be observed in the deck plate between the stringers.
GENERATE NECESSARY STIFFNESS, STRESS AND CONSISTENT LOAD MATRICES

GENERATE ADDRESS LOCATION FOR ALL FREEDOMS IN STRUCTURE

ASSEMBLE TOTAL STIFFNESS AND CONSISTENT LOADING. APPLY BOUNDARY CONDITIONS

SOLUTION OF SIMULTANEOUS EQUATIONS

CALCULATE STRESSES

FOLLOW CHART OF COMPUTER PROGRAM SHOWING VARIOUS OVERLAYS

TABLE 5-1
FLOW CHART OF COMPUTER PROGRAM SHOWING VARIOUS OVERLAYS
### TABLE 5-2

DEFLECTIONS AND STRESSES UNDER UNIFORMLY DISTRIBUTED LOAD

<table>
<thead>
<tr>
<th>QUANTITY</th>
<th>METHOD</th>
<th>SPAN</th>
<th>STATION A</th>
<th>STATION B</th>
<th>STATION C</th>
<th>STATION D</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u \times 10^0$</td>
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<td>0</td>
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<td>1.48</td>
<td>1.48</td>
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<td>-1.33</td>
<td>-1.33</td>
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<td>50</td>
<td>-1.34</td>
<td>-1.34</td>
<td>-1.34</td>
<td>-1.34</td>
</tr>
<tr>
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<td>4.29</td>
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<td>50</td>
<td>4.32</td>
<td>4.32</td>
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</tbody>
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**FP**: Folded Plate Theory  
**FE**: Finite Elements
### Table 5-3

**Deflections and Stresses under Vertical Concentrated Load at Midspan**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Method</th>
<th>Span</th>
<th>Station A</th>
<th>Station B</th>
<th>Station C</th>
<th>Station D</th>
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<tbody>
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<td>1.45</td>
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<td>-5.96</td>
<td>-6.01</td>
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<td>50</td>
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<td>-6.00</td>
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<td>-1.00</td>
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</table>

FP: Folded Plate Theory  
FE: Finite Elements
### CONCENTRATED LOAD AT MID SPAN

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<tr>
<th>QUANTITY</th>
<th>METHOD</th>
<th>SPAN</th>
<th>STATION A</th>
<th>STATION B</th>
<th>STATION C</th>
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</tr>
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<td>Stress in deck $\times 10^0$</td>
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<td>1.86</td>
<td>6.37</td>
</tr>
</tbody>
</table>

FP: Folded Plate Theory  
FE: Finite Elements

### TABLE 5-4
DEFLECTIONS AND STRESSES UNDER HORIZONTAL CONCENTRATED LOAD AT MIDSPAN
$E = 30000 \text{ kips/sq.ft.}$

$\nu = 0.15$

**Figure 5.1** Orthotropic Deck with Four Open Stiffeners
FIGURE 5.2 IDEALIZATION OF ORTHOTROPIC DECK WITH FOUR STIFFENERS USING SHELL ELEMENTS THROUGHOUT

48 NODES
35 ELEMENTS

X
Y
Z
FIGURE 5.3 ORTHOTROPIC DECK WITH 3 OPEN STIFFENERS

FIGURE 5.4 IDEALIZATIONS OF THE DECK PLATE
Figure 5.5: Convergence of central deflection with grid refinement.

Folded Plate Theory

Finite element with eccentric beams
FIGURE 5.6 VARIATION OF $N_x$ IN THE DECK PLATE
**FIGURE 5.7 VARIATION OF BENDING MOMENTS IN THE BEAMS**

- $M_x$, lb.in
- **Folded Plate Theory**
- **Finite element**

Legend:
- Solid line: Folded Plate Theory
- Dotted line: Finite element

Graph shows the variation of bending moments ($M_x$) in the beams as a function of distance ($y$) from 0 to 16 units.
FIGURE 5.8 VARIATION OF $M_x$ AND $M_y$ IN THE DECK PLATE AT DISTANCE $x = 14 2/3"$
FIGURE 5.9 SINGLE CELLS CLOSED STIFFENERS OF REF.52
FIGURE 5.10  FINITE ELEMENT IDEALIZATION OF HALF THE MODEL
FIGURE 5.11  FINITE ELEMENT IDEALIZATION OF QUARTER MODEL
Finite element

Experiment

MID-SPAN DEFLECTION

LOAD (Kips)

SPECIMEN A

SPECIMEN B
Finite element

Specimen A

Specimen B

Stress scale: 1 cm = 10 ksi

FIGURE 5.13 LONGITUDINAL STRESSES ON THE OUTSIDE OF WEBS
FIGURE 5.14 LONGITUDINAL STRESS VARIATION AT THE MIDDLE OF CROSS-SECTION
FIGURE 5.15 ORTHOTROPIC DECK WITH SIX CLOSED STIFFENERS
Rubber pads 3" x 3" x 1.5"
Young's modulus = 3300 psi

Deck plate

Cross beam

Edge beam

FIGURE 5.16  OVERALL VIEW OF ORTHOTROPIC DECK WITH SIX CLOSED STIFFENERS
LOAD CASE 1: 11\(\frac{1}{4}\) Tons

LOAD CASE 2: 11\(\frac{1}{4}\) Tons

LOAD CASE 3: 10 Tons

LOAD CASE 4: 10 Tons
FIGURE 5.18  FINITE ELEMENT IDEALIZATION OF ORTHOTROPIC DECK WITH SIX CLOSED STIFFENERS
FIGURE 5.19 DEFLECTIONS UNDER LOAD CASE 1
FIGURE 5.20  DEFLECTIONS UNDER LOAD CASE 2
FIGURE 5.21 DEFLECTIONS UNDER LOAD CASE 3

DEFLECTIONS, in

Finite element
FIGURE 5.22 DEFLECTIONS UNDER LOAD CASE 4
FIGURE 5.23 LONGITUDINAL MEMBRANE STRESSES - LOAD CASE 1

Membrane stresses
in deck plate

Finite element

Experiment

STRESSES AT BOTTOM OF STIFFENERS (ksi)

0
0

4

8

12

16

20
FIGURE 5.24  LONGITUDINAL MEMBRANE STRESSES - LOAD CASE 2

Membrane stresses in deck plate

Finite element

Experiment

STRESSES AT BOTTOM OF STIFFENERS (T/in²)

-4
0
4
8
12
16
20
FIGURE 5.25 LONGITUDINAL MEMBRANE STRESSES - LOAD CASE 3
Membrane stresses in deck plate

Finite element

Experiment

FIGURE 5.26 LONGITUDINAL MEMBRANE STRESSES - LOAD CASE 4
Positive stresses below deck plate or on outside of stiffener wall

Figure 5.27. Transverse bending stresses - Load Case 1
CHAPTER 6

INVESTIGATION OF RIB SPACING IN ORTHOTROPIC DECKS

6.1 INTRODUCTION

The various orthotropic decks reviewed in the AISC design manual show that, in general, for decks with closed stiffeners, the spacing of the troughs is about 24 inches, while for open stiffeners, the commonly used spacing is usually 12 inches. Very few decks have been designed with a larger rib spacing and these had a deck plate thickness of up to 1 inch.

The purpose of this Chapter is to describe an investigation of the effect of increasing the rib spacing for orthotropic decks with either closed or open stiffeners, the objective being a reduction in the construction cost.

The tertiary stress system in the deck plate (System III), as defined in Chapter 1, becomes important for large rib spacing and must be investigated. These stresses are usually neglected in the design procedure. For closed stiffeners, a typical trapezoidal shape is investigated for a panel with the floor beams 15 feet apart. While for the open stiffeners, an inverted T-section is investigated with the cross-beams 6 or 9 feet apart. For a given floor beam location, the rib spacing is increased from the currently used values. Furthermore, the total cross-
sectional area of the deck, under consideration, was kept approximately constant by either increasing the thickness of the deck plate itself or sometimes the stiffener walls.

Since the purpose of this Chapter is to study the behaviour of orthotropic decks with regard to rib spacing, the stresses arising from System I are not considered. In reality, complete interaction between all stress systems exists. However, to obtain a value for System I, all relevant data pertaining to the superstructure is required. Therefore, it is more appropriate to consider the bridge deck as a completely independent unit since the design is governed by the wheel loads over the deck.

From the discussion above, it is, therefore, sufficient to consider in the finite element idealization of a given panel, a sufficient number of stiffeners across the deck interacting with the cross-beams at the ends of the panel. It is felt that this idealization adequately represents the secondary and tertiary stress systems within the loaded region of the deck. The effect of continuity in the panels is partly accounted for by correctly idealizing the cross-beams as eccentric stiffeners where the torsion and the bending rigidities, about the weak axis, provide adequate restraint against the free movement at the end of the panels.

Two loading cases are studied in every deck arrangement analysed. In the first case, the load is placed between two stiffeners, while in the second case, the load is centred over
the rib. Only one truck wheel load of 12 kips is placed at mid-span of the panels. The area of contact of the load with the deck plate is 22x12 inches, as recommended by the AISC code. A 30% impact factor is also included, thus yielding a pressure of 59 psi.

In accordance with the current design provisions for steel deck bridges, the System I and System II stresses considered separately, should each be smaller than the allowable stress of 29.9 ksi. For superposition of Systems I and II, the allowable stress is 34.1 ksi. These values are based on low-alloy steel with yield stress of 52 ksi. The reason for increasing the allowable stress when superimposing the stress systems is due to the unlikelihood of developing full design values of stress System I. Also, tests on deck panels have shown a factor of safety of 10 against collapse. Examples in reference 6 (Chapter 11) show that for a 3-span bridge, the stresses arising from System I are of the order of -10 ksi at mid-span and +10 ksi over the supports.

6.2 SPACING OF TRAPEZOIDAL STIFFENERS

A typical closed stiffened deck resting longitudinally on immovable supports is shown in Fig.6.1. The spacing of the stiffeners is taken as 24 in. centre to centre with a deck plate thickness of 3/8 in. The spacing of ribs is then increased but
the stiffener shape is unchanged. The total cross-sectional area is kept constant by either increasing the thickness of the deck plate or the stiffener walls, as noted in the figure. This has the advantage of keeping the deck weight unchanged. The cross-beams are inverted T-sections placed 15 feet apart, and are also shown in Fig.6.1.

6.2.1 Deflection Study

The deflection profile for a 1.3x12 kip load placed between two stiffeners is shown in Fig.6.2. The deflection relative to the stringer walls is \(1/650\)th of the span. Figure 6.3 shows the deflection profile when the rib spacing is increased to 28.5 in. The deflection is \(1/222\)th of the span when the deck plate-thickness \(t_p\) is \(3/8\) in. while the stiffener wall-thickness \(t_s\) is .3 in. However, when \(t_p = .416\) in. and \(t_s = .25\) in., the deflection reduces to \(1/275\)th of the span. Examination of Fig.6.4 confirms that when the spacing is increased further to 34.2 in., then the deck plate has a more pronounced influence on the deflections than the torsional rigidity of the stiffeners. This conclusion is at least true for a load placed between two stiffeners.

For the second loading case, in which a load of the same magnitude is placed directly over a stiffener, examination of Figs.6.5, 6.6 and 6.7 shows that the deflection decreases with increasing rib spacing. This is to be expected since the span
between the stiffener walls is kept constant and a stiffening
effect, arising from either the deck plate or the stiffener
walls, is introduced through the increased thickness.

A comparison of these deflections, using the proposed
AISC\(^6\) formula, is shown in Table 6-1. It must be pointed out
that the empirical formula underestimates the deflections in
most of the cases investigated here.

6.2.2 Direct Stresses (System II)

The direct stress variation in the span direction is
shown in Fig.6.8 for a directly loaded stiffener. These maxi-
mum membrane stresses occur at the bottom of the stiffeners
at mid-span. The values are seen to vary between 8 and 12 ksi
(tensile). The corresponding membrane compression in the deck
plate itself is much smaller (about -2 ksi), especially when
t\(_p\) is increased. The influence of the cross-beams is seen by
the reversal of the stress signs near the end of the panel.

The stress variation at the bottom of the stiffeners for
loading case 1 is not shown since such a load case produces
lower stresses in the stiffeners.

It can be concluded from these results that the stiffener
spacing has little influence on the magnitude of the direct
membrane stresses due to bending (System II) providing the
stresses from System I are not considered. In fact, considera-
tion of the latter would lead to totally different values for the final stresses in the ribs.

6.2.3 Local Tertiary Bending Stresses (System III)

The tertiary stresses (System III) which arise due to the local bending of the deck plate under a wheel load, and are ignored in current design practice, can no longer be neglected if the stiffener spacing is increased. Indeed, in some cases, these stresses can be critical.

In Figure 6.9, the transverse bending stresses are shown for a load placed directly over a stiffener. As the rib thickness is increased, the maximum transverse bending stress in the deck plate is seen to decrease. Clearly, if the deck plate thickness $t_p$ was increased, rather than $t_s$, a further reduction in the stresses would have resulted. In any case, this loading condition is not the most critical.

Figure 6.10 shows the transverse bending stresses when the load is placed in position 1. With $t_p$ constant at 3/8 in. and $t_s$ varied, high stresses approaching yield are obtained for a rib spacing of 34.2 inches. Despite the increase in $t_s$ higher bending stresses are obtained in the stiffener walls as the spacing is increased. Hence, the feasibility of increasing the rib spacing for economic reasons by changing $t_s$ only, rather than $t_p$ does not seem possible on account of the high tertiary stresses developed.
On the other hand, if the rib spacing is compensated by a thicker deck plate, then much lower tertiary stresses are obtained even for a rib spacing of 34.2 in., as shown in Fig. 6.11.

It must be pointed out that these transverse bending stresses produce longitudinal components as shown in Fig. 6.12. These stresses arise, not only due to Poisson's ratio effects, but also due to the curvature in the longitudinal direction. Fortunately, the deck plate itself has very little direct stresses arising from System II and, therefore, the combination of the membrane and bending stresses (see Table 6-2) even for the case of 34.2 in. spacing is still below the recommended working stress level, providing $t_p = .467$ inch.

The finite element grid used in these studies is similar to the one shown in Fig. 5.18, except for the critical loading case 1 where the deck plate joining the two stiffeners was divided into four rather than two elements. It is important to note that the membrane and bending stresses (Systems II and III) calculated in this study, are interactive contrary to the usual design assumptions where all the stress systems are uncoupled.

6.3 SPACING OF INVERTED T-SECTIONS

A study of the effect of rib spacing for the case of inverted T-sections is described in this section. Although such stiffeners are usually torsionally weak, they have a definite
advantage in the fabricating procedure. This advantage stems from the fact that welding to the deck plate can be performed on both sides of the webs, while for closed stiffeners, welding can be performed on the outside face only and, therefore, require more elaborate inspection and testing during fabrication.

A typical panel analysed is shown in Fig. 6.13 with the ribs 12 in. apart and the cross-beams (also shown in Fig. 6.13) placed 6 feet or 9 feet apart. The minimum deck plate thickness of 3/8 in. is used. Rib spacing of 15 and 18 inches is also considered and, correspondingly, the deck plate thickness is changed to .469 in. and .563 in. respectively.

6.3.1 Deflection Study

The deflection profiles for loading cases 1 and 2 are shown in Figs. 6.14, 6.15 and 6.16. The critical loading case is number 1 where the load is placed between two stiffeners. The deflection of the deck plate is 1/570, 1/580 and 1/623th of the span between the ribs when these are at 12, 15 and 18 inch spacing respectively and the cross-beams 6 feet apart. For cross-beams at 9 foot intervals, the corresponding deflections are 1/600, 1/590 and 1/600th of the span. It is seen that the deflection value is not constant as predicted by the AISC design formula, which gives a value of 1/425th of the span. In fact, contrary to the closed stiffeners case, the deflections
as calculated by the design formula, are now overestimated.

The use of orthotropic plate theory for the problem in hand would have produced totally erroneous values for the deflections, especially for load case 2 where the maximum deflection does not correspond to the location of the loaded rib.

A further investigation of the maximum transverse curvature of the deck plate at the rib junction shows that, for various rib and floor beam spacing, these values vary between $1.51 \times 10^{-3}$ and $4.28 \times 10^{-3}$. The important conclusion that can be reached is that while for wider rib spacing the relative deflections can increase despite the thicker deck plate, the curvatures always decrease. This is very desirable since not only the local tertiary stresses would be lower in magnitude but the possibilities of cracks in the wearing surface are reduced. However, experimental tests are required to investigate the maximum permitted curvature in the deck plate for different wearing surfaces, and the introduction into design codes of a criterion based on curvatures, rather than deflections, would then be possible.

6.3.2 Direct Stresses (System II)

A plot of the bending moment for the directly loaded rib (Load case 2) is shown in Fig. 6.17. The maximum moment of
56 kip. in. is obtained in the rib, excluding the deck plate, when the spacing is 15 inches. This gives rise to a tensile bending stress at the bottom of the stiffeners. However, due to the eccentricity of the ribs, an axial force is developed in the stiffeners as shown in Fig. 6.18. Although the bending moments, for the central rib, are roughly constant for a given cross-beam spacing, the axial forces vary quite significantly, depending on the spacing of the ribs, cross-beams and the deck plate thickness. The stresses arising from these axial forces tend to increase the total stress at the bottom of the ribs. In any case, the maximum direct stress never exceeds 7 ksi.

The compressive internal forces $N_x$ in the deck plate itself are also shown in Fig. 6.18. The advantage of calculating separately the plate and stiffener stresses by the finite element method is emphasized once more since no reference to the effective width concept is made.

The stresses arising from System I are not considered since these depend on the superstructure configuration. However, a superposition of such stresses (of order ±10 ksi) to the combined stresses arising from System II and III is still well below the allowable stress.

6.3.4 Local Tertiary Bending Stresses (System III)

The most important stresses in the study under consideration are the transverse bending stresses in the deck plate. These
are shown in Figs. 6.19, 6.20 and 6.21 for various rib and cross-beam spacings. The highest stress obtained is 25.4 ksi over the stiffeners when the spacing is 12 in. and the cross-beams 6 feet apart. This reduces to 23.1 and 20.8 ksi for wider rib spacing. Further reduction of the tertiary stresses are obtained for wider cross-beam spacing.

It is interesting to note that the transverse bending stresses are confined to the area of load application. This is partly due to the weak torsional rigidity of the stiffeners accompanied by a relatively short distance between the cross-beams.

It can, therefore, be concluded that the open stiffener deck does not fully utilize the interaction between the various components to shed the load sideways, as experienced with the closed stiffeners up to a certain extent. This leads to a reduction in the transverse stresses. In fact, the results shown in Figs. 6.19 to 6.21 suggest that further economy would result from even greater cross-beam spacing than the maximum considered here.

Finally, a 50% increase rib and cross-beam spacing can be safely achieved by simply increasing the deck plate thickness from 3/8 in. to 9/16 in. This, of course, brings considerable saving in the fabricating and welding cost involved, while the total weight of the steel deck would be approximately the same.
6.4 SUMMARY OF THE RESULTS

For convenience and quick reference to the various stress components in the deck plate, these are tabulated in Table 6-2 for the trapezoidal stiffeners. The following conclusions can be drawn:

1) The membrane compressive stresses (System II) in the deck plate in the longitudinal direction are always negligible irrespective of loading, plate thickness or stiffener spacing.

2) The maximum membrane stresses (System II) at the bottom of the stiffeners, when it is directly loaded, varies between 10 and 12 ksi irrespective of the spacing or deck plate thickness.

3) The longitudinal bending stresses in the stiffener walls are unimportant and can be neglected.

4) The bending stresses in the deck plate itself can be significant for high rib spacing especially under load case 1.

5) Stresses arising from Systems II and III are interactive. If the stresses for System I are assumed of the order of ±10 ksi, then superposition of all stress systems still yields stresses below the recommended allowable design values despite the large rib spacing.

The corresponding results arising from the study on open stiffeners is shown in Table 6-3. The following conclusions can be drawn:
1) The membrane stresses (System II) in the deck plate are negligible despite the rib or cross-beam spacing and loading conditions considered.

2) The longitudinal bending stresses in the deck plate do not vary a great deal with rib spacing for load case 1 and, in fact, decrease with large cross-beam spacing. This decrease is mainly due to a transfer of strain energy from the deck plate to ribs when the latter are allowed higher deflections.

3) The stresses at the bottom of the stiffeners are highest for load case 2 but never exceed the value of 7 ksi.

4) Superposition of stresses arising from System I would yield stresses far below the allowable level despite the relatively wide rib spacing of 18 in. and cross-beams at 9 foot intervals.

In conclusion, it can be seen from Tables 6-2 and 6-3 that the present design methods can lead to conservative answers for System II when designing orthotropic decks. This is partly due to the design assumptions involved and the lack of interaction between Systems II and III. A more realistic approach, based on the results obtained in this Chapter, is to design orthotropic decks on the basis of System III, rather than System II. This would be particularly useful if the rib spacings are increased for economic reasons. The deflection criterion presently used
for the deck plate could also be replaced by a curvature criterion which is directly related to the local bending stresses, but involves the thickness. Furthermore, curvatures in the deck plate could lead to a more accurate picture of the behaviour of the wearing surface if such values are determined experimentally for various kinds of wearing surfaces.
### TABLE 6-1

<table>
<thead>
<tr>
<th></th>
<th>Load Case 1</th>
<th></th>
<th>Load Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w_B^{(F.E.)}$</td>
<td></td>
<td>$w_A^{(F.E.)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a$, inches</td>
<td>13</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>$b$, inches</td>
<td>11</td>
<td>15.5</td>
<td>21.2</td>
</tr>
<tr>
<td>$s$, inches</td>
<td>24</td>
<td>28.5</td>
<td>34.2</td>
</tr>
<tr>
<td>$t_p$, inches</td>
<td>.375</td>
<td>.375</td>
<td>.375</td>
</tr>
<tr>
<td>$t_s$, inches</td>
<td>.250</td>
<td>.300</td>
<td>.375</td>
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<td></td>
</tr>
<tr>
<td></td>
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<td>b</td>
<td>b</td>
</tr>
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<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>$w_B^{(AISC)}$</td>
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</tr>
<tr>
<td></td>
<td>b</td>
<td>b</td>
<td>b</td>
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<tr>
<td></td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td>$w_A^{(F.E.)}$</td>
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</tr>
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<td>a</td>
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<tr>
<td></td>
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<tr>
<td>$w_A^{(AISC)}$</td>
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<tr>
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<td>a</td>
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<td></td>
<td>a</td>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

* Results obtained from Computer Program

** Results computed using the AISC design formula (Ref. 6)

#### Load Case 1

![Diagram of Load Case 1]

#### Load Case 2

![Diagram of Load Case 2]
### LOAD CASE 1

<table>
<thead>
<tr>
<th>STRESS</th>
<th>LOCATION</th>
<th>24&quot;</th>
<th>28.5&quot;</th>
<th>34.2&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_M$</td>
<td>a</td>
<td>-2.1</td>
<td>-1.6</td>
<td>-1.2</td>
</tr>
<tr>
<td>$\sigma_B$</td>
<td>a</td>
<td>±8.0</td>
<td>±12.0</td>
<td>±20.3</td>
</tr>
<tr>
<td>sum</td>
<td>a</td>
<td>-10.1</td>
<td>-13.6</td>
<td>-21.5</td>
</tr>
</tbody>
</table>

### LOAD CASE 2

<table>
<thead>
<tr>
<th>STRESS</th>
<th>LOCATION</th>
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<th>28.5&quot;</th>
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</thead>
<tbody>
<tr>
<td>$\sigma_M$</td>
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<td>-2.1(-5)</td>
<td>-2.0</td>
<td>-1.8</td>
</tr>
<tr>
<td>$\sigma_B$</td>
<td>b</td>
<td>±14</td>
<td>±11.6</td>
<td>±9.4</td>
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<tr>
<td>sum</td>
<td>b</td>
<td>-16.1</td>
<td>-13.6</td>
<td>-11.2</td>
</tr>
<tr>
<td>$\sigma_M$</td>
<td>c</td>
<td>10.7(14.3)</td>
<td>11.3</td>
<td>11.8</td>
</tr>
<tr>
<td>$\sigma_B$</td>
<td>c</td>
<td>±.5</td>
<td>±.4</td>
<td>±.2</td>
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<tr>
<td>sum</td>
<td>c</td>
<td>11.2</td>
<td>11.7</td>
<td>12.0</td>
</tr>
</tbody>
</table>

**Note:** Figures in bracket are obtained from AISC manual using design charts and formulae.

**TABLE 6-2**

<p>| | |</p>
<table>
<thead>
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<tbody>
<tr>
<td>COMPARISON OF STRESSES FOR CLOSED STIFFENED DECK</td>
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</tr>
</tbody>
</table>
### Table 6-3

#### COMPARISON OF STRESSES FOR INVERTED T-SECTIONS WITH VARIOUS SPACINGS

<table>
<thead>
<tr>
<th>LOADING</th>
<th>STRESS</th>
<th>LOCATION</th>
<th>RIB SPACING (k.s.i.)</th>
<th>RIB SPACING</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>σ&lt;sub&gt;H&lt;/sub&gt;</td>
<td>a</td>
<td>12&quot;</td>
<td>15&quot;</td>
</tr>
<tr>
<td></td>
<td>σ&lt;sub&gt;B&lt;/sub&gt;</td>
<td>a</td>
<td>±8.3</td>
<td>±8.9</td>
</tr>
<tr>
<td></td>
<td>sum</td>
<td>a</td>
<td>±10.0</td>
<td>±10.3</td>
</tr>
<tr>
<td></td>
<td>σ&lt;sub&gt;S&lt;/sub&gt;</td>
<td>c</td>
<td>3.3</td>
<td>3.6</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ&lt;sub&gt;H&lt;/sub&gt;</td>
<td>b</td>
<td>-2.4(-3.7)</td>
<td>-2.3</td>
<td>-1.9</td>
</tr>
<tr>
<td>σ&lt;sub&gt;B&lt;/sub&gt;</td>
<td>b</td>
<td>±7.2</td>
<td>±6.3</td>
<td>±4.5</td>
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<tr>
<td>sum</td>
<td>b</td>
<td>-9.6</td>
<td>-8.6</td>
<td>-6.4</td>
</tr>
<tr>
<td>σ&lt;sub&gt;S&lt;/sub&gt;</td>
<td>c</td>
<td>3.8(5.9)</td>
<td>4.4</td>
<td>4.5</td>
</tr>
</tbody>
</table>

**Note:** Figures in bracket are obtained using AISC design formulae.

#### Longitudinal membrane stresses

#### Longitudinal bending stresses

#### Direct bending stresses at bottom of stiffeners
Cross beams, 15' apart

**FIGURE 6.1** CLOSED STIFFENERS WITH VARYING RIB SPACING
FIGURE 6.2 DEFLECTIONS FOR LOAD BETWEEN TWO STIFFENERS.
RIB SPACING 24"

$t_p = .375''$

$t_s = .250''$
FIGURE 6.3 DEFLECTIONS FOR LOAD BETWEEN TWO STIFFENERS.
RIB SPACING 28.5"
FIGURE 6.4  DEFLECTIONS FOR LOAD BETWEEN TWO STIFFENERS.
RIB SPACING 34.2".
FIGURE 6.5 DEFLECTIONS FOR LOAD OVER A STIFFENER.
RIB SPACING 24"
FIGURE 6.6 DEFLECTIONS FOR LOAD OVER A STIFFENER.
RIB SPACING 28.5"

\[ t_p = 0.375", \quad t_s = 0.30" \]

\[ t_p = 0.416", \quad t_s = 0.25" \]
FIGURE 6.7 DEFLECTIONS FOR LOAD OVER A STIFFENER.
RIB SPACING 34.2"
FIGURE 6.8  SPANWISE VARIATION OF LONGITUDINAL DIRECT STRESSES AT BOTTOM OF STIFFENER - LOAD CASE 2. (LOAD OVER STIFFENER).
FIGURE 6.9 TRANSVERSE BENDING STRESSES FOR \( t_p = 0.375'' \) WITH VARIOUS RIB SPACING - LOAD CASE 2

- 24" SPACING
  - 1.3 X 12 kips
  - \( t_s = 0.250'' \)
  - 28.9 ksi

- 28.5" SPACING
  - 1.3 X 12 kips
  - \( t_s = 0.3'' \)
  - 27.2 ksi

- 34.2" SPACING
  - 1.3 X 12 kips
  - \( t_s = 0.375'' \)
  - 25.6 ksi
FIGURE 5.10 TRANSVERSE BENDING STRESSES FOR $t_p = .375''$
WITH VARIOUS RIB SPACING - LOAD CASE 1
FIGURE 6.11  TRANSVERSE BENDING STRESSES FOR $t_g = .25''$ WITH VARIOUS RIB SPACING - LOAD CASE 1
FIGURE 6.12   LONGITUDINAL VARIATION OF BENDING STRESSES IN DECK PLATE. LOAD CASE 1
FIGURE 6.13 OPEN STIFFENERS WITH VARIOUS RIB SPACING
FIGURE 6.14 DEFLECTIONS FOR RIB SPACING AT 12"

LOAD CASE 1

LOAD CASE 2

Cross beams at 6 ft.

Cross beams at 9 ft.

1.3 X 12 kip (22 X 12")
LOAD CASE 1

1.3X12 kip (22X12"")

Cross beams at 6 ft.

Cross beams at 9 ft.

LOAD CASE 2

1.3X12 kip (22X12"")

FIGURE 6.15 DEFLECTIONS FOR RIB SPACING AT 15"

-0.01 -0.02 -0.03 -0.04 -0.05

in
FIGURE 6.16 DEFLECTIONS FOR RIB SPACING AT 18"

LOAD CASE 1

- Cross beams at 6 ft.
- Cross beams at 9 ft.

LOAD CASE 2

- 1.3 X 12 kip

1.3 X 12 kip (22 X 12"

in
9' cross beam spacing
6' cross beam spacing

1.3 x 12 kip

12 in. rib spacing

15 in. rib spacing

18 in. rib spacing
FIGURE 6.18 DISTRIBUTION OF AXIAL FORCES IN DECK PLATE AND RIBS - LOAD CASE 2
*Bending moment (kip.in.) in stiffeners about their local centroids. 6' cross beams.

• Bending moment (kip.in.) in stiffeners about their local centroids. 9' cross beams.

Transverse bending stresses in deck plate (ksi)
Numbers in brackets refer to cross beam spacing of 9'. Other numbers refer to cross beam spacing of 6'.

**Figure 6.19** VARIATION OF TRANSVERSE BENDING STRESSES IN DECK PLATE AND LONGITUDINAL MOMENTS IN STIFFENERS. RIB SPACING 12".
FIGURE 6.20 VARIATION OF TRANSVERSE BENDING STRESSES IN DECK PLATE AND LONGITUDINAL MOMENTS IN STIFFENERS. RIB SPACING 15".

REFER TO FIG. 6.19 FOR LEGEND.
FIGURE 6.21 VARIATION OF TRANSVERSE BENDING STRESSES IN DECK PLATE AND LONGITUDINAL MOMENTS IN STIFFENERS. RIB SPACING 18".

REFER TO FIG.6.19 FOR LEGEND.
CHAPTER 7

GEOMETRIC NONLINEARITIES

7.1 INTRODUCTION

All static tests on steel deck panels, reported in the AISC manual, show that the actual ultimate capacity of a directly loaded rib is considerably greater than the computed values.

The tests have also shown that the limit of a purely elastic behaviour of a loaded rib is higher than predicted. This is not surprising since the use of first-order theory in the design manual is conservative due to the many inherent assumptions. The stresses at the bottom of the ribs are overestimated by at least 35% if the finite element method is considered "exact", as shown in Tables 6-2 and 6-3. Hence, assuming first-order theory throughout the loading range, yield stresses are bound to be predicted at a much lower load using the AISC design formula.

Another factor to be investigated in this Chapter is the nonlinear load-deflection relationship arising from geometric nonlinearities within the elastic range. Such effects would cause the stresses to increase at a slower rate than the loads and, consequently, the actual load required to reach the yield point may be higher still.
A particularly important case to be considered here is the closed-stiffener deck with a rib spacing of 34.2 in. and the load placed exactly between two stiffeners. This was seen, in Chapter 6, to give rather high transverse bending stresses using linear theory.

A nonlinear finite element analysis of the deck plate can be performed treating the deck plate as an isotropic continuum between the two stiffeners. Immediately, a problem arises as to whether the boundary conditions are to be considered as simple or fixed supports at the junction of the stiffener. On the other hand, a nonlinear analysis of the complete deck is not really justified since the region of nonlinearities are confined to the loaded area.

A compromise is to simulate in the analysis the closed stiffeners on either side of the loaded region, assuming linear behaviour in the rest of the panel.

The rectangular shell element described earlier for linear analysis is, unfortunately, not suitable for analysing geometric nonlinearities. The reason being that in a step-by-step incremental analysis, the element is distorted at each increment and redefinition of the nodal points lying in one plane becomes difficult. This is not a very serious drawback since the stiffness of the newly distorted quadrilateral can be approximated to the original rectangle. However, a transformation of the distorted element to a global system of axes would require the evaluation of the direction cosines between the global axes and
the distorted plane which obviously leads to some difficulties as mentioned by Von Riesemann.

A triangular shell is, therefore, more suitable for non-linear analysis. But if the uncoupled membrane and bending components satisfy the conformity requirements separately by use of different order polynomials, these effects, when superimposed to represent the large deflection case, complete compatibility is lost. Therefore, using the conforming 12 degrees of freedom triangle with mid-side normal slopes, as developed in Chapter 2, to represent the bending action, a triangular plane stress partner based also on a cubic polynomial is developed. However, complete compatibility at the element interface is not guaranteed. Nevertheless, it is felt that such an element would be more suitable than a constant strain triangle especially for representation of a three-dimensional model.

7.2 TRIANGULAR SHELL ELEMENT

As mentioned above, a triangular shell element is more suitable for geometrically nonlinear problems if at each load increment the current geometry is used. For the element considered, the bending and membrane contributions are first uncoupled in the formulation as described below.

* Shell element
7.2.1 Plate Bending: Elastic Stiffness Matrix

The 12 degrees of freedom (Fig. 7.1) triangle, based on a cubic polynomial for each subregion, has been fully described in Chapter 2. Introducing the suffix \( E \) to denote the elastic part, then

\[ R = K_E r \] (7.1)

where \( K_E \) is given in equation (2.54) and the deflection vector \( r \) is defined in (2.21). The vector \( R \) is the corresponding force vector.

7.2.2 Plate Bending: Geometric Stiffness Matrix

The concept of geometric stiffness matrices was first introduced by Turner et al. The physical interpretation of this matrix arises from consideration of second-order terms in the strain/displacement relations. Thus, in general,

\[
\begin{align*}
\varepsilon_{\xi} &= -z \frac{\partial^2 w}{\partial \xi^2} + \frac{1}{2} \frac{\partial w}{\partial \xi}^2 \\
\varepsilon_{\eta} &= -z \frac{\partial^2 w}{\partial \eta^2} + \frac{1}{2} \frac{\partial w}{\partial \eta}^2 \\
\varepsilon_{\xi\eta} &= 2z \frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial w}{\partial \xi} \frac{\partial w}{\partial \eta}
\end{align*}
\] (7.2)

where, during the imposition of this strain, the in-plane initial
stresses give rise to a potential energy $V_\varepsilon$ such that

$$V_\varepsilon = \frac{1}{2} \int \left( \sigma_\xi w_\xi^2 + \sigma_\eta w_\eta^2 + \tau_\xi \eta w_\xi w_\eta + \tau_\xi \eta w_\eta w_\xi \right) dV$$  \hspace{1cm} (7.3)

Evaluation of the potential energy for a typical subregion of the triangle in Fig. 2.1 is obtained using the same technique adopted for the evaluation of the strain energy in Section 2.3.2. From (2.2)

$$w_\xi = \sum_{i=1}^{10} \alpha_i \cdot m_i \cdot n_i$$  \hspace{1cm} (7.4)

Therefore,

$$w_\xi^2 = \sum_{i=1}^{10} \sum_{i=1}^{10} \alpha_i \alpha_j \cdot m_i \cdot m_j \cdot n_i \cdot n_j$$  \hspace{1cm} (7.5)

Rewriting (7.3) in quadratic form

$$V_\varepsilon = \frac{1}{2} \alpha^t k' \alpha$$  \hspace{1cm} (7.6)

and, integrating over the thickness and the subregion area yields

$$k'_{i,j} = t(\sigma_\xi m_i m_j F(m_i + m_j, n_i + n_j) + \sigma_\eta n_i n_j F(m_i + m_j, n_i + n_j - 2)$$

$$+ \tau_\xi \eta (m_i n_j + m_j n_i) F(m_i + m_j - 1, n_i + n_j - 1))$$  \hspace{1cm} (7.7)

If the values of $\sigma_\xi$, $\sigma_\eta$ and $\tau_\xi \eta$ are allowed a linear variation between the vertices $i$, $j$ and $k$ of the subregion, that is,
where the constants $\beta_x$, $\beta_y$ and $\beta_{xy}$ are similar to relations (2.42) except that the stresses at the vertices replace the thicknesses. Substituting (7.8) into (7.7) yields

\[
\begin{align*}
\sigma_{\xi} &= \sum_{k=1}^{3} \beta_{x_k} \xi_{i_k} m_{i_k} n_{i_k} \\
\sigma_{\eta} &= \sum_{k=1}^{3} \beta_{y_k} \xi_{i_k} m_{i_k} n_{i_k} \\
\tau_{\xi\eta} &= \sum_{k=1}^{3} \beta_{xy_{k}} \xi_{i_k} m_{i_k} n_{i_k}
\end{align*}
\]

(7.8)

Combining the potential energy contribution from the other sub-regions yields the total potential energy $V_{eT}$ of the triangle. Hence,

\[
V_{eT} = \frac{1}{2} \bar{a}^T \bar{K} \bar{a}
\]

(7.10)

where $\bar{a}$ has been defined in (2.51) and
where the superscripts refer to subregions.

The vector of constants $\vec{a}$ is expressed in terms of the displacements $r$ by relation (2.30). Using this relation in (7.10) yields the desired geometric stiffness

$$K_G = G^t \bar{k}' G$$

Other alternatives of obtaining the same geometric stiffness matrix have been proposed by Kapur and Hartz and Przemieniecki for elastic stability analysis; however, the in-plane stresses within the element were assumed constant. A recent survey on the subject, by Gallagher, can be found in ref.86.

More recently, second-order geometric stiffnesses were introduced by Mallet and Marcal. These matrices produce coupling between the in-plane and bending actions in $K_G$. A detailed discussion of the various meaning of the second-order effects is given by Martin. However, these second-order effects which are of importance for postbuckling analysis or snap-through problems, are not considered here.

Combination of (7.1) and (7.12) yields the total plate bending contribution in the element. Thus

$$(K_E + K_G)r = R$$
7.2.3 **Plane Stress Element**

The plane stress element used in conjunction with the plate bending element has 12 degrees of freedom: two displacements and an in-plane rotation at each vertex and tangential displacements at the mid-side nodes, as shown in Fig. 7.2. The introduction of the mid-side displacement has the effect of producing a linear stress distribution which is certainly a desirable feature.

The derivation of the element stiffness matrix is contained in Appendix A. The cantilever problem described in Section 3.4.1 is used again as a means of evaluating the element accuracy. This example is also presented in Appendix A.

The stresses in the element are calculated at the vertices and the centroid of the triangle. These stresses are then transformed to the subregion local axes of the plate bending for the evaluation of the geometric stiffness matrix.

7.2.4 **Transformation to Global Axes**

The same procedure adopted in Section 4.2 is used again to transform the nodal degrees of freedom to a global system of axes. The mid-side translations and rotations remain in the local system. The combination of both membrane and bending systems yield a stiffness matrix of order $24 \times 24$. 
7.3 **INCREMENTAL LINEAR ANALYSIS**

The equilibrium equations of the assembled elements for the complete structure are of the form

\[(\bar{K}_E + \bar{K}_G)\bar{F} = \bar{R}\]  

(7.14)

where \(\bar{K}_E\) is the elastic contribution from both membrane and bending actions and \(\bar{K}_G\) is the geometric stiffness matrix which is dependent on the unknown stress level. However, by applying the load in increments, equation (7.14) can be written as

\[(\bar{K}_E + \bar{K}_G)\Delta\bar{F} = \Delta\bar{R}\]  

(7.15)

where \(\Delta\bar{r}\) represents incremental values of the displacements under incremental loading \(\Delta\bar{R}\). These incremental displacements serve the dual purpose of updating the nodal geometry and calculating the stresses required for the formulation of \(\bar{K}_G\) in the next load increment. Therefore, any change in the load direction is automatically accounted for. Other iterative solution techniques such as Newton-Raphson's have been summarized by Zienkiewicz [27] and Oden [89].

The linear incremental approach is, unfortunately, dependent on the size of each step. But, if sufficiently small steps are taken, the errors involved are minimized and convergence to the true solution would be obtained.
7.4 EVALUATION OF THE ELEMENT PERFORMANCE

To test both the element behaviour and the incremental approach, a simply supported square plate on immovable supports with a pressure load is analysed. The plate side is 16 inches and its thickness 0.1 inch. The finite element results are compared with those obtained originally by Levy\textsuperscript{90} and also reported by Murray & Wilson\textsuperscript{91}.

The uniform load was varied from 0 to 10 psi in 8 unequal increments. The load vs. central deflection is shown in Fig.7.3. Note the "shifted" results obtained by the finite element method due to the initially "large" first increment of 0.5 psi. This is further reflected in the plot of the bending and membrane stresses as shown in Figs.7.4 and 7.5. However, the errors involved in comparison with small deflection theory are quite acceptable.

7.5 INVESTIGATION OF GEOMETRIC NONLINEARITIES IN CLOSED STIFFENED DECKS

The geometric nonlinearities for the orthotropic deck with trapezoidal stiffeners at 34.2 in. spacing is investigated in this section. Two loading conditions are studied: a standard 1.3x12 kip load placed between two stiffeners and exactly over a stiffener.
7.5.1 Load between Two Stiffeners

When the load is placed between two stiffeners, the deck plate can be treated as an isotropic continuum as shown in Fig. 7.6. The boundary conditions can be treated as either simply-supported or clamped at the junction of the plate with the stiffener. Figure 7.7 shows the load-deflection curves for a 1.3×12 kip load. Note the pronounced nonlinearities for the simply-supported continuum when the deck plate thickness is taken as 3/8 inch. The clamped condition exhibits very little nonlinearity under the normal working load. As expected, the results of the linear 3D analysis, using the rectangular shell, fall in between these two conditions. The results of Fig. 7.7 show the same pattern for a deck plate thickness of .467 inch. The nonlinearities are less pronounced due to small deflections under working loads in this case.

To overcome the problem of boundary conditions, the closed stiffeners on either side of the continuum are now included in the finite element model. An incremental analysis was performed on a three-dimensional basis taking the load up to 1.3×24 kips. The results for the central deflections are shown in Fig. 7.9. The transverse bending stresses in the deck plate are shown in Fig. 7.10. It is interesting to note two important points: First, the linear analysis, using triangular shells, yielded results fairly close to the more accurate rectangular shell described earlier. Secondly, for the case where the deck plate is restricted to .375 in., the transverse bending stresses exceed the yield
stress of 52 ksi for a load of 24 kips, while for a plate thickness of .467 in. this value could be acceptable but the nonlinearities are less pronounced.

A more important variation is the transverse membrane stresses; for a linear analysis, a negligible compressive stress is obtained. However, this stress quickly changes to a tensile value under working loads as shown in Fig.7.11.

The incremental analysis was not carried any further for the reason that the material nonlinearities were not considered in this work.

7.5.2 Load over a Stiffener

A standard truck wheel load is placed exactly over a trapezoidal stiffener with a triangular finite element mesh, simulating both the deck plate and the stiffener. The deck plate was assumed supported by the adjacent stiffeners and the cross-beams. Under such a loading condition, the stiffener is working in direct bending only. A 1.3×48 kip load is applied in 8 equal increments and the results are outlined below.

The transverse bending stresses in the deck plate, exactly at mid-span and halfway between the stiffener walls, reached a value of 60 ksi without nonlinearity. The longitudinal bending stresses at the same point reached 32 ksi. The linear variation under such high loads is due to thick deck plate (.467 in.) and
the rather small span between the stiffener walls (13 in.). This geometric configuration did not allow large deflections to occur for the range of load considered. The tensile membrane stresses at the bottom of the trough were 44 ksi. This value is in very close agreement with the answer obtained using the rectangular shell in Chapter 6. Again, the overall trough deflections (spanning between 15 ft. cross-beams) were small. On the other hand, the membrane stresses in the deck plate, which are compressive, are seen to decrease with increasing load as shown in Fig.7.12.

7.6 DISCUSSION OF THE RESULTS

The orthotropic deck with widely spaced closed stiffeners can exhibit some nonlinearities under the normal working load if the load is placed between the two stiffeners. However, upon increasing the load further, the local bending stresses reach the yield point at the extreme fibers only of the deck plate. At the same time, the compressive membrane stress in the deck plate changes to a tensile value. A redistribution of the stresses occurs under the wheel load and even with the formation of a hinge at the point of maximum bending stresses, the rather high reserve in membrane strength in the deck plate will allow much higher loads to be applied before complete failure. The high reserve in membrane strength is further enhanced by the large deflection occurring locally under wheel load. Furthermore, failure of the deck plate is localized to the region of
load application with the remainder of the structure remaining intact.

The case of directly loaded stiffeners is even less critical under high loads since the deck span between the stiffener walls is rather small.

It can, therefore, be concluded that a study of a second-order theory for orthotropic decks is of prime importance for evaluation of ultimate strength of the deck plate. Such an investigation would require division of the finite element into equal sections across its thickness and evaluation of the energy in each section depending on the stress level and accounting for material nonlinearities. In essence, this would have the effect of reducing the flexural rigidity of the deck plate at each load increment and thus producing "large" deflections where the membrane strength would be utilized fully. However, such an investigation is beyond the scope of this work.

Finally, the analysis of geometric nonlinearities in orthotropic decks with the present design criteria is not justified unless these are changed to include an investigation of ultimate strength. However, with present day computer analyses of this kind using finite elements could be very costly.
FIGURE 7.1  TRIANGULAR PLATE BENDING ELEMENT (12 D.O.F.)

FIGURE 7.2  TRIANGULAR PLANE STRESS ELEMENT (12 D.O.F.)
FIGURE 7.3 CENTRAL DEFLECTION OF A UNIFORMLY LOADED SQUARE PLATE
Figure 7.4 Bending Stresses in a Uniformly Loaded Square Plate

- Levy Solution
- Finite element
- Small deflection theory
FIGURE 7.5 MEMBRANE STRESSES IN A UNIFORMLY LOADED SQUARE PLATE
FIGURE 7.6  FINITE ELEMENT IDEALIZATION OF DECK PLATE BETWEEN TWO STIFFENERS FOR TWO DIMENSIONAL NONLINEAR ANALYSIS.
Figure 7.7: Deflection at point a. 2D Nonlinear analysis using triangles. 3D Linear analysis using rectangles.

Deck plate thickness = .375"
Stiffener wall thickness = .375"

LOAD INCREMENT = 1.0 x 1.3 x 12 kips
FIGURE 7.8 DEFLECTION AT POINT a.  2D NONLINEAR ANALYSIS USING TRIANGLES.
3D LINEAR ANALYSIS USING RECTANGLES

LOAD INCREMENT = 0.1 x 1.3 x 12 kips

- S.S. Plate - 2D analysis
- C. Plate - 2D analysis
- Linear 3D analysis

Deck plate thickness = 0.467"
Stiffener wall thickness = 0.250"
TOTAL LOAD = $\Delta P \times 1.3 \times 12$ kips

- $t_p = t_s = .375''$
- $t_p = .467''$, $t_s = .25''$

Finite element mesh
Figure 7.10 Transverse bending stresses at point a. A three-dimensional nonlinear analysis.
Linear analysis

\[ \Delta P \]

\[ 2.0 \]

\[ 1.0 \]

\[ 0.5 \]

\[ -2 \]

\[ 0 \]

\[ 2 \]

\[ 4 \]

\[ 6 \]

\[ 8 \]

\[ 10 \]

\[ \text{ksi} \]

Finite element mesh

\[ t_{p} = t_{s} = .375" \]

\[ t_{p} = .467", t_{s} = .25" \]

**FIGURE 7.11** TRANSVERSE MEMBRANE STRESSES AT a. A THREE-DIMENSIONAL NONLINEAR ANALYSIS - LOAD CASE 1
FIGURE 7.12 MEMBRANE STRESSES AT POINT a. A THREE-DIMENSIONAL NONLINEAR ANALYSIS - LOAD CASE 2
CHAPTER 8

CONCLUSION

8.1 IDEALIZATION

The idealization of orthotropic bridge decks with either open or closed ribs using the finite element method, has been presented. For a linear elastic analysis, rectangular shells were used to simulate the deck plate. The ribs were simulated using also rectangular shells or eccentric beam elements. Compatibility of deformation was insured between the various types of elements. The results obtained from a computer program compared favourably with classical or experimental results.

Among the advantages displayed by the method over existing solutions, the orthotropic deck is not idealized as an equivalent continuum and, therefore, approximate flexural and torsional rigidities need not be calculated. Furthermore, eccentricity of both the ribs and cross-beams is easily accounted for. It is felt that the finite element method has been put to its most powerful use with orthotropic deck analysis since the stresses are calculated individually within each part of the deck. Also, interaction between various stress systems is automatically achieved.
8.2 BEHAVIOUR: CLOSED RIBS

The theory was used to investigate the effect of rib spacing in orthotropic decks. For the specific case of trapezoidal closed ribs with 15 foot spacing of cross-beams, the following conclusions can be drawn:

1) The deflected shape of the panel is not a "smooth" curve as always obtained when using orthotropic plate theory. In fact, very different deflection profiles are obtained depending on the load position.

2) Local deflections under a standard wheel load are higher using this method in comparison to the empirical formula of the design manual.

3) For relatively large rib spacings, it is preferable to increase the deck plate thickness, rather than the stiffener walls, since the plate flexural rigidity plays a more important role than the torsional rigidity of the ribs under a local wheel load.

4) The membrane stresses in the deck plate arising from System II are very small. The highest stresses at the bottom of the ribs are lower than the computed values using the design formula.

5) High transverse bending stresses can be obtained in the deck plate especially for wide rib spacing. These are usually neglected in the design procedure providing the deck plate deflections do not exceed 1/300th of the span.
6) The longitudinal bending stresses are confined to the region of load application and do not spread spanwise in the deck plate.

7) Geometric nonlinearities are only apparent in special cases when the load is applied between two stiffeners with large spacings. A three-dimensional nonlinear analysis using triangular shells showed an increase in deflection and bending stresses at a lower rate than the load, providing the deflections reach a magnitude of about half the deck plate thickness.

8) Membrane compressive stresses in the deck plate switch to tensile using large deflection theory. The high reserve of membrane strength in the deck plate can, therefore, be utilized when investigating ultimate strength.

8.3 BEHAVIOUR: OPEN RIBS

The case of longitudinal ribs made of inverted T-sections was also investigated. The following conclusions can be drawn:

1) Deflections under a standard wheel load are smaller than the corresponding values using the AISC design formula. The deflection values are a function of both rib and cross-beam spacing.

2) The membrane stresses in both the deck plate and at the bottom of the stiffeners are small and did not exceed a maximum value of 7 ksi even for the cross-beams at 9 foot intervals.
and the rib spacing at 18 inches.

3) Longitudinal and transverse bending stresses are the highest stresses obtained in the panel. The values decrease with increasing rib spacing due to the higher deck plate thickness used. These stresses decrease further with larger cross-beam spacing. Hence, the importance of interaction between various stress systems.

4) An increase of 50% in rib spacing to 18 inches and cross-beam spacing of 9 feet, results in an acceptable design with appreciable savings in manufacturing and welding cost.

5) Any kind of wearing surface can be used since the local plate deflections never exceed $1/300$th of the span between ribs using the given thickness.

8.4 COMPARISON OF CLOSED AND OPEN STIFFENERS

The bridge deck stiffened by closed ribs has a slightly better load-distributing capacity than open ribs. However, the difficulties involved in precise fabrication techniques and complicated field splices of the closed ribs might outweigh that advantage.

Closed ribs can have floor beams at 15 foot intervals while for open ribs, 6 foot intervals are usually used. However, the study in Chapter 6 clearly shows that cross-beams at 9 foot
intervals can be used with inverted T-sections, resulting in a comparatively lighter deck.

The open rib depth and thickness may be varied as required in the various parts of the bridge deck. Also, all rib surfaces at the bottom of the deck plate are accessible for inspection and maintenance during the lifetime of the structure. Furthermore, the welding in open ribs is achieved on both sides of the web, while for closed stiffeners, the welding is limited to one side only. This is certainly more economical but more elaborate inspection is required to insure proper welding to the deck plate.

The tertiary bending stresses are lower in the open rib in comparison to the closed ribs especially for wider rib spacing. However, a thicker deck plate is required in the first case.

Smaller localized deflections result in open ribs which can be advantageous with regard to wearing surface.

While these comparisons are valid for the conditions under which they were obtained, the behaviour of other open or closed rib shapes would indicate similar trends.

8.5 **SUGGESTIONS FOR FUTURE WORK**

Little effort is required, in some instances, to extend the work presented in this thesis. Possible extensions are as
follows:

1) The free vibration analysis of orthotropic decks with closed-type ribs would require the calculation of the mass matrices and the solution of an eigenvalue problem. Due to the high number of degrees of freedom, a matrix reduction scheme would be required.

2) A study of the effect of other parameters entering into the problem such as various stiffener shapes within a given type.

3) The possibility of including the wearing surface into the finite element idealization of the deck plate and to investigate any relief in the local bending stresses and the curvatures if these are to replace the deflection criterion in the design.

4) The incorporation into the large deflection analysis, the effects of material nonlinearities for investigation of ultimate strength in orthotropic decks providing the vast numerical problem can be overcome.

5) Modification of the available computer program to make it design-oriented, based on System III stresses, with minimum input data. Alternatively, make extensive use of the present theory to create design charts where interaction between the stress systems is considered.

6) Optimization of the bridge deck weight and a detailed study of fabrication costs involved for open or closed stiffeners
as well as experimental studies to investigate maximum permissible curvatures for various kinds of wearing surfaces.
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FORMULATION OF A PLANE STRESS TRIANGLE
WITH IN-PLANE ROTATIONS

A.1 ELEMENT GEOMETRY

The element geometry is shown in Fig.A.1, with the local $\xi$ axis passing through nodes 1 and 2 and the perpendicular $\eta$ axis passing through node 3. The angles $\gamma_1$, $\gamma_2$ and $\gamma_3$ are positive in the anticlockwise directions with respect to sides 1-2, 2-3 and 3-1 respectively and the $\xi$ axis. The geometric calculations are basically the same as the case of the triangular bending element and are shown in Table 2-1.

A.2 STIFFNESS MATRIX

A full cubic function for the $u$ and $v$ displacements involve 20 constants, given by the column $a$ where

$$u = \sum_{i=1}^{10} a_i \xi^i \eta^i$$  \hspace{1cm} (A.1)

$$v = \sum_{i=11}^{20} a_i \xi^i \eta^i$$  \hspace{1cm} (A.2)

These constants are evaluated in terms of 20 degrees of freedom.
as shown in Fig. A.2. The matrix C relating the constants to the nodal geometry is automatically generated and inverted. Hence,

\[ a = C^{-1} \delta_1 \]  

(A.3)

where

\[ \delta_1 = \{u_1 v_1 u_2 v_2 \ldots u_{10} v_{10}\} \]  

(A.4)

The strain energy calculation is identical to the case of the rectangle developed in Section 3.3.2 except that the integration function \( F(m,n) \) is now considered over the area of the triangle and is given by (2.49). The stiffness matrix is, therefore, given by

\[ R_1 = K_1 \delta_1 \]  

(A.5)

where

\[ K_1 = C^{-1} \hat{\tau} k C^{-1} \]  

(A.6)

and \( k \) is given by (3.27).

Static condensation of node 10 yields the following relation:

\[ R_2 = K_2 \delta_2 \]  

(A.7)

where

\[ \delta_2 = \{u_1 v_1 u_2 v_2 \ldots u_9 v_9\} \]  

(A.8)

and \( R_2 \) the corresponding force vector.
The same development, using different local axes and lengthy algebraic expressions is given by Holand. This element exhibits very high accuracy, however, it cannot serve as a companion to the plate bending one due to the existence of nodes 4 to 9. The development below is mainly concerned in redefining these freedoms at nodes 4 to 9 in terms of more suitable ones.

The first step is to transform the degrees of freedom at nodes 4 to 9 to coincide with the directions of the sides. Hence,

\[ \delta_2 = \phi \delta_3 \]  

(A.9)

where

\[ \delta_3 = \{u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3 \ u'_4 \ v'_4 \ u'_5 \ v'_5 \ldots u'_9 \ v'_9\} \]  

(A.10)

Figure A.3 shows the transformation \( \phi \) and the displacement vectors it relates.

The second step is to evaluate the tangential displacements only, along the sides, in terms of the corner degrees of freedom and a new midside displacement \( u_{ij} \) along the side \( ij \). This, in effect, reduces the cubic variation of the \( u \) displacement along the side to a quadratic one. Consider a typical side 1-2 of length \( S_{12} \). The displacement \( u' \) along 1-2 is uniquely given by:
\[ u' = (1-3\zeta+2\zeta^2)u'_1 + (-\zeta+2\zeta^2)u'_2 + (4\zeta-4\zeta^2)u_{12} \]  \hspace{1cm} (A.11)

\[ u'_1 \quad u_{12} \quad u'_2 \quad u'_1 \quad u'_3 \quad u'_5 \quad u'_2 \]
\[ \zeta=0 \quad \zeta=\frac{1}{2} \quad \zeta=1 \quad \zeta=0 \quad \zeta=\frac{1}{3} \quad \zeta=\frac{2}{3} \quad \zeta=1 \]

Therefore, evaluating \( u'_4 \) and \( u'_5 \) in terms of \( u'_1, u'_2 \) and \( u_{12} \) yields

\[ u'_4 = \frac{1}{9} (2u'_1-u'_2+8u_{12}) \]  \hspace{1cm} (A.12)

\[ u'_5 = \frac{1}{9} (-u'_1+2u'_2+8u_{12}) \]  \hspace{1cm} (A.13)

Expressing now \( u'_1 \) and \( u'_2 \) in terms of \( u_1, v_1 \) and \( u_2, v_2 \) and repeating the same procedure for the other sides of the triangle, yields a transformation matrix of the form

\[ \delta_3 = \psi \delta_4 \]  \hspace{1cm} (A.14)

where

\[ \delta_4 = \{u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3 \ v'_4 \ v'_5 \ v'_6 \ v'_7 \ v'_8 \ v'_9 \ u_{12} \ u_{23} \ u_{31} \} \]  \hspace{1cm} (A.15)

The matrix \( \psi \) and the deflection vectors it relates are shown in Fig.A.4.

The displacements \( v'_4 \) to \( v'_9 \) are now expressed in terms of the nodal degrees of freedom if a cubic function is assumed along the sides. Considering a typical side 1-2 with a cubic function defined in terms of in-plane rotations, then
Evaluation of \( v'_4 \) and \( v'_5 \) in terms of \( v'_1, v'_2, \theta_1 \) and \( \theta_2 \) yields

\[
v'_4 = \frac{1}{27} \left( v'_1 + 4S_{12} \theta_1 + 7v'_2 - 2S_{12} \theta_2 \right)
\]

(A.17)

\[
v'_5 = \frac{1}{27} \left( 7v'_1 + 2S_{12} \theta_1 + 20v'_2 - 4S_{12} \theta_2 \right)
\]

(A.18)

Relating \( v'_1 \) and \( v'_2 \) in terms of \( u_1, v_1, u_2 \) and \( v_2 \) and repeating the same procedure for the other sides of the triangle, yields the desired transformation:

\[
\delta_5 = G \delta_5
\]

(A.19)

where

\[
\delta_5 = \{u_1, v_1, \theta_1, u_2, v_2, \theta_2, u_3, v_3, \theta_3, u_{12}, u_{23}, u_{31}\}
\]

(A.20)

and the matrix \( G \) is shown in Fig.A.5.

The stiffness matrix \( K_5 \) corresponding to \( \delta_5 \) is obtained using relations (A.7), (A.9), (A.14) and (A.19). Hence
The advantages of having in-plane rotations in the triangular element are obvious. However, the element is not conforming due to the reduction in polynomial order through the transformation matrix of relation (A.14).

The mid-side displacements can be eliminated if a linear function is used in (A.11) rather than a quadratic one. The displacements $u_4'$ and $u_5'$ are, therefore, linear combinations of $u_1'$ and $u_3'$. Hence

$$u_4' = \frac{1}{3} (2u_1' + u_2') \quad (A.22)$$

$$u_5' = \frac{1}{3} (u_1' + 2u_2') \quad (A.23)$$

The modified transformation $\psi'$ relating $\delta_3$ to $\delta_4'$ is given in Fig.A.6. All other matrices remain unchanged except for the $G$ matrix where the last three rows and columns are no longer required. A $9 \times 9$ stiffness matrix is, therefore, obtained where two displacements and an in-plane rotation are defined at each node. However, this element with only 9 degrees of freedom is too stiff and, therefore, the 12 degrees of freedom element with mid-side displacements is used in the nonlinear analysis.

A.3 EVALUATION OF THE ELEMENT ACCURACY

The cantilever problem already discussed in Section 3.4.1
is used to evaluate the element accuracy. Comparison is made between the cases where the midside degrees of freedoms are retained or not. Fig.A.7 shows the results of both elements using three different idealizations. The advantages of introducing midside displacements not only improves the convergence of deflections but also stresses. The elements described in this section are not as accurate as the rectangle of Chapter 3 but, in any case, they are superior to the constant strain triangle.
Figure A.1 Element Geometry

Figure A.2 Degrees of freedom corresponding to a full cubic expansion
\[ \phi = \begin{bmatrix} I_2 & \phi_1 \\ I_2 & \phi_1 \\ I_2 & \phi_1 \end{bmatrix} \]

\[ \phi_i = \begin{bmatrix} \cos \gamma_i & -\sin \gamma_i \\ \sin \gamma_i & \cos \gamma_i \end{bmatrix} \begin{bmatrix} u'_i \\ v'_i \end{bmatrix} \quad i=1,2,3 \]

**FIGURE A.3**

TRANSFORMATION MATRIX \( \phi \)
\[ \psi = \frac{1}{9} \]

\[
\begin{bmatrix}
9 \\
9 \\
9 \\
9 \\
9 \\
2C_1 & 2S_1 & -C_1 & -S_1 & 9 \\
-C_1 & -S_1 & 2C_1 & 2S_1 & 8 \\
2C_2 & 2S_2 & -C_2 & -S_2 & 9 \\
-C_2 & -S_2 & 2C_2 & 2S_2 & 8 \\
-C_3 & -S_3 & 2C_3 & 2S_3 & 9 \\
2C_3 & 2S_3 & -C_3 & -S_3 & 8 \\
\end{bmatrix}
\]

\[ S_i = \sin \gamma_i \]

FIGURE A.4
\[
G = \frac{1}{27}
\]

\[
G = \begin{bmatrix}
27 & 27 & 27 \\
20S_1 & 20C_1 & 4S_{12} & -7S_1 & 7C_1 & -2S_{12} \\
-7S_1 & 7C_1 & 2S_{12} & -20S_1 & 20C_1 & -4S_{12} \\
-20S_2 & 20C_2 & 4S_{23} & -7S_2 & 7C_2 & -2S_{23} \\
-7S_2 & 7C_2 & 2S_{23} & -20S_2 & 20C_2 & -4S_{23} \\
-20S_3 & 20C_3 & -2S_{31} & -7S_3 & 7C_3 & 2S_{31} \\
-20S_3 & 20C_3 & 4S_{31} & -7S_3 & 7C_3 & 2S_{31}
\end{bmatrix}
\]

\[C_i = \cos \gamma_i\]

\[S_i = \sin \gamma_i\]

\[S_{ij} = \text{length of side } i\text{-}j\]

**FIGURE A.5**

TRANSFORMATION MATRIX G
\[ \psi' = \frac{1}{3} \]

\[
\begin{bmatrix}
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
2C_1 & 2S_1 & C_1 & S_1 & C_1 & S_1 & 2C_1 & 2S_1 & 2C_1 & 2S_1 \\
C_1 & S_1 & 2C_1 & 2S_1 & 2C_2 & 2S_2 & C_2 & S_2 & 3 \\
C_2 & S_2 & 2C_2 & 2S_2 & 2C_3 & 2S_3 & 3 \\
C_3 & S_3 & 2C_3 & 2S_3 & 3 \\
2C_3 & 2S_3 & C_3 & S_3 & 3 \\
\end{bmatrix}
\]

\[ c_i = \cos \gamma_i \]
\[ s_i = \sin \gamma_i \]

**FIGURE A.6**

TRANSFORMATION MATRIX \( \psi' \)
FIGURE A.7 CONVERGENCE OF TIP DEFLECTION FOR CANTILEVER TEST CASE

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<th>9 D.O.F.</th>
<th>12 D.O.F.</th>
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</thead>
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</tr>
<tr>
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<td>0.31473</td>
</tr>
<tr>
<td>0.30259</td>
<td>0.33158</td>
</tr>
<tr>
<td>EXACT</td>
<td>EXACT</td>
</tr>
<tr>
<td>0.3558</td>
<td>0.3558</td>
</tr>
</tbody>
</table>