Q-ALGEBRAS AND RELATED TOPICS

by

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ABSTRACT

A Q-Algebra is a Banach algebra which is isomorphic to a quotient of a uniform algebra by a closed ideal.

A.M. Davie [3] proved that: A Banach algebra B is a Q-algebra if and only if there exists a constant C > 0 such that if $x_1, x_2, \ldots, x_p \in B$ with $\|x_i\| \leq 1$ and $p$, a polynomial which is homogeneous of degree n, we have

$$\|p(x_1, x_2, \ldots, x_p)\| \leq C^n \sup \{ |p(z_1, z_2, \ldots, z_p)| : z_i \in |z_i| \}$$

In this work we examine this condition under different types of Banach algebras. In particular the author proves that for a commutative 2-summing algebra a similar condition is necessary ( theorem 5.3 ).
Q-ALGEBRAS AND RELATED TOPICS

PAR

Ryuji Yamaguchi

Résumé

Une Q-algèbre est une algèbre de Banach qui est isomorphe à un quotient par un idéal fermé d'une algèbre uniforme. A.M. Davie [3] a prouvé que :

L'algèbre de Banach est une Q-algèbre si et seulement s'il existe une constante $C > 0$ tel que, si $x_1, x_2, \ldots, x_p \in B$ avec $\|x_i\| \leq 1$ et $P$, un polynôme homogène du degré $n$, on a

$$\|P(x_1, x_2, \ldots, x_p)\| \leq C^n \sup\{|P(z_1, z_2, \ldots, z_p)| : z_i \in \mathbb{C} \mid |z_i| < 1\}$$

Dans ce travail nous examinons cette condition dans des différentes types des algèbres des Banach. En particulier, l'auteur prouve que pour une algèbre 2-sommaire commutative une condition semblable est nécessaire (Théorème 5.3).
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INTRODUCTION

In 1969 at the Symposium of function algebras and rational approximation held at the University of Michigan, N. Th. Varopoulos raised the following question: When is a Banach algebra, a quotient of a uniform algebra? Today such algebras are called Q-algebras.

The first answer came from Brian Cole, who proved that every Q-algebra is isomorphic to a subalgebra of $B(H)$, the algebra of all bounded linear operators on a Hilbert space $H$, for some Hilbert space $H$. With the Hilbert space methods initiated by Cole, J. Wermer was able to prove certain bounds for complex polynomials acting on Q-algebras. Bernard later found a proof which did not rely on Hilbert space techniques. A. M. Davie furthered this result. He proved, in fact, that this boundedness of polynomials is an intrinsic necessary and sufficient condition for a Banach algebra to be a Q-algebra. By applying Littlewood's inequality on a finite tensor algebra, Davie proceeded to prove a few other equivalent conditions and hence showed that the question is essentially a quantitative one.

Varopoulos saw the importance of the role played by injective tensor norms in the study of Q-algebras. He defined an injective algebra to be a Banach algebra that has
the bounded multiplication map with respect to the injective tensor norm. He then proved that every commutative injective algebra is a $\mathcal{Q}$-algebra.

We have other algebras defined in terms of tensor norms. The $p$-summing algebras are such examples. Charpentier [2] proved that every commutative $1$-summing algebra is a $\mathcal{Q}$-algebra. Whether every commutative $2$-summing algebra is a $\mathcal{Q}$-algebra or not is an open question. The Author however, proves that commutative $2$-summing algebras and $\mathcal{Q}$-algebras have a number of similar properties.

The theory of $\mathcal{Q}$-algebras has yet another extension. Lumer extended Cole's result to operator algebras. Varopoulos proved that for a compact space $X$, $C(X)$ is an operator algebra. The consequence of this is that every $\mathcal{Q}$-algebra is an operator algebra.

Currently research is being carried out to find just which operator algebras are $\mathcal{Q}$-algebras and also to obtain better bounds for polynomials acting on $\mathcal{Q}$-algebras.

In this rapid development, a number of important theorems appeared without proofs and many others with only outlines of proofs. But often, proofs of those theorems were not trivial and they included techniques useful to the study of $\mathcal{Q}$-algebras. Two main aims of this paper are to expose those techniques in full detail and to be an introductory text to the theory of $\mathcal{Q}$-algebras.
In Chapter 1 we prove Cole's theorem and in Chapter 2 we discuss a number of important inequalities some of which form criterions for Q-algebras in Chapter 3.

In Chapter 4 we discuss injective algebras and in Chapter 5 we discuss p-summing algebras from a Q-algebraic point of view.

In Chapter 6 we give examples of Q-algebras using techniques useful to the study of Q-algebras. And in the final chapter operator algebras are discussed.

The only new developments to the study of Q-algebras comes in Chapter 5 where p-summing algebras are examined from a Q-algebraic point of view.

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CHAPTER 1

In this chapter we prove the theorems in $\mathbb{Q}$-algebras using Hilbert space methods. After Chapter 1 we do not use Hilbert space techniques until Chapter 7, and some of the results in this chapter can be proven without the use of such methods as we will see in Chapter 2. In Chapter 7 however, we will see that the implications of those theorems go beyond the theory of $\mathbb{Q}$-algebras. They are, in fact, theorems on certain algebras of operators of Hilbert spaces called operator algebras.

1.1 Definition A closed subalgebra of $C(X)$ is called a uniform algebra on $X$, where $C(X)$ denotes the algebra of all continuous functions on a compact set $X$ with the uniform norm. A Banach algebra is called a $\mathbb{Q}$-algebra if it is quotient of a uniform algebra by a closed ideal in the uniform algebra.

The following theorem was the first theorem to appear on $\mathbb{Q}$-algebras. It is due to Brian Cole[15].

1.2 THEOREM

Every $\mathbb{Q}$-algebra is a subalgebra of $B(H)$ for some Hilbert space $H$. 
Proof:

Let $B = A/I$ where $A$ is a uniform algebra and $I$ a closed ideal in $A$. Choose any probability measure $\mu$ on $X$ and let $H_\mu$ be the closure of $A$ in $L^2(\mu)$. For each $f \in A$, let $T_f$ be the multiplication operator by $f$. That is $T_f \in B(H_\mu)$ and $T_f(g) = fg$ for $g \in H_\mu$.

Let $I$ denote the closure of $I$ and $I^l$, the orthogonal complement of $I$ in $H_\mu$. Let $P$ be the orthogonal projection of $H_\mu$ on $I^l$ and finally put $S_f = P \circ T_f$, the composition of $T_f$ and $P$.

Now we claim that $S_f$ depends only on the coset class of $f$ in $A/I$. For if $f \in I$ then $fg \in I \forall g \in A$. Furthermore, if $f \in I$ then $wg \in H_\mu$, $fg \in I$. Hence if $f \in I$,

$$S_f(g) = P \circ T_f(g) = P(fg) = 0 \forall g \in H_\mu$$

which proves our claim.

We now want to show that the map $[f] \mapsto S_f$ is a homomorphism from $A/I$ into $B(H_\mu)$. So we must show that $S_{f+g} = S_f + S_g$ and $S_{fg} = S_f S_g$.

$$S_{f+g}(h) = P \circ T_{f+g}(h) = P(fh+gh) = P(fh) + P(gh)$$

$$= S_f(h) + S_g(h) \quad \forall h \in H_\mu$$

To prove $S_{fg} = S_f S_g$ it suffices to show that $P T_f(I-P) = 0$ since then $P T_f T_g = P T_f P T_g$ where $I(g) = g$. 


But now \((I-P)(h) = h-P(h) \in \mathcal{I} \quad \forall h \in H_{\mu}\). Hence \(f \cdot (I-P)(h) \in \mathcal{I}\) therefore we have \(PT_f(I-P)(h) = 0 \quad \forall h \in H_{\mu}\), \(\forall f \in A\). Hence \(PT_f(I-P) = 0\) and so \(f \mapsto S_f\) is a homomorphism from \(A/I\) into \(B(H_{\mu})\).

Q.E.D.

We remark here that

\[(1.3) \quad \|S_f\|_{\text{op}} \leq \|f\|_{A/I}\]

where \(\|\|_{\text{op}}\) denotes the operator norm and \(\|\|_{A/I}\) the quotient norm. This is so because

\[\|S_f(h)\| = \|S_g(h)\| \leq \|P(gh)\| \leq \|g\| \|h\| \quad \text{for all } g \in A\]

such that \([g] = [f]\) i.e. \(f-g \in \mathcal{I}\).

By (1.3) the isomorphism \(f \mapsto S_f\) is a bounded map.

One may easily show that \(A/I\) is actually isomorphic to a subalgebra of \(B(H)\) for some Hilbert space \(H\). Cole proceeded to show that for each \(f \in A\) we can choose a probability measure \(\mu\) such that the equality in (1.3) holds. From this we can construct a Hilbert space \(H\) such that \(A/I\) is isometrically isomorphic to a closed subalgebra of \(B(H)\).

\[(1.4) \quad \text{Lemma}\]

For each \([f] \in A/I\) there exists a probability measure on \(X\) such that \(\|S_{[f]}\|_{\text{op}} = \|[f]\|_{A/I}\).
Proof:

Without loss of generality let $\| [f] \|_{A/I} = 1$. By the Hahn-Banach theorem there exists a bounded linear functional $L$ on $A$ with $L = 0$ on $I$, $\| L \| = 1$ and $L(f) = 1$. Hence there is a complex measure $\nu$ on $X$ with $\| \nu \| = 1$,

$$\int f d\nu = 1 \quad \text{and} \quad \int g d\nu = 0 \quad \forall g \in I.$$ 

Choose any $g_n \in I$ such that $\| f - g_n \|_1 \to 1$ and let $g$ be a weak* cluster point of $\{ f - g_n \}$ in $L^\infty(\mathbb{R})$. Then $\| g \|_{L^\infty} \leq 1$. 

$\int (f - g_n) d\nu = 1$ \quad so \quad $\int g d\nu = 1$. Since $\| \nu \| = 1$ and $\| g \| \leq 1$, this implies $gd\nu \geq 0$ and $|g| = 1$ a.e. $-d|\nu|$. Hence,

$$gd\nu = d|\nu| \quad \text{so} \quad d\nu = g|\nu|.$$ 

Put $\lambda = |\nu|$ thus $\lambda$ is a probability measure.

If $h \in I$ then

$$\int hgd|\nu| = \int hd\nu = 0.$$ 

Thus $g \perp I$ in $H_{\lambda}$. Furthermore it follows from the definition of $g$ that $f - g \in I$ in $H_{\lambda}$. But $f = g \oplus (f - g)$ and $P(f) = g$

where $P$ is the projection on $I$ in $H_{\lambda}$.

Hence $S_f(1) = P(f) = g$ and so we have $\| S_f \|_{H_{\lambda}} = \| g \|_{H_{\lambda}} = 1$ where the first $\|,\|_{H_{\lambda}}$ is the operator norm on $H_{\lambda}$ and the second $\|,\|_{H_{\lambda}}$ is the Hilbert space norm of $H_{\lambda}$.

Therefore $\| S_f \|_{OP} = \| [f] \|_{A/I}$. 

Q.E.D.
Now for each $[f] \in A/I$ we choose $\lambda$ such that the statement of Lemma (1.4) holds and let $\Lambda$ be the collection of all $\lambda$'s chosen this way.

Define $\bigoplus_{\lambda \in \Lambda} H_{\lambda}$ to be the direct sum of $H_{\lambda}$ where an element $\bigoplus_{\lambda \in \Lambda} f_{\lambda} \in \bigoplus_{\lambda \in \Lambda} H_{\lambda}$ has $f_{\lambda} = 0$ for all but finitely many $\lambda$. We define the norm on $\bigoplus_{\lambda \in \Lambda} H_{\lambda}$ by

$$
\left\| \bigoplus_{\lambda \in \Lambda} f_{\lambda} \right\|^{2} = \sum_{\lambda \in \Lambda} \left\| f_{\lambda} \right\|^{2}_{H_{\lambda}}.
$$

Let $H$ be the closure of $\bigoplus_{\lambda \in \Lambda} H_{\lambda}$ with respect to this norm. It is obvious that $H$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle_{H}$ where

$$
\langle \bigoplus_{\lambda \in \Lambda} f_{\lambda}, \bigoplus_{\lambda \in \Lambda} g_{\lambda} \rangle_{H} = \sum_{\lambda \in \Lambda} \langle f_{\lambda}, g_{\lambda} \rangle_{H_{\lambda}}
$$

Define $\tilde{S}[g] \in B(H)$ for each $[g] \in A/I$ as follows:

Let $\bigoplus_{\lambda \in \Lambda} f_{\lambda} \in H$. Then $f_{\lambda} \in H_{\lambda}$ for all $\lambda \in \Lambda$. Now $\tilde{S}[g]$ takes $f_{\lambda}$ to $\tilde{S}[g](f_{\lambda})$ where $\tilde{S}[g] \in B(H_{\lambda})$ is defined as in the theorem (1.2). That is

$$
\tilde{S}[g] \left( \bigoplus_{\lambda \in \Lambda} f_{\lambda} \right) = \bigoplus_{\lambda \in \Lambda} \tilde{S}[g](f_{\lambda}).
$$

Then we have

$$
\left\| \tilde{S}[g] \left( \bigoplus_{\lambda \in \Lambda} f_{\lambda} \right) \right\|_{H} \leq \sum_{\lambda \in \Lambda} \left\| f_{\lambda} \right\|_{H_{\lambda}}^{2} \left\| [g] \right\|_{A/I}
$$

by (1.3). Thus the isomorphism $[g] \in A/I \rightarrow \tilde{S}[g] \in B(H)$ is bounded.
On the other hand for each \([g] \in \Lambda/I\), there is \(\lambda_0\) and 
\(f_{\lambda_0}\) with \(\|f_{\lambda_0}\|_H = 1\) such that \(\|S_{[g]}(f_{\lambda_0})\|_H \geq (1-\varepsilon)\) 
\(\|[g]\|_{\Lambda/I}\) Let \(\lambda \in \Lambda\) such that \(f_{\lambda} = 0\) if \(\lambda \neq \lambda_0\)
and \(f_{\lambda} = f_{\lambda_0}\) if \(\lambda = \lambda_0\), then \(\|S_{[g]}(\theta f_{\lambda})\| \geq (1-\varepsilon)\) 
\(\|[g]\|\Lambda/I\) and \(\|\theta f_{\lambda}\|_H = \|f_{\lambda}\|_{H_{\lambda_0}}\). Hence we have \(\|S_{[g]}\|_{B(H)} = 1\).

A consequence of the above is the following theorem.

(1.5) THEOREM

A \(Q\)-algebra is isometrically isomorphic to a closed subalgebra of \(B(H)\) for some Hilbert space \(H\).

Proof:

The map \([g] \in \Lambda/I \rightarrow S_{[g]} \in B(H)\) is obviously an isomorphism since \([g] \mapsto S_{[g]}\) is. The isometry of the map has already been proven. Q.E.D.

We now prove an inequality due to J. Wermer [15].

(1.6) THEOREM

Let \(B\) be a \(Q\)-algebra and fix \(x \in B\). If there is a constant \(C\) such that \(\|x^n\| \leq C\), \(n = 0, 1, 2, \ldots\). Then for every polynomial \(P\) we have \(\|P(x)\| \leq C^2 \sup\{ |P(z)|: \varepsilon \leq C \}
\(|z| = 1\).
Proof:

Fix $P(x)$ and put $y = P(x)$. Choose a probability measure $\lambda$ by lemma (1.4) such that $\|S_y\|_\lambda = \|y\|_B$, where $\|,\|_\lambda$ is the operator norm of $H_\lambda$. By thm.(1.2) and the inequality (1.3) we have

$$\|S(x)^n\|_\lambda = \|Sx^n\|_\lambda \leq \|x^n\|_B < C.$$  

We now define a new inner product $\langle,\rangle$ in $H_\lambda$ by

$$\langle \xi, \eta \rangle = \text{mean} \left( \langle S^n_x \xi, S^n_x \eta \rangle \right)$$  

where "mean" is any invariant mean (see Greenleaf [5]) and $(,)_{\lambda}$ the old inner product defined by $\lambda$. (note.

$H_\lambda = \text{closure of } A \text{ in } L^2(\lambda)$). It is clear that $\langle,\rangle$ is an inner product and $S_x$, a unitary operator in the new inner product. In addition we have

$$\frac{1}{C} \| \xi \|_{\text{old}} \leq \| \xi \|_{\text{new}} \leq C \| \xi \|_{\text{old}} \forall \xi \in H_\lambda$$  

because

$$\langle \xi, \xi \rangle = \text{mean} \left( \langle S^n_x \xi, S^n_x \xi \rangle \right) \leq C^2 \langle \xi, \xi \rangle$$  

and

$$\langle \xi, \xi \rangle < C^2 \langle S^n_x \xi, S^n_x \xi \rangle \forall \xi \in H_\lambda$$  

hence

$$\langle \xi, \xi \rangle < C^2 \text{mean} \left( \langle S^n_x \xi, S^n_x \xi \rangle \right) = C^2 \langle \xi, \xi \rangle.$$
If \( S \in B(H) \) then the operation norm of \( S \) in the new inner product satisfies

\[
\| S \|_{\text{new}} \leq \frac{\| S(h_1) \|_{\text{old}}}{\| h_1 \|_{\text{new}}} \leq \frac{C \| S(h_1) \|_{\text{old}}}{\| h_1 \|_{\text{old}}} = C^2 \| S \|_{\text{old}}
\]

and also

\[
\| S \|_{\text{new}} \geq \frac{1}{C} \| S(h_2) \|_{\text{old}} \geq \frac{1}{C^2} \| S \|_{\text{old}}
\]

where \( h_1, h_2 \in H \). Thus we have

\[
\frac{1}{C^2} \| S \|_{\gamma, \lambda} \leq \| S \|_{\text{new}} \leq C^2 \| S \|_{\gamma, \lambda}
\]

Then

\[
\| P(x) \|_B = \| S \|_{\gamma, \lambda} \leq C^2 \| S \|_{\text{new}}
\]

But also by Theorem (1.1) \( S_y = S_p(x) = P(S_x) \).

Since \( S_x \) is unitary in the new inner product we have

\[
\| S \|_{\text{new}} = \| P(S_x) \|_{\text{new}} \leq \max_{|z| = 1} |P(z)|.
\]

Therefore

\[
\| P(x) \|_B \leq C^2 \max_{|z| = 1} |P(z)|
\]

Q.E.D.
CHAPTER 2

In the proof of Theorem (1.6) the Hilbert space methods were used. But similar results can be obtained without the use of such techniques. The next theorem is such an example. It is due to Bernard [3].

(2.1) THEOREM

Let $G$ be a multiplicative group of elements in the $\mathbb{Q}$-algebra $B$ whose norms are bounded by the constant $C$. Then $\forall g_1, \ldots, g_n \in G$ and a complex polynomial $P$ with

$$||P|| = \max_{|x_i| \leq 1} |P(z_1, \ldots, z_n)| \leq 1$$

we have $||P(g_1, \ldots, g_n)||_B \leq C^2$.

Proof:

Let $B = A/I$ where $A$ is a uniform algebra on $X$ and $I$ a closed ideal in $A$ and $\pi: A \to B$ a canonical homomorphism. Let $\epsilon > 0$ be an arbitrary constant.

In order to see what is happening more clearly we first prove the theorem for $n = 1$.

Let $g \in G$ and $P(z) = \sum_{m=-M}^{M} P_m z^m$ choose $a_m \in A$ such that

$$||a_m|| \leq (1+\epsilon)C$$

and $\pi(a_m) = g^m$. Let $\Delta = \{ z \in \mathbb{C} : |z| = 1 \}$ with the normalized Haar measure $\mu$. For each positive integer $N$, define $F_N: X \times \Delta \to \mathbb{C}$ by $F_N(x, t) = \sum_{r=-N}^{N} a_r(x) e^{-irt}$. 
Then \( \sup_{x \in X} \| (F_N(x,t))^2 \|_{L^2(\Delta)} \)
\[
= \sup_{x \in X} \sum_{r=-N}^{N} |a_r(x)|^2
\]
\[
(\#) \leq (1+\epsilon)^2 C^2 (2N+1)
\]

Let \( b_N(x) = \frac{1}{2N+1} \int \sum_{m=-M}^{M} P_m e^{int} (F_N(x,t))^2 dt \)
\[
\text{then } b_N(x) = \frac{1}{2N+1} \sum_{m=-M}^{M} P_m a_r(x) a_s(x)
\]
\[
\text{so } \pi(b_N) = \sum_{m=-M}^{M} P_m g_m^r g_m^s
\]
\[
\text{when } |m| \leq 2N.
\]

Also \( \sup_{x \in X} |b_N(x)| \leq (1+\epsilon)^2 C^2 \) by (\#)

But \( \frac{2N+1 - |m|}{2N+1} \to 1 \) as \( N \to \infty \).

Thus we have \( \pi(b_N) \to \sum_{|m| \leq M} P_m g_m = P(g) \) as \( N \to \infty \).

This completes the proof for \( n = 1 \).
For \( n > 1 \) we use the same idea that was used in the proof for \( n = 1 \) but the notations will be a little more complicated.

First \( \Delta^n \) will mean \( \Delta \times \Delta \times \ldots \times \Delta \), a Cartesian product of \( n \) copies of \( \Delta \). This time \( n \) will mean the normalized Haar measure on \( \Delta^n \). Let \( a, b, c \in \mathbb{Z}_n \) so \( a = (a_1, \ldots, a_n) \) etc. and define \( \| a \| = \max |a_i| \) so when \( N \geq \deg P \) we can write

\[
\prod(z_1, \ldots, z_n) = \sum_{\alpha \in \mathbb{Z}_n^n} P(z_1, \ldots, z_n) \prod_{j=1}^{n} z_j^{a_j}.
\]

For a fixed polynomial \( P \) of \( n \)-variables there exists a finite \( S \subseteq \mathbb{Z}^n \) such that

\[
\prod(z_1, \ldots, z_n) = \sum_{\alpha \in S} P(z_1, \ldots, z_n) \prod_{j=1}^{n} z_j^{a_j}.
\]

with \( P \neq 0 \) for \( a \in S \). (Note. \( S \) is unique).

Now let \( g_1, g_2, \ldots, g_n \in G \) and \( P \) a complex polynomial of \( n \) variables with \( \| P \|_{\infty} \leq 1 \). Choose \( \{ u_a \} \) \( a \in \mathbb{Z}_n^n \) A such that

\[
\prod(u_a) = g_1 \cdot \ldots \cdot g_n
\]

and \( \| u_a \|_{\infty} \leq (1+\epsilon)C \) for all \( a \in \mathbb{Z}_n^n \).

For each positive integer \( N \) define \( F_N: X \times \Delta^n \rightarrow C \) by

\[
F_N(x, t_1, t_2, \ldots, t_n) = \sum_{\alpha \in \mathbb{Z}_n^n} u_\alpha(x) e^{-i\alpha_1 t_1 - i\alpha_2 t_2 - \ldots - i\alpha_n t_n}
\]

where the sum is taken over all \( a \in \mathbb{Z}_n^n \) such that

\[
\| a \| = \max |a_i| \leq N.
\]

We now show as in the case for \( n = 1 \), \( F_N \in C(X; L^2(\Delta^n)) \).

(The continuity of \( F_N \) is obvious).
Let $\sup_{x \in X} \left\| (F_n(x, t_1, t_2, \ldots, t_n))^2 \right\|_{L^2(\Delta n)}$

$= \sup_{x \in X} \left( \sum_{\alpha \in \mathbb{Z}^n, \left\| \alpha \right\| < N} |u_{\alpha}(x)|^2 \right)$

$\leq (1+\varepsilon)^2 C^2 (2N+1)^n$

Let $b_n(x) = \frac{1}{(2N+1)^n} \sum_{\alpha \in \mathbb{Z}^n} \Delta_n^{i \alpha} \Phi_{\alpha}^{i \alpha t_1 \ldots i \alpha t_n} \left( F_n(x, t_1, \ldots, t_n)^2 d_n(t_1, \ldots, t_n) \right)$

where $S$ is the finite set associated with $P$. When $N$ is large enough (for example, $N > \| \alpha \|$ for all $\alpha \in S$) we have

$b_n(x) = \frac{1}{(2N+1)^n} \sum_{\alpha \in S} \beta + \gamma = \alpha, \left\| \beta \right\|, \left\| \gamma \right\| \leq N \sum_{\alpha \in \mathbb{Z}^n} \frac{\Phi_{\beta} \Phi_{\gamma} u_{\beta}(x) u_{\gamma}(x)}{\alpha \in S}$

where $\beta + \gamma = (\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n)$. The sum is taken over all $\beta, \gamma \in \mathbb{Z}^n$ such that $\left\| \beta \right\| \leq N, \left\| \gamma \right\| \leq N$ and $\beta + \gamma = \alpha$ and over all $\alpha \in S$.
So \( \pi(b_N) = \frac{1}{(2N+1)^n} \sum_{\alpha \in S} \prod_{\beta + \gamma = \alpha, \|B_{\beta\gamma}\| \leq N} P_{a_{\alpha}} \pi(u_{\beta}) \pi(u_{\gamma}) \)

\[
= \sum_{\alpha \in S} \left( \prod_{i=1}^{n} \frac{2N+1-|a_i|}{2N+1} \right) a_1 a_2 \ldots a_n g_1 g_2 \ldots g_n
\]

But since \( S \) is finite, \( \prod_{i=1}^{n} \frac{2N+1-|a_i|}{2N+1} \to 1 \) as \( N \to \infty \)
for each \( \alpha \), hence \( \pi(b_N) \to P(g_1, \ldots, g_n) \) as \( N \to \infty \).

Also by (*2) we have \( \|\pi(b_N)\|_B \leq (1+\varepsilon^2)C^2 \).
Since \( \varepsilon \) was arbitrary we have
\[
\|P(g_1, \ldots, g_n)\|_B \leq C^2
\]
Q.E.D.

Next theorem is a partial generalization of the theorem of J. Wermer (Thm. 1.6) by Varopoulos [10].

(2.2) **THEOREM**

Let \( B \) be a Q-algebra, \( c > 0 \) and \( \alpha > 0 \) are two constants and \( b \) some element of \( B \) such that \( \|b^n\|_B \leq c n^\alpha \) for all \( n > 1 \).
Then for any polynomial of the form \( P(z) = \sum_{n=1}^{N} a_n z^n \) and any \( \varepsilon > 0 \) we have
\[
\|P(b)\|_B \leq C_1 \sup_{|z| \leq 1} \left| \sum_{n=1}^{N} a_n n^{2\alpha + \varepsilon} z^n \right|
\]
where \( C_1 \) is a constant that depends only on \( C, \alpha \) and \( \varepsilon \).
For the proof of this theorem we need the following Proposition (due to Varopoulos).

(2.3) Proposition

Let $H$ be a Hilbert space and let $T$ be a bounded operator on $H$ such that $\|T^n\| \leq Cn^a$ for all $n \geq 1$ where $C > 0$ and $a > 0$. Then for every $\varepsilon > 0$ and every $P(z) = \sum_{n=1}^{\infty} a_n z^n$ we have

$$\|P(T)\| \leq C_2 \sup_{|z| \leq 1} \left| \sum_{n=1}^{N} a_n n^{2a+\varepsilon} z^n \right| = C_2 P(2a+\varepsilon)$$

where $C_2$ is a constant that depends only on $C, a$, and $\varepsilon$.

Now with this proposition and Lemma (1.3) the proof of the Theorem (2.2) can take exactly the same course that the proof of the theorem (1.6) took. So we only give a proof of the proposition.

Proof of Prop. (2.3)

First we prove the case when $a$ is such that $0 < a < \frac{1}{2}$, and $2a + \varepsilon < 1$. The proof for this case contains the idea that was used in the proof of Thm. (2.1).

Before we start our proof we make the following observations:

(2.4) Define $A_n^\beta$ by $(1-x)^{-\beta-1} = \sum_{n=0}^{\infty} A_n^\beta x^n$, $\beta \neq -1, -2, \ldots, |x| < 1$.

Then (i) $A_n^{\beta+\gamma+1} = \sum_{p+q=n} A_p^\beta A_q^\gamma$.

(ii) $A_n^\beta = \frac{n^\beta}{\Gamma(\beta+1)} [1+O(\frac{1}{n})]$
(i) is a trivial consequence of Cauchy Products.

For (ii) we observe that \( A_n^\beta = \frac{\Gamma(n+\beta+1)}{n! \Gamma(\beta+1)} \),
so we need only to show \( f_n(\beta) = \frac{\Gamma(n+\beta+1)}{n! n^\beta} \approx 1 + O\left(\frac{1}{n}\right) \)

First we show that for a large \( n \), \( f_n(\beta) \) is monotone increasing for

\[
f_n(\beta) = \frac{1}{n!} \int_0^\infty \frac{y^{n+\beta}}{n^\beta} e^{-y} dy
\]

\[
= \frac{1}{n!} \int_0^\infty (\frac{y}{n})^\beta y^n e^{-y} dy
\]

\[
= \frac{1}{n} \int_0^\infty x^\beta (nx)^n e^{-nx} dx
\]

\[
= \frac{n+1}{n!} \int_0^\infty x^{n+\beta} e^{-nx} dx
\]

Now by the Lebesgue Dominated Convergence Theorem

\[
\frac{d}{d\beta} f_n(\beta) = \frac{n+1}{n!} \int_0^\infty (n+\beta) x^{n+\beta-1} e^{-nx} dx
\]

\[
= (n+\beta) f_n(\beta-1) > 0 \quad \text{when } n \text{ is large.}
\]

If we assume that \( \beta \) is an integer, then

\[
f_n(\beta) = \frac{\Gamma(n+\beta+1)}{n! n^\beta} = \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \ldots \left(1 + \frac{\beta}{n}\right)
\]
and so by expanding we see that \( f_n(\beta) = 1 + 0 \left( \frac{1}{n} \right) \). But \( f_n(\beta) \) is monotone increasing function of \( \beta \) so if \( p < \beta < q \), \( p, q \in \mathbb{Z} \) and for a sufficiently large \( n \), \( f_n(p) \leq f_n(\beta) \leq f_n(q) \).

Therefore we have

\[
f_n(\beta) = 1 + 0 \left( \frac{1}{n} \right) \quad \text{for all admissible } \beta.
\]

Now we proceed to prove the proposition. Let us denote by \( T^* \) the adjoint operator of \( T \) and by \( \langle \cdot, \cdot \rangle \) the scalar product in \( H \). Define \( F \) and \( F^* : H \times \Delta \rightarrow H \) by

\[
F(h,t) = \sum_{n=0}^{\infty} A_n^{-\alpha-1/2(1+\varepsilon)} (T^n h) e^{i \int n t}
\]

\[
F^*(h,t) = \sum_{n=0}^{\infty} A_n^{-\alpha-1/2(1+\varepsilon)} (T^* h) e^{i \int n t}
\]

where \( A_n \) is defined as before.

We claim here that for a fixed \( h \in H \), \( F(h,t) \), \( F^*(h,y) \in L^2(\Delta; H) \) where \( L^2(\Delta; H) \) denotes the space of \( H \)-valued \( L^2 \)-functions on \( \Delta \). This is so since the Plancherel Theorem holds for a Hilbert space and with (2.4) (ii) we have

\[
\int |F(h,t)|^2 \, d\eta(t) = \sum_{n=0}^{\infty} |A_n^{-\alpha-1/2(1+\varepsilon)} (T^n h) e^{i \int n t}|^2 \, d\eta(t)
\]

\[
= \sum_{n=0}^{\infty} |A_n^{-\alpha-1/2(1+\varepsilon)} (T^n h)|^2
\]

\[
\leq K \sum_{n=0}^{\infty} \frac{n^{-2\alpha-1-\varepsilon}}{(n^{-\alpha-1/2(1+\varepsilon)})^2} \left( 1 + 0 \left( \frac{1}{n} \right) \right) n^{2\alpha}
\]

\[
\leq K_2 \sum_{n=0}^{\infty} (n^{-1-\varepsilon} + K_3 n^{-2-\varepsilon}) < \infty
\]
where $K, K_2$, and $K_3$ are constants and $n$ the normalized Haar measure on $\Delta$. Similar results hold for $F_\alpha$.

We also define $\phi(h,t) = \langle F_\alpha(h,t), F(h,t) \rangle$. Then clearly $\phi(h,t) \in L^1(\Delta)$, $\|\phi(h,t)\|_1 \leq C_1 \|h\|^2$ for some constant $C$.

Now define $P^\ast(e^{it}) = \sum_{n=1}^{\infty} (A_n^{-2\alpha - \epsilon})^{-1} a_n e^{int}$. Then

$$\left| \int P^\ast(e^{it}) \phi(h,t) d\eta(t) \right|$$

$$= \left| \int \left( \sum_{n=1}^{\infty} A_n^{2\alpha + \epsilon} a_n e^{int} \right) \left( \sum_{n=0}^{\infty} A_n^{-2\alpha - \epsilon} <T^n h, h > e^{int} \right) d\eta(t) \right|$$

$$= \left| \sum_{n=0}^{\infty} a_n <T^n h, h > \right|$$

$$= \left| \langle P(T)h, h \rangle \right|$$

But this quantity is $\leq C_1 \|h\|^2 \|P^\ast\|_\infty$ by the inequality (*).

Again by (2.4) (ii) we have $\|P\|_\infty \leq K_0 \|P_{2\alpha + \epsilon}\|$ where $K_0$ is a constant depending only on $\alpha$ and $\epsilon$. Thus the desired result follows.

We consider the case when $\alpha > \frac{1}{2}$. Varopoulos suggests that we use the classical results on Riesz potentials and Bernstein's theorem (see Zygmund [15]). But we directly prove the result using only the Plancherel Theorem.
For $\alpha > \frac{1}{2}$ or $\alpha + \varepsilon > \frac{1}{2}$ we want to show that

\[(\ast 2) \quad \sum_{n=1}^{N} |a_n| n^\alpha \leq C \quad \sup_{|z| \leq 1} \left| \sum_{n=1}^{N} a_n \frac{2\alpha + \varepsilon}{n} \right| z^n \]

Suppose $\sup_{|z| \leq 1} \left| \sum_{n=1}^{N} a_n n^{2\alpha + \varepsilon} z^n \right| \leq 1$.

Then

$$\int_{\Delta} \left| \sum_{n=1}^{N} a_n n^{2\alpha + \varepsilon} e^{int} \right|^2 dn(t) \leq 1.$$ 

Hence by the Plancherel Theorem

$$\left| \sum_{n=1}^{N} a_n \right|^2 n^{4\alpha + \varepsilon} \leq 1.$$ 

Now we sum only from $2^j$ to $2^{j+1} - 1$ then

$$\sum_{n=2^j}^{2^{j+1}-1} |a_n| n < \left( \sum_{n=2^j}^{2^{j+1}-1} |a_n|^2 \right)^{1/2} \left( \sum_{n=2^j}^{2^{j+1}-1} n^{2\alpha} \right)^{1/2}.$$ 

But from

$$\sum_{n=2^j}^{2^{j+1}-1} |a_n|^2 n^{4\alpha + 2\varepsilon} \leq 1$$

we have

$$\sum_{n=2^j}^{2^{j+1}-1} |a_n|^2 \leq 1.$$ 

Hence

$$\sum_{n=2^j}^{2^{j+1}-1} |a_n|^2 \frac{1}{2} \leq 2^{j(2\alpha + \varepsilon)}.$$ 

$$\sum_{n=2^j}^{2^{j+1}-1} |a_n|^2 \frac{1}{2} \leq 2^{j(2\alpha + \varepsilon)}.$$ 

$$\sum_{n=2^j}^{2^{j+1}-1} |a_n|^2 \frac{1}{2} \leq 2^{j(2\alpha + \varepsilon)}.$$
Substituting this inequality into the preceding set of inequalities we have

\[ 2^{j+1-1} \sum_{n=2^j} |a_n|^a \leq 2^{-j(2\alpha + \epsilon)} 2^{j/2} 2^{(j+1)a} = 2^a 2^j (1/2 - \alpha - \epsilon) \]

Hence

\[ \sum_{n=1}^N |a_n|^a \leq \sum_{j=0}^\infty 2^a 2^j (1/2 - \alpha - \epsilon) = C < \infty \]

since \( \alpha + \epsilon > \frac{1}{2} \) and \( C \) depends only on \( \alpha \) and \( \epsilon \).

Q.E.D.

Now the proof of the above theorem (when \( 0 \leq \alpha < \frac{1}{2} \)) suggests that we may be able to generalize the theory of Q-algebras to a larger class of algebras defined on a Hilbert space \( H \) using Cole's theorem. This indeed is the case. This larger class of algebras are called operator algebras.

We shall discuss operator algebras in Chapter 5 so here we only give the definition of an operator algebra. The definition is due to Varopoulos [12].

(2.5) Definition Let \( B \) be a Banach algebra. Then \( B \) is called an operator algebra if it can be identified topologically (i.e., up to norm equivalence) with \( \overline{B} \subset B(H) \) a closed subalgebra of \( B(H) \) of some Hilbert space \( H \).

Again we remark here that the Proposition (2.3) is really a Theorem on Operator algebras.
CHAPTER 3

In this chapter we discuss criteria for $Q$-algebras. But before we begin we make some remarks about the definition of a $Q$-algebra.

Let $B$ be a $Q$-algebra, then $B = A/I$ for some uniform algebra $A$ and a closed ideal $I$. Now we consider the algebra $B$ with an equivalent norm. Now this algebra, let us denote it by $\tilde{B}$, is naturally identified with $B$. The inequalities in Chapter 1 and 2 obviously holds only if we change the constants. Thus from now on we consider algebras $\tilde{B}$ which is norm equivalent to the $Q$-algebra $B$, also a $Q$-algebra. This is Davie's definition of a $Q$-algebra. Of course we note here that with this enlarged definition everything we have proved so far is still true with only minor modifications at the end of some of the proofs.

Now the first theorem which is due to I.G.Craw (see Davie [3]).

(3.1) THEOREM

Suppose $B$ is a commutative Banach algebra. $B$ is a $Q$-algebra if and only if there exist positive constants $M$ and $\delta$ such that:

(*1) whenever $X_1, X_2, \ldots, X_p \in B$ with $\|X_i\|_B \leq \delta$ and $P$ is a complex polynomial in $p$ variables with no constant term satisfying
\[ \|P\|_\infty = \text{Sup} \{ |P(z_1, \ldots, z_p)| : |z_i| \leq 1 \}, \]

then, \[ \|P(x_1, \ldots, x_p)\|_B \leq M. \]

Proof:

When \( B \) is a uniform algebra, the condition is trivially true. Thus it is also true for \( Q \)-algebras.

Conversely suppose \((*1)\) holds, then let \( A = \{x \in B : \|x\|_B \leq 1\} \) and \( X = D^\Lambda \), a Cartesian product of copies of the closed disc \( D \) in the complex plane, (one for each element of \( A \)) with the product topology.

Let \( A \) be the uniform closure in \( C(X) \) of the algebra \( A_0 \) of all polynomials in the coordinate functions \( \{\delta_x : x \in A\} \) with no constant terms. Define an algebra homomorphism \( T:A_0 \rightarrow B \) by \( T(\delta_x) = x \) and extending linearly and multiplicatively to \( A_0 \) (i.e. \( T(P(\delta_{x_1}, \ldots, \delta_{x_p})) = P(x_1, \ldots, x_p) \)).

Then \((*1)\) implies that \( T \) is bounded with respect to the uniform norm on \( A \). So extend to a bounded homomorphism \( T:A \rightarrow B \) which is clearly onto. Therefore \( B \) is isomorphic to \( A/I \) where \( I = \ker T \), which is a closed ideal in \( A \).

Q.E.D.

Now when \( B \) is a commutative Banach algebra, we have another equivalent statement of \( B \) being a \( Q \)-algebra. It is due to Varopoulos [9].

(3.2) THEOREM

Let \( B \) be a commutative Banach algebra. Then \( B \) is a \( Q \)-algebra if and only if there exists a constant \( C \) with the following property:
(*2) For any $p \geq 1$, a positive integer, any $x_1, \ldots, x_p \in B$ with $\|x_i\|_B \leq 1$, $i = 1, \ldots, p$ and $P(z_1, \ldots, z_p)$ a homogeneous polynomial of a positive degree of $p$ variables with $\|P\|_\infty \leq 1$ we have $\|P(x_1, \ldots, x_p)\|_B \leq C^{\deg P}$.

Proof:

If $B$ is a $Q$-algebra then the condition (*2) is clearly satisfied with $C$ such that $M \deg P < C^{\deg P}$.

Conversely suppose there is a constant $C$ with the property (*2). We want to show that (*1) of Theorem (3.1) holds with $M \neq 1$ and $\delta = \frac{1}{2C}$. Let $x_1, \ldots, x_p \in B$ such that $\|x_i\| \leq 1$ for all $i$ and $\|x_i\|_\infty < 1$.

Write $P = \sum_{n=1}^{N} P_n$ where $P_n$ is a homogeneous polynomial of degree $n$. But now consider $Q(t) = P(tz_1, \ldots, tz_n)$. By applying Cauchy's inequality we have $|Q^{(n)}(0)| = n!|P_n| \leq n! \|P\|_\infty$. Hence $\|P_n\|_\infty \leq 1$. By hypothesis we have $\|P_n(y_1, \ldots, y_p)\| \leq C^n$ for all $y_i$ with $\|y_i\| \leq 1$. But $\|x_i\| \leq \frac{1}{2C}$ and $P_n$ is homogeneous of degree $n$, so we have $\|P_n(x_1, \ldots, x_p)\| \leq \frac{1}{2^n}$. Therefore we have $\|P(x_1, \ldots, x_p)\| \leq 1$.

Q.E.D.

To deal with polynomial of several variables more effectively, Davie introduces finite tensor algebras. We however start with more general tensor algebras.
(3.3) **Definitions.** Let $E$ and $F$ be normed linear spaces and $E \otimes F$ be the algebraic tensor product. We define a projective norm $\| \cdot \|$ on $E \otimes F$ by

$$
\| \tau \| = \inf \left\{ \sum_{j=1}^{J} | \lambda_j | : \tau = \sum_{j=1}^{J} \lambda_j e_j \otimes f_j; \| e_j \| \leq 1, \| f_j \| \leq 1 \right\}
$$

where the infimum is taken over all such expressions of $\tau$.

We take the closure of $E \otimes F$ with respect to this norm, call it the projective tensor product and denote it by $E \hat{\otimes} F$ since it satisfies the universal mapping property as a Banach space (see Schwartz [8]).

We define an injective norm $\| \cdot \|$ on $E \otimes F$ by

$$
\| \psi \| = \| \sum_{j=1}^{J} \lambda_j e_j \otimes f_j \| = \sup \left\{ \left| \sum_{j=1}^{J} \lambda_j \langle u, e_j \rangle \langle \psi, f_j \rangle \right| : \| u \| \leq 1, \| v \| \leq 1 \right\}
$$

where $E'$ and $F'$ are the dual of $E$ and $F$ respectively.

We take the closure of $E \otimes F$ with respect to this norm, call it the injective tensor product of $E$ and $F$ and denote it by $E \hat{\otimes} F$ (Schwartz [8]).

We first prove a simple equality in the $\| \cdot \|$ norm

(3.4) $\| \psi \| = \sup \{ | \langle \psi, \tau \rangle | : \tau \in E \hat{\otimes} F', \| \tau \| \leq 1 \}$
That is: if \( \psi = \sum \lambda_j e_j \otimes f_j \), \( \tau \) is of the form \( \sum_{k} \mu_k u_k \otimes v_k \)
then \[ \|\psi\|_\infty = \sup \{ |\sum_{k} \mu_k (\sum_{j} \lambda_j <u_k,e_j><v_k,f_j>)|: \|\tau\|_+ \leq 1 \} \]

Now in (3.4) the inequality "\( \leq \)" is clear from the definition of \( \|,\| \) (For \( \|u \otimes v\|_F \leq 1 \) if \( \|u\|_F, \|v\|_F \leq 1 \)).

Suppose \( \|\tau\|_+ \leq 1 \), \( \tau = \sum_{k} \mu_k u_k \otimes v_k \) with

\[ \sum_{i=1}^{n} |\lambda_i| \leq 1 \] and \( \|u\|_B, \|v\|_F \leq 1 \) and

\[ |<\tau,\psi>| = |\sum_{k} \mu_k (\sum_{j} \lambda_j <u_k,e_j><v_k,f_j>)| \]

\[ \leq 1 \text{ if } \|\sum_{j=1}^{n} \lambda_j e_j \otimes f_j\| \leq 1. \]

Therefore we have the equality in (3.4).

What we have proved in (3.4) is that

(3.5) \( E \hat{\otimes} F \subseteq (E' \hat{\otimes} F')' \) where the inclusion is isometric.

The equality may fail in (3.5) even if \( E \) and \( F \) are reflexive.

The example, \( E = F = C(I) \) where \( I \) is the closed unit interval on the real line \( R \), provides a case when the inclusion in (3.5) is proper.

We now prove a lemma about symmetric tensor products.

(3.6) Lemma

If \( \tau = \sum_{i=1}^{n} \lambda_i b_i \otimes \ldots \otimes b_i \) symmetric in \( B \hat{\otimes} \ldots \hat{\otimes} B \)
then \[ \|\tau\|_\infty \leq (2e)^n \sup \{ |\sum_{i=1}^{n} \lambda_i <v,b_i>| \} \|v\|_B, \|v\|_F \leq 1 \]
Proof.

The proof of the lemma is an easy conclusion of the following equality: For all \( u_i \in B', b_i \in B', i = 1, \ldots, n \)

\[
\sum_{\sigma \in S_n} <u_{\sigma(1)}, b_1> \cdots <u_{\sigma(n)}, b_n>
\]

\[= (-1)^n \sum_{\Omega} (-1)^{|\Omega|} \sum_{\Omega} <\sum_{\Omega} u_j(\lambda), b_1> \cdots <u_{\lambda(n)}, b_n>\]

where \( \sum_{\Omega} \) is summed over all permutations of \( \{1, \ldots, n\} \), \( \sigma \in S_n \) ranges over all non-empty subsets of \( \{1, \ldots, n\} \), \( |\Omega| \) the cardinality and \( v_\Omega = \sum_{j \in \Omega} u_j \) (note \( S_n \) is the symmetric group on \( n \) elements).

To prove the above equality first we write the right hand side of the equality as follows

\[= (-1)^n \sum_{\Omega} (-1)^{|\Omega|} \sum_{\Omega} <\sum_{\Omega} u_j(\lambda), b_1> \cdots <u_{\lambda(n)}, b_n>\]

where \( \sum_{\Omega} \) ranges over all maps \( \Omega : \{1, 2, \ldots, n\} \to \Omega \)

But if we first sum over \( \Omega \) such that \( \Omega = \lambda \) where \( \lambda \) is a map \( \lambda : \{1, \ldots, n\} \to \{1, \ldots, n\} \), that is

\[= (-1)^n \sum_{\lambda} (-1)^{|\Omega|} \sum_{\lambda} <u_{\lambda(\Omega)}(1), b_1> \cdots <u_{\lambda(n)}, b_n>\]

\[= (-1)^n \sum_{\lambda} \sum_{\lambda} \text{sgn}(\lambda \cdot \Omega) <u_{\lambda(\Omega)}(1), b_1> \cdots <u_{\lambda(n)}, b_n>\]

\[= (-1)^n \sum_{\lambda} \left( \sum_{\lambda} \text{sgn}(\lambda \cdot \Omega) \right) <u_{\lambda(\Omega)}(1), b_1> \cdots <u_{\lambda(n)}, b_n>\]
where the first \((\Sigma \lambda)\) is summed over all maps \(\lambda: \{1, \ldots, n\} \to \{1, \ldots, n\}\).

Suppose \(\lambda(\{1, \ldots, n\}) = \{k_1, \ldots, k_m\} = K\) where \(k_i\)'s are distinct elements then for each \(\Omega\) such that \(\theta_\Omega = \lambda\) for some \(\theta_\Omega\), \(\Omega \supseteq K\). Conversely, for each \(\Omega\) with \(\Omega \supseteq K\) there exists \(\theta_\Omega\) such that \(\theta_\Omega = \lambda\). In fact there are \(\binom{n-k}{|\Omega|-k}\) of them. By considering the sign of \(\theta_\Omega\) \((\text{sgn } \theta_\Omega = (-1)^{|\Omega|})\) we have (1) if \(k \neq n\) then

\[
\Sigma_{\lambda=\theta_\Omega} \text{sgn}(\theta_\Omega) \langle u_{\theta_\Omega}(1), b_1 \rangle \cdots \langle u_{\theta_\Omega}(n), b_n \rangle
\]

\[
= (-1)^k \left( 1 - \binom{n-k}{1} + \binom{n-k}{2} \cdots \right) \langle u_\lambda(1), b_1 \rangle \cdots \langle u_\lambda(n), b_n \rangle
\]

\[
= (-1)^k (1 - 1)^{n-k} \langle u_\lambda(1), b_1 \rangle \cdots \langle u_\lambda(n), b_n \rangle
\]

\[
= 0
\]

(2) and if \(k = n\) then \(\Omega = \{1, \ldots, n\}\) and \(\lambda \in S_n\) hence

\[
\Sigma_{\lambda=\theta_\Omega} \text{sgn}(\theta_\Omega) \langle u_{\theta_\Omega}(1), b_1 \rangle \cdots \langle u_{\theta_\Omega}(n), b_n \rangle
\]

\[
= \langle u_\lambda(1), b_1 \rangle \cdots \langle u_\lambda(n), b_n \rangle \quad \text{where } \lambda \in S_n\n\]

Therefore we have
(-1)^n \sum_{\Omega} (-1)^{|\Omega|} \sum_{\lambda=\theta_{\Omega}} \text{sgn}(\theta_{\Omega}) <u_{\lambda(1)}, b_1> \cdots <u_{\lambda(n)}, b_n> \\
= (-1)^n \sum_{\sigma \in S_n} <u_{\sigma(1)}, b_1> \cdots <u_{\sigma(n)}, b_n>

as claimed.

Now with this equality and symmetry of \( \varepsilon \) we have

\[
\sum_{i=1}^{n} \lambda_i <u_i, b_i> \cdots <u_n, b_n> = \frac{(-1)^n}{n!} \sum_{\Omega} (-1)^{|\Omega|} <u_{\Omega}, b_1> \cdots <u_{\Omega}, b_n>
\]

where \( \Omega \) and \( \nu_{\Omega} \) as before. But then \( \|v\|_{B^1} \leq |\Omega| \leq n \)
and since there are \( 2^n - 1 \) possible choise of \( \Omega \) we have

\[
\|t\| \leq \frac{2^n n^n}{n!} \sup_{\|v\|_{B^1} \leq 1} |\sum \lambda_i <v_i, b_i> \cdots <v_n, b_n>|
\]

The desired result follows from this and the inequality

\[
2^n n^n / n! \leq (2e)^n
\]

Q.E.D.

Now we look at finite tensor algebras. Let \( K_p \) be a
set \( K_p = \{1, 2, \ldots, p\} \) and \( K^p_p = K_p \times \cdots \times K_p \). We put the
discrete topology on \( K_p \). Then it is obvious that

\[
C(K_p^p) = K_p^\infty, \quad C(K_p) \otimes \cdots \otimes C(K_p) = C(K_p^p) = C_p^n
\]
where \( C_p^n \) is the \( n \)-dimensional vector space of all complex-valued functions on \( K_p^n \)
and \( C_p = C_p^1 \) (see any algebra book).
If \( a \in C^n_p = C(K_p) \otimes \ldots \otimes C(K_p) \) then

\[
\|a\|_\phi = \inf \{ \sum_{r=1}^{N} |\lambda_r| : = \sum_{r=1}^{N} \lambda_r f_1^{(r)} \ldots f_n^{(r)}, \|f_i^{(r)}\| \leq 1 \}
\]

where the infimum was taken over all such representation of \( a \).

This in terms of the notations for finite algebras,

\[
\|a\|_\phi = \inf \{ \sum_{r=1}^{N} |\lambda_r| : a(\beta_1, \ldots, \beta_n) = \sum_{r=1}^{N} \lambda_r f_1^{(r)}(\beta_1) \ldots f_n^{(r)}(\beta_n) \}
\]

\( f_i^{(r)} \in C_p \) with \( |f_i^{(r)}(\gamma)| \leq 1 \forall \gamma \in K_p \)

for all \( 1 \leq i \leq n, 1 \leq r \leq N \).

We can identify \( C^n_p \) with its own linear space dual by setting \( \langle a, b \rangle = \sum_{\beta \in K_p^n} a(\beta) b(\beta) \) so we have

\[
\|a\|_\phi = \sup \{ |\langle b, a \rangle| : \|b\|_\phi \leq 1 \}
\]

\[
= \sup \{ |\langle b, a \rangle| : b \in C^n_p, \|b\|_\phi \leq 1 \}
\]

\[
= \sup \{ \| \sum_{\beta \in K_p^n} a(\beta) f_1^{(\beta_1)} \ldots f_n^{(\beta_n)} \| : c_i \in C_p, |f_i^{(\gamma)}| \leq 1 \forall \gamma \in K_p, \forall i \}.
\]
We remark that the estimates for those norms were obtained by Littlewood [3] for \( n = 2 \). He showed
\[
\|a\|_{\mathcal{B}} \leq 3^{1/2} \sup_{\beta_1 \in \mathcal{P}} \Sigma_{\beta_2 \in \mathcal{P}} |a(\beta_1, \beta_2)|^{1/2}
\]
and
\[
\|a\|_{\mathcal{B}} \leq 3^{1/2} \cdot 2^{3/4} \left( \Sigma_{\beta \in \mathcal{P}} |a(\beta)|^{4/3} \right)^{1/4}.
\]

We note here that the Lemma (3.6) says, in terms of finite tensor algebra, that if \( a \in C_p^n \) and is symmetric with respect to the permutations of the arguments \( \beta_1, \ldots, \beta_n \) then
\[
(3.7) \quad \|a\|_{\mathcal{B}} \leq (2e)^n \sup \{ |\Sigma a(\beta) f(\beta_1) \cdots f(\beta_n)| : f \in C_p, \|f\| \leq 1 \}
\]

Now we can prove the following lemma due to Davie [3].

(3.8) **Lemma**

Suppose \( B \) is a commutative Banach algebra with constant \( K \) such that:

- if \( x_1, \ldots, x_p \in B \) with \( \|x_i\| \leq 1 \) and \( a \in C_p^n \) with \( \|a\|_{\mathcal{B}} \leq 1 \)

then we have
\[
\| \Sigma_{\beta \in \mathcal{P}} a(\beta) x_{\beta_1} \cdots x_{\beta_n} \| \leq K^n.
\]

Then \( B \) is a \( Q \)-algebra.
Proof:

We prove that the condition \((**1)\) of Thm. (3.1) is satisfied with \(M = 1\) and \(\delta = (4eK)^{-1}\). That is, if \(x_1, \ldots, x_p \in B\) with \(\|x_i\| \leq (4eK)^{-1}\) and \(\|P\|_{\infty} \leq 1\), \(P(0, \ldots, 0) = 0\) then we want to show that \(\|P(x_1, \ldots, x_p)\|_B \leq 1\).

Write \(P = \sum_{n=1}^{N} P_n\) where \(P_n\) is homogeneous of degree \(n\).

Fix \(n\) for the moment. We can write

\[
P_n(z_1, \ldots, z_p) = \sum_{\beta \in \mathbb{C}^n} a(\beta) z_{\beta_1} \cdots z_{\beta_n}\]

where \(a \in \mathbb{C}^n\) and \(a\) is symmetric in \(\beta_1, \ldots, \beta_n\).

We claim that \(\|P_n\|_{\infty} \leq 1\) because if we let \(Q(t) = P(tz_1, \ldots, tz_p)\) and apply Cauchy's inequality we get

\[
|Q^{(n)}(0)| = |n! P_n(z_1, \ldots, z_n)| \leq n! \|P\|_{\infty} \text{ for all } (z_1, \ldots, z_n) \text{ with } |z_i| \leq 1 \text{ for all } i = 1, \ldots, n.
\]

But then that says

\[
\sum_{\beta \in \mathbb{C}^n} |a(\beta) z_{\beta_1} \cdots z_{\beta_n}| \leq 1 \text{ if } |z_i| \leq 1 \text{ for all } i = 1, \ldots, n.
\]

So by Lemma (3.7) we have \(\|a\|_{\infty} \leq (2e)^n\).

From the hypothesis and \(\|x_i\| \leq (4eK)^{-1}\) we have

\[
\sum_{\beta \in \mathbb{C}^n} a(\beta) x_{\beta_1} \cdots x_{\beta_n} \|_B \leq \frac{x^n(2e)^n}{(4eK)^n} = 2^{-n}.
\]
That is to say \( \| P_n(x_1, \ldots, x_p) \|_B \leq 2^{-n} \) for all \( n \).

Therefore we have \( \| P(x_1, \ldots, x_p) \|_B \leq 1 \)

Q.E.D.

We note here that the converse of the Lemma (3.8) is obviously true. Because if we suppose (*) of Thm. (3.1) holds and \( x_1, a \) as in the Lemma (3.7) put

\[
P(z_1, \ldots, z_p) = \sum_{\beta \in K^n} a(\beta) z_{\beta 1} \cdots z_{\beta n}
\]

and note that

\[
\| P \|_\infty \leq \| a \|_\infty \leq 1. \text{ So it is clear that } \| P(x_1, \ldots, x_p) \|_B \leq M^n
\]

and, put \( K \) to be any constant such that \( K^n > M^n \) \( \forall n \).

We also note that the following condition is equivalent to the condition in the Lemma (3.8).

There is a constant \( K > 0 \) such that if \( x_1, \ldots, x_p \in B \) with

\[
\| x \|_2 \leq 1 \text{ and } \phi \in A \text{ with } \| \phi \| \leq 1 \text{ and we define } \alpha \in C^n \text{ by }
\]

\[
\alpha(\beta) = \phi(x_{\beta 1}, \ldots, x_{\beta n}) \text{ then } \| \alpha \|_\infty \leq K^n.
\]

First suppose the above condition holds then \( \| a \|_\infty \leq 1 \) implies \( |< a, \alpha >| \leq K^n \). But

\[
|< a, \alpha >| = \| \sum_{\beta \in K^n} a(\beta) \alpha(\beta) \|_p
\]

\[
= \| \phi (\sum_{\beta \in K^n} a(\beta) x_{\beta 1}, \ldots, x_{\beta n}) \|_p.
\]
This is true for all \( \phi \in A' \) with \( \| \phi \| \leq 1 \) hence \( \| a(\beta) x_{\beta_1} x_{\beta_2} \cdots x_{\beta_n} \| \leq K^n \).

Conversely suppose the condition in the Lemma (3.8) is satisfied, then we have \( |\langle a, \alpha \rangle| \leq K^n \) for all \( a \) with \( \| a \| \leq 1 \).

But by (3.5) with the usual dimension argument \( E' \otimes F' = (E \otimes F)' \) when \( E, F \) of finite dimension. Thus we have \( \| a \| \leq K^n \).

By collecting them together we have the theorem due to Davie [3].

(3.9) THEOREM

Let \( B \) be a commutative Banach algebra then the following are equivalent.

(i) \( B \) is a \( Q \)-algebra
(ii) There is a constant \( K > 0 \) such that if \( x_1, \ldots, x_p \in B \) with \( \| x_i \| \leq 1 \) and \( a \in C^n_p \) with \( \| a \| \leq 1 \),
then \( \| \sum_{\beta \in K^n_p} a(\beta) x_{\beta_1} x_{\beta_2} \cdots x_{\beta_n} \|_B \leq K^n \).

(iii) There is a constant \( K > 0 \) such that if \( x_1, \ldots, x_p \in B \) with \( \| x_i \| \leq 1 \) and \( \phi \in B' \) with \( \| \phi \| \leq 1 \), we define \( a \in C^n_p \) by \( a(\beta) = \phi (x_{\beta_1} x_{\beta_2} \cdots x_{\beta_n}) \), then \( \| a \| \leq K^n \).
CHAPTER 4

In this chapter we discuss injective algebras which was first defined by Varopoulos [10,11].

(4.1) Definitions

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{A}$ is an injective algebra if the linear mapping induced by the algebra multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, where $m(x \otimes y) = xy$, $\forall x, y \in \mathcal{A}$ is continuous for the injective norm of the tensor product $\mathcal{A} \otimes \mathcal{A}$.

We shall soon prove that every commutative injective algebra is a $\mathcal{Q}$-algebra. But before that we need the following important criterion for injective algebras. It is due to Varopoulos [10].

(4.2) THEOREM

(i) Let $\mathcal{A}$ be a Banach algebra and suppose there exists some constant $C$ such that for any $f \in \mathcal{A}$ there exists $S, P_m, Q_n \in \mathcal{A}$ such that $|\langle s, f \rangle| > \|f\|$, $\sum_{n=1}^{\infty} \|P_n\| \|Q_n\| < C$ and

$$
\langle s, xy \rangle = \sum_{n=1}^{\infty} \langle P_n, x \rangle \langle Q_n, y \rangle \quad \forall x, y \in \mathcal{A}.
$$

Then $\mathcal{A}$ is an injective algebra.

(ii) Let $\mathcal{A}$ be a commutative Banach algebra, then $\mathcal{A}$ is an injective algebra if and only if there exists a constant $K$ such that for any $n \geq 1$ and any $s \in \mathcal{A}$ there exists $\mu \in M(\mathcal{A})$. 


some Radon measure such that for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$,
\[ \langle S, x_1, x_2, \ldots, x_n \rangle = \int \langle T, x_1 \rangle \langle T, x_2 \rangle \ldots \langle T, x_n \rangle \, d\mu(T) \]
and $\|\mu\| \leq K^n$ \text{(where $S \in \mathbb{R}^1$ denotes $S \in \mathbb{R}^n$ and $\|S\| < 1$)}.

**Proof of (i)**

Now we want to show that there exists $\delta$ such that

\[
\text{if } f = \sum_{i,j} a_{ij} x_i y_j \quad \| \sum_{i,j} a_{ij} x_i y_j \| \leq \delta
\]

then $\|f\| < 1$. Put $\delta = 1/C$ where $C$ is as in the hypothesis.

Thus we have the desired result.

**Proof of (ii)**

Consider the map $\mathbb{R} \otimes \mathbb{R} \otimes \ldots \otimes \mathbb{R} \rightarrow C(\mathbb{R}^1 \times \mathbb{R}^1 \times \ldots \times \mathbb{R}^1)$ defined by $x_1 \otimes x_2 \otimes \ldots \otimes x_n \mapsto (ax_1, ax_2, \ldots, ax_n)$ and extending it linearly

where $(ax_1, ax_2, \ldots, ax_n)(T_1, T_2, \ldots, T_n) = \langle T_1, x_1 \rangle \langle T_2, x_2 \rangle \cdots \langle T_n, x_n \rangle$.

Now if $x_i \neq y_i$ then there exists $T_i \in \mathbb{R}_1'$ such that

$\langle T_i, x_i \rangle \neq \langle T_i, y_i \rangle$. Hence $\mathbb{R} \otimes \ldots \otimes \mathbb{R} \not\subseteq C(\mathbb{R}^1 \times \mathbb{R}_1' \times \ldots \times \mathbb{R}_1')$
and $R^0 \otimes \cdots \otimes R$ is a linear subspace and with the injective norm $\| \cdot \|$ on $R^0 \otimes \cdots \otimes R$ we have an isometry. That is to say $R^0 \otimes \cdots \otimes R$ is a closed linear subspace of $C(R'_1 \times \cdots \times R'_1)$ with the induced norm.

Consider $\mathbf{m}: R^0 \otimes \cdots \otimes R \to R$ defined by $\mathbf{m}(x_1 \otimes \cdots \otimes x_n) = x_1 \cdots x_n$ and extending it linearly and $S: R \to C$ by $S(x) = \langle \mathbf{s}, x \rangle$. Then we have a linear functional $\text{som}$ on $R^0 \otimes \cdots \otimes R$ which is a linear subspace of $C(R'_1 \times \cdots \times R'_1)$ (note $\text{som}(\langle ax_1, \ldots, ax_n \rangle) = \langle s, x_1 \ldots x_n \rangle$.) map is bounded by $\| \mathbf{s} \| \| \mathbf{m} \| \leq K_1^n$ for $s \in R'_1$, so we can extend $\text{som}$ to the whole of $C(R'_1 \times \cdots \times R'_1)$ without affecting its norm. We now apply the Riesz Representation Theorem to obtain

\begin{align*}
(4.3) \quad \text{som}(x_1 \otimes \cdots \otimes x_n) &= \langle \mathbf{s}, x_1 \ldots x_n \rangle \\
&= \int_{R'_1 \times \cdots \times R'_1} (a x_1, \ldots, a x_n) \left( \mathbf{T}_1, \ldots, \mathbf{T}_n \right) d\mu(\mathbf{T}_1, \ldots, \mathbf{T}_n) \\
&= \int_{R'_1 \times \cdots \times R'_1} \langle \mathbf{T}_1, x_1 \rangle \cdots \langle \mathbf{T}_n, x_n \rangle d\mu(\mathbf{T}_1, \ldots, \mathbf{T}_n)
\end{align*}

where $\mu$ is a Radon measure with $\| \mu \| \leq K_1^n$.

Since we can express $z_1 z_2 \cdots z_n = \sum_{k} \lambda_k \left( \sum_{j=1}^{n} a_j k z_j \right)^n$ if we prove the theorem for $x_1 = x_2 = \cdots = x_n$ then we have
\[
\langle S, x_1 \ldots x_n \rangle = \langle S, \sum_{k=1}^{m} \lambda_k (\sum_{j=1}^{n} \alpha_{j,k} x_j)^n \rangle \\
= \sum_{k=1}^{m} \lambda_k \sum_{S_1} (\sum_{j=1}^{n} \alpha_{j,k} x_j)^n \\
= \sum_{k=1}^{m} \lambda_k \int_{\mathbb{R}^n} (\langle T, \sum_{j=1}^{n} \alpha_{j,k} x_j \rangle)^n \, d\mu(T) \\
= \int_{\mathbb{R}^n} \sum_{k=1}^{m} \lambda_k (\langle T, \sum_{j=1}^{n} \alpha_{j,k} x_j \rangle)^n \, d\mu(T) \\
= \int_{\mathbb{R}^n} \langle T, x_1 \rangle \ldots \langle T, x_n \rangle \, d\mu(T)
\]

Hence without loss of generality we may suppose \( x_1 = x_2 = \ldots = x_n \). But then \( x_1 \otimes \ldots \otimes x_n \) is clearly symmetric, so we use the equality and the inequality in the proof of Lemma (3.5) to write

\[
\langle T_1, x_1 \rangle \ldots \langle T_n, x_n \rangle = \frac{(-1)^n}{n!} \sum_{\Omega} (-1)^{|\Omega|} \langle v_{\Omega, x_1} \rangle \ldots \langle v_{\Omega, x_n} \rangle \\
= \frac{(-1)^n}{n!} \sum_{\Omega} (-1)^{|\Omega|} \langle v_{\Omega, x} \rangle^n
\]

and putting \( \frac{(-1)^n}{n!} \sum_{\Omega} (-1)^{|\Omega|} v_{\Omega, x} = \mu(T) \) we have

\[
\|w\| \leq K^n(2e)^n \leq K^n
\]

and for all \( x_1, \ldots, x_n \in \mathbb{R}, \langle s, x_1 \ldots x_n \rangle = \)

\[
\int_{\mathbb{R}^n} \langle T, x_1 \rangle \ldots \langle T, x_n \rangle \, d\mu(T) \\
Q.E.D.
\]
The consequence of the above equality is the following theorem due to Varopoulos [10].

\textbf{(4.4) THEOREM}

Every commutative injection algebra is a Q-algebra.

Proof:

Let \( P \) be a homogeneous polynomial of \( p \) variables of degree \( n \) and \( x_1, \ldots, x_p \in \mathbb{R}_1 \) then by Thm \((4.2) \) (ii) we have

\[
| <s_t P(x_1, \ldots, x_p)> | \leq \int_{\mathbb{R}_1} P(<T,x_1>, \ldots, <T,x_n>) \, d\mu(T)
\]

\[
\leq \|\mu\| \| P\|_\infty \leq x^n \| P\|_\infty
\]

Thus by the Lemma \((3.8) \) \( R \) is a Q-algebra.

Q.E.D.

Varopoulos [11] defines the following algebras and proves in fact that they are equivalent statements of a commutative injective algebra.

\textbf{(4.5) Definitions.} Let \( R \) be a commutative Banach algebra

(i) \( R \) is a DQ-algebra (a direct Q-algebra) if there exists a uniform algebra \( A \), a Banach algebra homomorphism \( h:A \rightarrow R \) and a linear map \( \ell:R \rightarrow A \) such that \( h\ell = \text{Id}_R \) where \( \text{Id}_R \) the identity map on \( R \).
(ii) $R$ is a WDO-algebra if there exists a uniform algebra $A$, an algebraic homomorphism $h: A \to R$ and a filter of linear maps $\ell_\gamma : R \to A$ such that $\lambda = \sup_{\gamma \in \Gamma} \| \ell_\gamma \| < \infty$.

$h \circ \ell_\gamma (x) \in x \forall x \in R$.

(4.6) **Theorem**

Suppose $R$ is a commutative Banach algebra. Then the following are equivalent:

(I): $R$ is an injective algebra

(DQ): $R$ is a DQ-algebra

(WDQ): $R$ is a WDQ-algebra

Proof: %

(DQ) $\Rightarrow$ (WDQ) is obvious. So we need only to prove

(WDQ) $\Rightarrow$ (I) and (I) $\Rightarrow$ (DQ).

(WDQ) $\Rightarrow$ (I):

Let $A, R, h$ and $\ell_\gamma$ as in the definition of a WDO-algebra. Also let $m_1 : R \times R \to R$ $m_2 : A \times A \to A$ be multiplication maps as before. We note here that since $A$ is a uniform algebra, $\| m_2 \| = 1$ because by definition

$\| x \otimes y \| = \sup_{u \in A, v \in A} \langle u, x \rangle \langle v, y \rangle \leq \| x \| \| y \| = \| xy \|$
The last equality holding since the norm is uniform and also by the Hahn-Banach Theorem for each \((x,y) \in \mathbb{R} \times \mathbb{R}\) there is \(T \in A_1^1\) such that \(|\langle T, x \rangle| = \|x\|\) and \(|\langle T, y \rangle| = \|y\|\) so \(\|x \otimes y\| \geq \|x\| \|y\| = \|xy\|\). We mention here that we prove the converse to the above later in the chapter.

For each element \(\tau = \sum_{j=1}^{J} x_j \otimes y_j \in \mathbb{R} \otimes \mathbb{R}\) we have

\[
(h \circ \gamma \circ \delta)(\tau) = \sum_{j=1}^{J} (h \circ \gamma(x_j))(h \circ \gamma(y_j)) + \sum_{\gamma \in \Gamma} x_j y_j = m_1(\tau)
\]

Hence \(\|m_1(\tau)\| \leq \lambda^2 \|h\| \|\tau\|\).

Therefore \(\mathbb{R}\) is an injective algebra.

\((I) \Rightarrow (DQ)\)

The proof is an extension of the idea that appeared in the proof of Thm (3.1).

Let \(K\) be a constant associated with the injectivity.

Let \(\wedge = \{x \in \mathbb{R} : \|x\| \leq (2K)^{-1}\}\), \(C^\wedge = \prod_{x \in \wedge} C_x\), the Cartesian product of \(\wedge\) copies of \(C\) and

\[
E = \{z = (z_x)_{x \in \wedge} \in C^\wedge; \ T \in \mathbb{R}^\wedge \text{'s.t. } z_x = \langle T, x \rangle, \ |z_x| \leq l \forall x \in \wedge \}
\]

a subset of a unit ball of \(C^\wedge\). We have,

\((1)\) if \(z = (z_x)_{x \in \wedge} \in E\) and \(|t| \leq 1\) then \(tz = (tz_x)_{x \in \wedge} \in E\).
Let $J$ be the set of all polynomials in the coordinate functions \( f_x(z) = z_x : x \in \Lambda \) on \( \mathbb{C}^\Lambda \). Let $J_E$ be the restriction of $J$ on $E$. We identify $J_E$ with an algebra of functions on $E$.

We put \( \|P\|_E = \sup_{z \in E} |P(z)| \) for $P \in J$, then \( \|P\|_E \) induces a norm on $J_E$. We let $A$ be the completion of $J_E$ with respect to this norm. Then $A$ is a uniform algebra on $E$.

For $P \in J$ we write $P = P_1 + P_2 + \cdots + P_S$ where $P_n$ is homogeneous of degree $n$ and $s = \deg P$. As in the proof of Thm.(3.2) by applying Cauchy's inequality on $P(tz)$ we have

\[
(2) \quad \|P_n\|_E \leq \|P\|_E \quad \forall P \in J \quad n = 1, \ldots, S.
\]

Define an algebraic homomorphism $H: J \rightarrow \mathbb{R}$ by

\[
H(f_x) = x \quad \forall x \in \Lambda
\]

and extending it linearly and multiplicatively. Clearly $L$ is surjective. We also have

\[
(3) \quad \|P\|_R \leq \|P\|_E \quad \forall P \in J.
\]

To show (3) we write $P = P_1 + \cdots + P_S$ and let us denote by $P_n(\langle T, x \rangle)$ the polynomial $P_n$ evaluated at a point $\langle T, x \rangle \in \mathbb{C}^\Lambda$. But by (4.5) of Thm. (4.4) we have

\[
|\langle T, HP_n \rangle| = \left| \int_{\mathbb{R}^1} P_n(\langle x, s \rangle) d\mu_n(s) \right|
\]
\[
= (2K)^{-n} \int_{R^n} P_n(<x,s/2K>) \, d\mu_n(T(s)) \leq (2K)^{-n} \|u\| \, \|P_n\|_E \leq 2^{-n} \, \|P_n\|_E.
\]

Hence with (1) and (2) we have \( \|HP\|_R \leq \|P\|_E \) \( \forall P \in J \).

So far we showed that there is a norm decreasing homomorphism \( H \) from a uniform algebra \( A \) onto \( R \). To complete the proof we need show that there is a linear map \( L: R \to A \) such that \( HoL = \text{Id}_R \).

Define \( L: A \to J_E \) by \( L(x) = f_x|E \) \( \forall x \in A \) and \( L: R \to A \) by \( L(0) = 0 \) and \( L(x) = 2K \|x\| \ell (\frac{x}{2K \|x\|}) \) if \( x \neq 0 \).

Then \( L \) is a linear map since \( z \in E \) if and only if \( z^\dagger = (<T,y>)_{y \in A} \) for some \( T \in R' \) so \( 2K \|x\| \int f_x \frac{x}{2K \|x\|} (z) = <T,x> = f_x(z) \)

and hence \( f_x|E + f_x|E = f_{x_1+x_2}|E \).

Also \( HoL(x) = x \) \( \forall x \in R \).

Q.D.E.

We remark here that in view of the above theorem it is now obvious that every uniform algebra is an injective algebra.

Next is a well known theorem.
Theorem

Let $A$ be a commutative Banach algebra. If $m : A \otimes A \to A$ defined by $m(x \otimes y) = xy$ has norm 1 and $A$ has an identity, then $A$ is a uniform algebra.

Proof:

Let $e \in A$ be the identity. By the Krein-Millman Theorem we have $\|x\|_A = \sup \{ |<s,x>| : s \in \text{extr}(A_1^*) \}$ where $\text{extr}(A_1^*)$ is the set of extreme points of a unit ball in $A'$ with weak *-topology.

Now for any $s \in A_1^*$ by the equation (4.3) of Prop. (4.2) there is a Radon measure $\mu \in M(A_1^* \times A_1^*)$ ($\|\mu\| = 1$) such that

\[
<s,xy> = \int_{A_1^* \times A_1^*} <p,x><q,y> \, d\mu(p,q)
\]

\[
= \int_{A_1^* \times A_1^*} \theta(p,q)d\|\mu\|(p,q)
\]

where $\theta(p,q)$ is a function of modulus 1.

We claim that for $s \in \text{extr}(A_1^*)$, $<s,xy> = <p,x><q,y>\theta(p,q)$ for some $p,q \in A_1^*$.

Suppose supp $(\mu) \subseteq A \cup B$ (disjoint union) then $\mu = \mu_A + \mu_B$, $\|\mu\| = \|\mu_A\| + \|\mu_B\|$ where $\mu_A(E) = \mu(A \cap E)$, $\mu_B(E) = \mu(B \cup E)$. 

Now then \(<s,x> = \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu(p,q)\)

\[= \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu_A(p,q)\]

\[= \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu_A(p,q)\]

\[= \frac{1}{\|\mu_A\|} \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu_A(p,q)\]

\[= \frac{1}{\|\mu_A\|} \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu_A(p,q)\]

\[= \frac{1}{\|\mu_A\|} \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu_B(p,q)\]

Let \(s_A, x = \int_{A_1' \times A_1'} <p,x><q,e> \, d\mu_A = \int_{A_1' \times A_1'} <p,x><q,e> \, \Theta(p,q) \, d\mu_A\)

and similarly for \(s_B\). Then \(\frac{s_A}{\|\mu_A\|} = \frac{s_B}{\|\mu_B\|} \in A_1'\) and

\[s = \|\mu_A\|(\frac{s_A}{\|\mu_A\|}) + \|\mu_B\|(\frac{s_B}{\|\mu_B\|})\]

If \(s\) is an extreme point this implies that the support of \(\mu\) is an atom so that the following equations hold.

1. \(<s,x> = <p,x><q,e> \, \Theta(p,q)\>
2. \(<s,y> = <p,e><q,y> \, \Theta(p,q)\>
3. \(<s,e> = <p,e><q,e> \, \Theta(p,q)\>
4. \(<s,xy> = <p,x><q,y> \, \Theta(p,q)\>

So we have
\[ <s,x><s,y> = <\theta,P,x><\theta,Q,y><P,e><Q,e> \quad \{\theta(P,Q)\}^2 \]
\[ = <s,xy><s,e>. \]

But \(<s,e> \neq 0\) since \(<s,e> = 0\) implies \(s = 0\).

Thus define \(\phi(x) = \frac{s,x}{<s,e>}\) then \(\phi \in A'\) and \(\phi(xy) = \phi(x)\phi(y)\),
\(\phi(e) = 1\) that is to say \(\phi\) is in \(m_A\), the spectrum of \(A\).

( note. \(\phi(xy) = \phi(x) \phi(y)\) for
\[ \frac{s,xy}{<s,e>} = \frac{s,x}{<s,e>} \cdot \frac{s,y}{<s,e>}. \]

Since \(|<s,x>| \leq |<s,e>| |\phi(x)|\) we have
\[ \|x\|_A = \sup_{s \in \text{extr}(A_1')} |<s,x>| \]
\[ \leq \sup_{s \in \text{extr}(A_1')} |<s,e>| \sup_{\phi \in m_A} |\phi(x)|. \]

Therefore \(\|x\|_A \leq \|x\|_A\).

This proves that \(A\) is isometrically isomorphic to \(\hat{A}\), the Gelfand transform of \(A\). Hence \(A\) is a uniform algebra.

We remark here that there is an extension of this theorem.

A net \(\{e_\alpha\}\) of elements of \(A\) is said to be an approximate identity if for each \(x \in A\) \(e_\alpha x + x\) in \(A\). Suppose \(A\) has an approximate
identity \{ e_\alpha \} in stead of an identity, and further suppose that \( \sup_\alpha \| e_\alpha \|_A \leq 1 \). Then the theorem still holds.

The proof involves constructions of certain subalgebras of \( A \). We omit the proof here.
CHAPTER 5

In this chapter we examine p-summing algebras. Charpentier [2] proved that every commutative 1-summing algebra is a Q-algebra. On the other hand every commutative 2-summing algebra is an operator algebra (Tonge [9]). So the underlying question throughout this chapter is: Is every commutative 2-summing algebra a Q-algebra? We show in this chapter that commutative 2-summing algebras share many properties with Q-algebras but the author was unable to determine whether the statement is true or not.

We examined p-summing algebras only from the Q-algebraic point of view. More general studies are found in the paper by Lindenstrass and Pelczynski [6].

(5.1) Definitions A Banach algebra $B$ is said to be p-summing ($1 \leq p \leq \infty$) if the map $\ell^q(\mathcal{B}) \mathcal{B} + \ell^q(\mathcal{B})'$ defined by $b \otimes \phi \rightarrow b \cdot \phi$ is bounded where $b \in B$, $\phi \in \mathcal{B}'$, $b \cdot \phi \in \mathcal{B}'$ with $\langle b \cdot \phi, a \rangle = \langle \phi, ab \rangle$ $\forall a \in B$ and $\phi$ is the conjugate exponent of $P$.

The dual formulation of the above definition is:
if $\phi \in \mathcal{B}'$, let $\widetilde{\phi}: \mathcal{B} + \mathcal{B}'$ be defined by $\langle \widetilde{\phi}(b), a \rangle = \langle \phi, ab \rangle$ $\forall a, b \in B$

Then $B$ is p-summing if and only if there exists some constant $K$ such that

$$\| \phi \| \mathcal{L}(\ell^p(\mathcal{B}), \ell^p(\mathcal{B}')) \leq K \| \phi \|_{\mathcal{B}'}$$
An immediate consequence of the above definition is the following theorem due to Charpentier [2].

(5.2) **THEOREM**

Every commutative 1-summing algebra is a Q-algebra.

**Proof.**

Let $\mathcal{C}$ be the 1-summing norm of the multiplication on $B$. We verify the condition (iii) of Thm. 3.9 i.e. we want to show that there is a constant $K$ such that for all $x_1, \ldots, x_p \in B$, $\phi \in B^*$, and $a \in C^n_p$ defined by $a(\phi) = \phi(x_{\beta_1} \cdots x_{\beta_n})$ then $\|a\| \lesssim K^n$

when $n = 1$ it is trivial that $\|a\| \lesssim 1$.

when $n = 2$ we claim that $\|a\| \lesssim C^{n-1}$

because $\|x_{\beta_2} \cdots x_{\beta_n}\| \lesssim \|x_{\beta_2} \cdots x_{\beta_n}\|$.

and

$\langle (x_{\beta_2} \cdots x_{\beta_n}) \phi, x_{\beta_1} \rangle = \langle \phi, x_{\beta_1} x_{\beta_2} \cdots x_{\beta_n} \rangle = a(\phi)$.

Taking $k$ to be a number satisfying $\max(1, C^{n-1}) \leq k^n$ for $n = 0, 1, 2, \ldots$ we have the desired result.

We show now that polynomials acting on commutative 2-summing algebras satisfy some similar boundedness conditions as polynomials acting on Q-algebras.
First we show that a very similar statement to that of Bernard's theorem (Thm. 2.1) holds for commutative 2-summing algebras. The next theorem is due to the author.

(5.3) THEOREM

Suppose B is a commutative 2-summing algebra with a constant M and \( g_1, g_2, \ldots, g_p \in B \) such that \( \| g_1 \cdots g_p \|_B \leq C \) for some constant C for all \( k_i \in \mathbb{Z}, i = 1, 2, \ldots, p \). Then for any complex polynomial \( P \) of \( p \)-variables we have

\[
\| P(g_1, \ldots, g_p) \|_B \leq C^2 M \| P \|_{\infty} \text{ where } \| P \|_{\infty} = \sup \{ |P(z_1, \ldots, z_p)| : |z_i| \leq 1 \}
\]

Proof.

First we prove the statement for \( p = 1 \). As in the proof of Bernard's theorem we let \( \frac{1}{2N+1} \sum_{|n| \leq N} \hat{g}_n \) for each positive integer \( N \). Now if we assume \( \| F_N^2 \|_B \otimes L^1 \leq C^2 M \) then as in the proof of Bernard's theorem by writing

\[
P(t) = \sum_{|m| \leq K} P_m e^{int} \quad \text{(here } P_m \in B) \quad \text{we have for all } N > K
\]

\[
\int_\Delta P(t) F_N^2(t) \, dn(t) = \sum_{|n| \leq K} \frac{2N+1-|n|}{2N+1} P_m g^n
\]

where \( \Delta \) the unit circle in the complex plane and \( n \) the normalized Haar measure on \( \Delta \) so as \( N \to \infty \).

\[
\int_\Delta p(t) F_N^2(t) \, dn(t) + P(g)
\]
And for all \( N \).

\[
\| \int P(t) F_N^2(t) d\eta(t) \|_B < C^2 M \| P \|_\infty
\]

Therefore it remains to show that \( \| F_N^2 \|_B \otimes L^1 \leq C^2 M \forall N \).

\[
F_N^2 = \sum_{|n| \leq 2N} \frac{2N+1-|n|}{2N+1} g^n_{\text{int}}
\]

and

\[
\| F_N^2 \|_B \otimes L^1 = \sup_{u \in A_1^i} \left| \sum_{|n| \leq 2N} \frac{2N+1-|n|}{2N+1} u^n_{\text{int}} \right| \leq C^2 M
\]

We shall show that for each \( n \in A_1 \)

\[
\sum_{|n| \leq 2N} \frac{2N+1-|n|}{2N+1} |u^n_{\text{int}}| \leq C^2 M
\]

Fix \( N \) and \( u \in B_1^i \). Let \( G \) be a sequence in \( B \) whose \( n \)th slot \( (G)_n \) is

\[
(G)_n = \begin{cases} 
  g^n & \text{if } |n| \leq n \\
  0 & \text{if } |n| > N
\end{cases}
\]

and let \( \tilde{u} : \ell^2 \otimes B \rightarrow \ell^2 (B') \) be as in Definition (5.1).

So \( (\tilde{u}(G))_n = g^n \cdot u \) if \( |n| \leq N \) where \( <g^n u, b> = <u, b g^n> \forall b \in B \).

We take the convolution \( \ast \) of \( \tilde{u}(G) \) and \( G \) in \( \ell^2 \)

\[
\tilde{u}(G) \ast G = \left( \sum_{|n| \leq N} \left< g^n \cdot u, (G)_{m-n} \right> \right)_m
\]

hence
(\tilde{u}(G)^*G)_m = \begin{cases} 
(2N+1-|m|)<u,g^m>, & \text{if } |m| \leq 2N \\
0, & \text{if } |m| > 2N 
\end{cases}

But \( \|G\|_2 \otimes B \leq C (2N+1)^{1/2} \) and since \( B \) is 2-summing with a constant \( M \), \( \|u\| \leq M \|u\| \leq M \) so \( \|\tilde{u}(G)\|_2 \leq MC(2N+1)^{1/2} \).

Hence \( \|\tilde{u}(G)^*G\|_1 = \sum_{|n| \leq 2N} (2N+1-|n|) \|u,g^n\| \leq C^2 M (2N+1) \)

which is what we wanted. Now it follows that \( \|F_N^* \otimes \|_{L^1} \leq C^2 M \) \( \forall N \).

For \( p > 1 \) we do essentially the same. Let \( P \) and \( g_1, \ldots, g_p \in B \) as in the statement of the theorem. Let \( \alpha \in \mathbb{Z}^P \), \( \alpha = (a_1, \ldots, a_p) \) and \( \|\alpha\| = \max |a_i| \). Recall that there exists a unique set \( S \subseteq \mathbb{Z}^P \) such that \( P(z_1, \ldots, z_p) = \sum_{\alpha \in S} P_{\alpha} z_1^{a_1} \cdots z_p^{a_p} \) where \( P_{\alpha} \neq 0 \) \( \forall \alpha \in S \).

Define \( F_N(x, t_1, t_2, \ldots, t_p) = (2N+1)^{-p/2} \)

\[ \sum_{\alpha \in \mathbb{Z}^P, \|\alpha\| \leq N} \epsilon^{a_1 \cdot \cdot \cdot a_p} \epsilon^{i a_1 t_1} \cdots \epsilon^{i a_p t_p} \]

If we assume \( \|F_N^* \otimes \|_{L^1(\Delta^P)} \leq C^2 M \) where \( \Delta^P = \Delta \times \cdots \times \Delta \)

with the normalized Haar measure \( \eta \) then for a large enough \( N \) we have
\[
\int_{\Delta^p} \left( \sum_{a \in S} \exp \left( \sum_{i=1}^{2N+1} |a_i| \right) \prod_{i=1}^{2N+1} g_1^{a_1} g_2^{a_2} \ldots g_p^{a_p} \right) \rho(x, t_1, t_2, \ldots, t_p)^2 d\eta(t_1, t_2, \ldots, t_p)
\]

Since \( S \) is finite, the above function converges to \( P(g_1, g_2, \ldots, g_p) \) as \( N \to \infty \). It is also bounded by \( C^2M \| P \|_\infty \), hence the desired result follows.

Therefore it remains to show that for all \( N > 0 \) we have

\[
\| F^2_{N} \|_{B \otimes L^1(\Delta^p)} \leq C^2M
\]

\[
F^2_{N} = \sum_{\beta \in \mathbb{Z}^p_{\| \beta \| \leq 2N}} \prod_{i=1}^{2N+1} \frac{2N+1 - |\beta_i|}{2N+1} g_1^{\frac{\beta_1}{\| \beta \|}} g_2^{\frac{\beta_2}{\| \beta \|}} \ldots g_p^{\frac{\beta_p}{\| \beta \|}} e^{i\beta t^p}
\]

\[
\| F^2_{N} \|_{B \otimes L^1(\Delta^p)} = \sup_{\| \beta \| < 2N} \left| \sum_{\beta \in \mathbb{Z}^p_{\| \beta \| \leq 2N}} \left( \prod_{i=1}^{2N+1} \frac{2N+1 - |\beta_i|}{2N+1} \right) e^{i\beta t^p} \right|
\]

\[
\| F^2_{N} \|_{B \otimes L^1(\Delta^p)} = \sup_{\| \beta \| < 2N} \left| \sum_{\beta \in \mathbb{Z}^p_{\| \beta \| \leq 2N}} \left( \prod_{i=1}^{2N+1} \frac{2N+1 - |\beta_i|}{2N+1} \right) e^{i\beta t^p} \right|
\]

We shall show that for each \( u \in B_1^1 \)

\[
\sum_{\| \beta \| < 2N} \left| \sum_{\beta \in \mathbb{Z}^p_{\| \beta \| \leq 2N}} \left( \prod_{i=1}^{2N+1} \frac{2N+1 - |\beta_i|}{2N+1} \right) e^{i\beta t^p} \right| \leq C^2M
\]

\[
\int_{\Delta^p} \left( \sum_{a \in S} \exp \left( \sum_{i=1}^{2N+1} |a_i| \right) \prod_{i=1}^{2N+1} g_1^{a_1} g_2^{a_2} \ldots g_p^{a_p} \right) \rho(x, t_1, t_2, \ldots, t_p)^2 d\eta(t_1, t_2, \ldots, t_p)
\]
For this we put a group structure on $Z^p = Z \times Z \times \ldots \times Z$.

The operation we impose is the coordinate-wise addition. i.e. $\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_p + b_p)$. Furthermore we put a discrete topology on $Z^p$. Now let $G$ be a $l^2(Z^p;B)$-function which takes a value $(G)\in B$ at $\alpha \in Z^p$ as follows.

$$(G)_\alpha = \begin{cases} g_1 g_2 \ldots g_p & \text{if } \|\alpha\| = \max |a_i| \leq N \\ 0 & \text{if } \|\alpha\| > N \end{cases}$$

But then

$$\|G\|_{l^2(Z^p) \otimes B} = \sup_{u \in B} \|\langle u, g_1 g_2 \ldots g_p \rangle\|_{l^2(Z^p)} \leq C(2N+1)^{P/2}$$

Let $\tilde{u}: l^2(B) \rightarrow l^2(B')$ be defined as in Def. (5.1)

Then $(\tilde{u}(G))_\alpha = \begin{cases} < g_1 g_2 \ldots g_p, u > & \text{if } \|\alpha\| \leq N \\ 0 & \text{if } \|\alpha\| > N \end{cases}$

where $< g_1 g_2 \ldots g_p, u, b > = < u, bg_1 g_2 \ldots g_p > \forall b \in B$.

We take the convolution product $\ast$ of $\tilde{u}(G)$ and $G$ in $l^2(Z^p)$

$\tilde{u}(G) \ast G = (\sum_{\|\alpha\| \leq N} < g_1 \ldots g_p, (u, (G)_{\alpha - \beta} >)_{\beta}$

hence

$$(\tilde{u}(G) \ast G)_\beta = \begin{cases} \sum_{i=1}^{p} 2N+1 - |\beta_i| < u, g_1 g_2 \ldots g_p > & \text{if } \|\beta\| \leq 2N \\ 0 & \text{if } \|\beta\| > 2N \end{cases}$$
Since B is 2-summing with a constant M we have
\[ \|\tilde{u}\| \leq M \text{, } \|\tilde{u}(G)\|_{L^2(\mathbb{Z};B')} \leq MC(2N+1)^{p/2} \]
But also \[ \|G\|_{L^2(\mathbb{Z};B)} \leq C(2N+1)^{p/2} \]
Hence \[ \|\tilde{u}(G)\|_{L^2(\mathbb{Z};B')} \leq MC^2(2N+1)^p \]

Thus we have
\[ \|\delta\| < 2N \]
from which it follows trivially that
\[ \|F_N\|_{\mathbb{B} \otimes L^1(\Delta^p)} \leq C^2M \]

This completes our proof.

We summarize what we have done in the above proof.
For \( p = 1 \) we found a function \( F_N \) such that \[ \|F_N\|_{L^1(\Delta^p) \otimes \mathbb{B}} \leq C^2M \]
for all \( N \) and the \( n \)th Fourier coefficients \( \hat{F}_N(m) \) of \( F_N \) converge to \( g^m \) for each \( m \) such that \( \hat{p}(m) \neq 0 \). For \( p > 1 \) we replaced \( \mathbb{Z} \) by \( \mathbb{Z}^p \) and used the Fourier coefficients on the discrete abelian group \( \mathbb{Z}^p \). To prove \[ \|F_N\|_{L^1(\Delta^p) \otimes \mathbb{B}} \leq C^2M \]
we found \( G \in L^2(\mathbb{Z}^p;B) \) such that
\[ \|G\|^2_{L^2(Z^p,\mathcal{B})} \leq \|G\|_{L^2(Z^p) \otimes \mathcal{B}} \leq C(2N+1)^{p/2} \]

and

\[ \mathbf{i} \langle \tilde{u}(G) \# G \rangle_{\mathcal{B}} \langle \mathbf{v}, e^{i \mathbf{1} t_1} \ldots e^{i \mathbf{p} t_p} \rangle = \langle u \odot v, P^2 \rangle_{\mathcal{B}}, \]

\[ \|\mathbf{g}\| < 2N \]

Then we used the 2-summingness of \( \mathcal{B} \) to obtain the \( L^2 \) bound for \( \tilde{u}(G) \).

We note that in the proof, coefficients of the polynomials need not be complex numbers but can also be in the algebra \( \mathcal{B} \). Hence we have

(5.4) Corollary

Suppose \( \mathcal{B} \) is a commutative 2-summing algebra with a constant \( M \) and \( g_1, \ldots, g_p \in \mathcal{B} \) such that \( \|g_1 g_2 \ldots g_p\|_\mathcal{B} \leq C \)

for some constant \( C \) for all \( k_i \in \mathbb{Z} \ i = 1, 2, \ldots, p \), then for any \( \mathcal{B} \)-valued polynomial \( P \) of \( p \)-variables (i.e., polynomial whose coefficients are in \( \mathcal{B} \)) we have

\[ \|P(g_1, \ldots, g_p)\|_{\mathcal{B}} \leq C^2 M \|P\|_\infty \]

where \( \|P\|_\infty = \sup_{\|z\|_1 \leq 1} \|P(z_1, \ldots, z_p)\|_{\mathcal{B}} \)

and the supremum is taken over all complex numbers \( z_i \) with \( |z_i| \leq 1 \).

The proof of this corollary is exactly the same as that of the theorem. The thing to realize is that \( \int P(t) f_N^2(t) \, dn(t) \)
converges to \( P(g_1, \ldots, g_p) \) in \( B \) and is bounded by \( C^2 M \| P \| \infty \)
when \( P \) is either a complex or a \( B \)-valued polynomial.

Now can we use the techniques used above to extend
Bernard's theorem for complex polynomials acting on \( Q \)-algebras
to the theorem for \( B \)-valued polynomials (polynomials
whose coefficients are in the \( Q \)-algebra) with the same norm
for polynomials as in the preceding corollary?

The answer is no. Charpentier [2] proves the following
theorem.

(5.5) \textbf{THEOREM}

(i) Let \( B \) be a commutative Banach algebra and suppose
there exists \( K > 0 \) such that whenever \( b_1, \ldots, b_n \in B, \)
\( x_1, \ldots, x_n \in B \) then,

\[
\| \sum_{i=1}^{n} b_i x_i \|_B \leq K \sup \{ \| \sum_{i=1}^{n} z_i b_i \|_B : z_i \in C, |z_i| \leq 1 \}.
\]

Then \( B \) is a \( Q \)-algebra.

(ii) The condition is not necessary for \( B \) to be a \( Q \)-algebra.

Now we come back to 2-summing algebras. We remarked
earlier that every 2-summing algebra is an operator algebra
hence Prop.(2.3) holds in 2-summing algebra. That is:
(5.6) If $B$ is a commutative 2-summing algebra and $b \in B$ such that $\|b^n\|_B \leq Cn^\alpha$ for all $n > 1$ for some constants $C$ and $\alpha > 0$. Then for any polynomial of the form

$$P(z) = \sum_{n=1}^{N} a_n z^n$$

and $\epsilon > 0$ we have

$$\|P(b)\|_B \leq K \sup_{\|z\|_1 < 1} |\sum_{n=1}^{N} a_n z^{2\alpha + \epsilon} n^n|$$

where $K$ is a constant that depends only on $C, \alpha$ and $\epsilon$.

We remark here that we can prove this result directly, using the techniques used in the proof of Thm. (5.3).

We now prove the bounds for Banach valued polynomials on commutative $p$-summing algebras along the line of Davie’s criterions for $Q$-algebras. (Thm. 3.1 and Thm. 3.2)

The next two theorems are due to the author. The proof of the next theorem was suggested by S.W. Drury.

(5.7) **Theorem**

Suppose $B$ is a commutative $p$-summing algebra. Then there exists a constant $C$ such that for any $B$-valued homogeneous polynomial $P$ of $m$ variables and $b_1, b_2, \ldots, b_m \in B$ we have

$$\|P(b_1, b_2, \ldots, b_m)\|_B \leq C_{\deg P} \|P\|_\infty \|b\|^n$$

where

$$\|P\|_\infty = \sup \{\|P(z_1, \ldots, z_m)\|_B : \sum_{i=1}^{m} |z_i|_1^q \leq 1\}$$

and

$$\|b\| = \left(\sum_{i=1}^{m} \|b_i\|_B^q \right)^{1/q},$$

where $q$ is the conjugate exponent of $p$. 
Proof:

We write \( P(z_1, \ldots, z_m) = \sum a_\beta z_1^{\beta_1} \cdots z_m^{\beta_m} \) where \( n = \deg P \) and \( a_\beta \) is symmetric then by Lemma (3.6)

\[
\|a_\beta\| P = \sum_{\beta} \|P\| \leq (2e)^n \|P\|_{\infty}
\]

Now consider \( b_\beta = b_{\beta_1} b_{\beta_2} \cdots b_{\beta_n} \) where \( 1 \leq \beta_i \leq m \) we can extend \( \beta \) to \( \mathbb{Z}^n \) by putting \( b_i = 0 \) if \( i \leq 0 \) or \( i > m \). Then clearly we have

\[
\|b_\beta\| \leq \|q(B)\| \cdots \|q(B)\| < \|b\|_n
\]

Let \( \phi \in B_1 \) and define inductively as in Definition (5.1) the map \( \|q(B)\| \cdots \|q(B)\| \). It is obvious that the map is bounded by \( M^n \). It takes

\[
b_{\beta_1} b_{\beta_2} \cdots b_{\beta_n} \rightarrow (b_{\beta_1} b_{\beta_2} \cdots b_{\beta_n} \phi)
\]

and

\[
\|b_{\beta_1} b_{\beta_2} \cdots b_{\beta_n} \phi\| \leq M^n \|b\|_n
\]

Now we have

\[
(b_{\beta_1} \cdots b_{\beta_n} \phi)(a_\beta) = \sum_{\beta} b_{\beta_1} b_{\beta_2} \cdots b_{\beta_n} \phi(a_\beta)
\]

\[
= \sum_{\beta} \langle \phi, a_\beta b_{\beta_1} b_{\beta_2} \cdots b_{\beta_n} \rangle
\]

\[
= \langle \phi, P(b_1, b_2, \ldots, b_m) \rangle
\]

and

\[
|\langle b_{\beta_1} \cdots b_{\beta_n} \phi(a_\beta) \rangle| = |\langle \phi, P(b_1, \ldots, b_m) \rangle| \leq (2e)^n M^n \|b\|_n \|P\|_{\infty}
\]

for all \( \phi \in B_1 \).
Hence $\|B(b_1, b_2, \ldots, b_m)\|_{B} \leq (2e)^n M^n \|b\|^n \|P\|_{\infty}$.

Putting $C = 2eM$ we have the desired result.

Q.E.D.

The next theorem is an immediate consequence of the above. It is due to the author.

(5.8) **THEOREM**

Suppose $B$ is a commutative $p$-summing algebra. Then there is a constant $\delta > 0$ such that for any $B$-valued polynomial $P$ of $m$ variables without constant term and $\|P\|_{\infty} \leq 1$, and $b_1, b_2, \ldots, b_m \in B$ with $\|b\| = \left\{ \sum_{i=1}^{m} \|b_i\|^q \right\}^{1/q} \leq \delta$

we have $\|P(b_1, b_2, \ldots, b_m)\|_{B} \leq 1$ where $\|P\|_{\infty}, \|b\|$ defined as in Thm.(5.7).

The proof of the above theorem is exactly like that of Thm.(3.2) hence is omitted.

We remark here that neither above two theorems nor Thm.(5.3) is enough to prove that every commutative 2-summing algebra is a $Q$-algebra. On the other hand, the author did not find a counter example to such a statement. Thus the question still remains open.
CHAPTER 6

In this chapter we discuss theorems relating Banach algebras to Q-algebras or injective algebras and from those theorems as well as criterions of Chapter 3 we draw examples of Q-algebras, non Q-algebras, and of injective algebras.

For the first theorem we require some knowledge of intermediate algebras. We briefly state the definitions and some elementary theorems due to A.P. Calderon [1].

(6.1) Definitions. An interpolation pair of Banach algebras is a pair of complex Banach algebras $B^0$ and $B^1$ continuously embedded in a complex topological algebra $V$ in such a way that the subspace $B^0 + B^1 \subset V$ of $V$ is a subalgebra of $V$. This subspace with the norm

$$\|x\|_{B^0+B^1} = \inf\{\|y\|_{B^0} + \|z\|_{B^1} : x = y + z\}$$

where the infimum is taken over all pairs $y \in B^0, z \in B^1$ such that $x = y + z$. Then $B^0 + B^1$ becomes a Banach algebra.

Next we consider $F(B^0, B^1)$ the space of $B^0 + B^1$ valued function $f(z)$ defined in the strip

$\{z = s + it : 0 \leq s \leq 1\}$, continuous and bounded with respect to the norm of $B^0 + B^1$, analytic in $\{z = s + it : 0 < s < 1\}$ and such that $f(it) \in B^0$ is $B^0$-continuous and tends to zero as $|t| \to \infty$, and that $f(1 + it) \in B^1$ is $B^1$-continuous and tends to zero as $|t| \to \infty$. With the norm
\[ \|f\|_F = \max \{ \sup_{t} \|f(it)\|_0, \sup_{t} \|f(1+it)\|_1 \} \]

F becomes a Banach algebra.

For every real number \( s, 0 \leq s \leq 1 \) we consider the subspace \( B_s = [B^0, B^1]_s \) of \( B^0 + B^1 \) defined by
\[ B_s = \{ x : x = f(s), f \in F(B^0, B^1) \} \]
and introduce the norm
\[ \|x\|_s = \|x\|_{B_s} = \inf \{ \|f\|_F : f(s) = x \} . \]

Then \( B_s \) is a Banach algebra continuously embedded in \( B^0 + B^1 \).

Furthermore if \( N_s \) denotes the subalgebra of \( F(B^0, B^1) \) defined by
\[ N_s = \{ f \in F : f(s) = 0 \} . \]

Then \( N_s \) is a closed ideal in \( F \) and \( F(B^0, B^1)/N_s \) is isometrically isomorphic to \( B_s \). We call \( B_s \), the intermediate algebra between \( B^0 \) and \( B^1 \).

We now state our first theorem due to Varopoulos [10].

(6.2) THEOREM

Let \( R^0 \) and \( R^1 \) be two Q-algebras that form an interpolation pair. Then for all \( 0 \leq s \leq 1 \), the intermediate algebras \( R_s \) is also a Q-algebra.

Proof:

Since \( R_s = F(R^0, R^1)/N_s \), all we need is to show that \( F \)

is a Q-algebra.

We show that the criterion for Q-algebra in Thm. (3.2) is satisfied, that is there exists a constant \( C > 0 \) such that if
If \( n = 1, \ldots, p \) and \( P \) a homogeneous polynomial of \( p \)-variables we have \( \| P(f_1, \ldots, f_p) \| \leq C^{\deg P} \| P \|_{\infty} \), where

\[
\| P(f_1, \ldots, f_p) \|_F = \sup_{\tau} \{ \| P(f_1(\tau), \ldots, f_p(\tau)) \|_B^0 \}
\]

and

\[
\| f_n \|_F = \sup \{ \| f_n(\tau) \|_B^0, \| f_n(l+it) \|_B^1 \}.
\]

But \( R^0 \) and \( R^1 \) are \( Q \)-algebras so there are constants \( C_0 \) and \( C_1 \) such that for all \( \tau \)

\[
\| P(f_1(\tau), \ldots, f_p(\tau)) \|_B^0 \leq C_0^{\deg P} \| P \|_{\infty}
\]

\[
\| P(f_1(l+it), \ldots, f_p(l+it)) \|_B^1 \leq C_1^{\deg P} \| P \|_{\infty}
\]

The result immediately follows.

Q.E.D.

A theorem of intermediate spaces states that for all \( 1 \leq \alpha < \beta < \infty \), \( L^\alpha \) and \( L^\beta \) from an interpolation pair with \( V = L^\infty \) and the intermediate spaces are all spaces \( L^\gamma \), \( \alpha \leq \gamma \leq \beta \).

Now we claim that \( L^1 \) and \( L^\infty \) are \( Q \)-algebras, so that for all \( p, 1 \leq p < \infty \), \( L^p \) is a \( Q \)-algebra.
Clearly $l^\infty$ is a Q-algebra since $l^\infty$ is isometrically isomorphic to $C(X)$ where $X$ is the Stone-Cech compactification of $Z$ with the discrete topology, or we can view $X$ as the maximal ideal space of $l^\infty$.

To prove $l^1$ Q-algebra we again need tensor algebras.

For finite tensor algebras we had Littlewood's inequality which states

(i) if $a \in C_p^2$

\[ \| \bigotimes a \| \leq 3 \frac{1}{2 \sqrt{2}} \sup_{\beta_1, \beta_2} \left( \sum_{\beta_1, \beta_2} |a(\beta_1, \beta_2)|^2 \right)^{1/2} \]

(ii) if $a \in C_p^n$ then

\[ \| \bigotimes a \| \leq 3 \frac{n-1}{2 \sqrt{2^n}} \sup_{\beta_1, \beta_2, \ldots, \beta_n} \left( \sum_{\beta_1, \beta_2, \ldots, \beta_n} |a(\beta_1, \ldots, \beta_n)|^2 \right)^{1/2} \]

In terms of general tensor algebras the above inequalities state in fact that $l_p^\infty(l_2^2) \subseteq l_p \bigotimes \hat{l}_p^\infty$, or we can write $C(F, l_p^2) \subseteq C(F) \bigotimes l_p^\infty$ where $F$ is finite.
If $a \in C(F) \hat{\otimes} l^\infty$ then define $a_N$ to be the restriction of $a$ on $C(F) \hat{\otimes} l_N^\infty$. Clearly $\|a_N\|_{\hat{\otimes}} \to \|a\|_{\hat{\otimes}}$ as $N \to \infty$. Therefore we also have $l_p^\infty(t^2) \leq l_p^\infty \otimes l^\infty$ or equivalently $C(F, l^2) \subseteq C(F) \hat{\otimes} l^\infty$ where $F$ is finite.

With these inequalities DAVIE [3] showed that for all $p, 1 \leq p \leq 2$, $l^p$ is a $Q$-algebra by verifying the condition (iii) of Theorem (3.9). But for us it suffices to show that $C(F, l^1) \subseteq C(F) \hat{\otimes} l^\infty$ is true since we need only to prove that $l^1$ is a $Q$-algebra. The next Proposition was pointed out to me by S.W. Drury.

(6.3) **PROPOSITION**

Suppose $F$ is finite, then $C(F, l^1) \subseteq C(F) \hat{\otimes} l^\infty$.

**Proof:**

It is enough to prove that extreme points of a unit ball in $C(F, l^1)$ have $\|\| \hat{\otimes} \| \leq 1$.

Suppose $f \in C_1(F, l^1)$, that is $\sup \{ \sum_{n=1}^{\infty} |f(n, m)| \} = 1$.

Then for some $n_0$, $\sum_{m=1}^{M} |f(n_0, m)| = 1$. Suppose there is more than one integer $m$ such that $|f(n, m)| > 0$. Let $m_0$ be one of them. Then define $g_0$ and $g_1$ by

$$g_0(n, m) = \begin{cases} 
  \frac{f(n, m)}{|f(n, m)|} & \text{if } n = n_0, \ m = m_0 \\
  0 & \text{if } n = n_0, \ m \neq m_0 \\
  f(n, m) & \text{elsewhere}
\end{cases}$$
\[
g_1(n,m) = \begin{cases} 
0 & \text{if } n = n_0, \ m = m_0 \\
\frac{f(n,m)}{\sum_{k \neq m_0} |f(n,k)|} & \text{if } n = n_1, \ m \neq m_0 \\
f(n,m) & \text{elsewhere} 
\end{cases}
\]

Then it is clear that \( \|g_0\|_{C(F,\ell^\infty)} = \|g_1\|_{C(F,\ell^\infty)} = 1 \)
and \( f = |f(n,m_0)|g_0 + \left( \sum_{k \neq m_0} |f(n,k)| \right)g_1 \). This \( f \) is a convex combination of \( g_0 \) and \( g_1 \) where \( g_0, g_1 \in C_1(F,\ell^\infty) \).

Therefore we conclude that if \( f \notin \text{extr} \{C_1(F,\ell^\infty)\} \)
(Extreme points of the unit ball in \( C(F,\ell^\infty) \)) then \( f \) must be of the following form. For each \( n \) \( (f(n,m))_{m=1}^{\infty} = C(n) e_{f(n)} \)
where \( |C(n)| = 0 \) or 1, and \( e_k \in \ell^\infty \) has 1 in \( k \)-th place and 0 elsewhere. We note that \( e_k \in \text{extr}(\ell^\infty) \) and \( \tilde{f}: F \to \mathbb{Z} \) is well defined (by the preceding arguments.)

Take \( f \in C_1(F,\ell^\infty) \). We shall show that \( \|f\|_{\ell^\infty} < 1 \).

Let \( G = \{e_0, \ldots, e_{M-1}\} = \{e_{f(n)}\} \in F \) and we put on \( G \), \( \mathbb{Z}(M) \) structure. (We can put \( \mathbb{Z}(M) \) structure in anyway we like.) With the discrete topology, \( G \) is a compact abelian group.

Let \( \omega = e^{2\pi i/M} \), a \( M \)-th root of unity. Define characters \( x_r \) by \( x_r(n) = \omega^{rn} \) and

\[ A(G) = \{ \phi = \sum_r \sigma_r x_r : \|\phi\|_A = |\sigma_r| < \infty \} \]
Consider the map \( \mathcal{C}(G) \to \mathcal{C}(G \times G) \) defined by \( \phi \to \phi \)
where \( \phi(x,y) = \phi(x-y) \). We claim that if \( \|\phi\|_A \leq 1 \)
then \( \|\phi\|_\Theta \leq 1 \), because we have

\[
\phi(x,y) = \phi(x-y) = \sum_r a_r x_r(x-y) = \sum_r a_r x_r(x) x_r(y),
\]

so it follows from \( \|\phi\|_A \leq 1 \) that \( \|\phi\|_x \leq 1 \).

Define \( k_{\{0\}}(x) = \frac{1}{M} \sum_{r=0}^{M-1} x_r(x) \in \mathcal{C}(G) \). This function
is, in fact, a characteristic function of \( \{0\} \) because

\[ k_{\{0\}}(0) = 1 \quad \text{and if } x \neq 0, \]

\[ k_{\{0\}}(x) = \frac{1}{M} \sum_{r=1}^{M-1} \omega^r = \frac{1}{M} \sum_{r=1}^{M-1} \omega^r = 0. \]

Let \( K_d \in \mathcal{C}(G) \otimes \mathcal{C}(G) \) be defined by \( K_d(x,y) = k_{\{0\}}(x-y) \)
which is the characteristic function of the diagonal of \( G \times G \) with \( \|K_d\|_\otimes = 1 \).

Now we come to a point. We have

\[
f(n,m) = C(n) e^{-f(n)} = C(n) K_d(f(n),m).
\]

But \( K_d \in \mathcal{C}(G) \otimes \mathcal{C}(G) \) with \( \|K_d\|_\otimes = 1 \), hence

\[
K_d(f(n),m) = \sum_r a_r h_1(f(n)) h_2(m) \quad \text{where}
\]

\[
h_1, h_2 \in \mathcal{C}(G), \quad \|h_1\| = \|h_2\| = 1 \quad \text{and} \quad \sum_r |a_r| \leq 1.
\]

Thus we can write

\[
f(n,m) = \sum_r a_r C(n) h_1(f(n)) h_2(m) \quad \text{where}
\]

\[
C(n) h_1(f(n)) \in \mathcal{C}(F) \text{ with } \|C \cdot (h_1 \circ f)\| \leq 1.
\]
\[
\begin{align*}
\text{Therefore } f \in C(F) \otimes C(G) = C(F) \otimes \ell ^{\infty} \text{ and } \|f\| \leq 1.
\end{align*}
\]

Q.D.E.

We write the equation we obtained in the proof above

\begin{equation}
(6.4) \text{ If } f \in C(F, l^1) \text{ then } f \in C(F) \hat{\otimes} \ell ^{\infty} \text{ and } \|f\| \leq \|f\|_{C(F, l^1)}.
\end{equation}

Later we show how Varopoulos uses essentially the same ideas to prove that \( l^1 \) is an injective algebra. But for now let us proceed to prove that \( l^1 \) is a \( \mathbb{Q} \)-algebra.

\begin{equation}
(6.5) \text{ THEOREM}
\end{equation}

\( l^1 \) is a \( \mathbb{Q} \)-algebra.

Proof:

We verify the condition (iii) of Thm. (3.9). That is, we prove there is a constant \( K > 0 \) such that if \( x_1, \ldots, x_p \in l^1 \) with \( \|x_i\| \leq 1 \), \( \phi \in \ell \) with \( \|\phi\| \leq 1 \) and \( a \in \mathbb{C}^n \), defined by

\[
\alpha(\beta) = \phi(x_{\beta_1} \cdot x_{\beta_2} \cdots x_{\beta_m}) \text{ then } \|a\| \leq K^h.
\]

We write \( x_{\beta_i} = (x(\beta_{i,m}))_{m=1}^{\infty} \) and \( \phi = (\delta_{m})_{m=1}^{\infty} \) and so

\[
(1) \quad \alpha(\beta) = \phi(x_{\beta_1} \cdots x_{\beta_m}) = \sum_{m=1}^{\infty} b_m x(\beta_{1,m}) \cdots x(\beta_{n,m}).
\]
Now if \( x(r, m) \in C(K_p, \mathbb{L}) \) with \( \sup_r \| x_r \| < 1 \)

i.e. \( \sup \left( \sum_{m=1}^{\infty} x(r, m) \right) < 1. \)

Then by (6.4) we have \( x(r, m) \in C(K_p, \mathbb{L}) \) and \( \| x_r \| < 1. \)

That is

\[
(2) \quad x(r, m) = \sum_{t_r=1}^{s} \lambda_{t_r} f_{t_r}(r) y_{t_r}(m)
\]

where \( \sum_{t_r=1}^{s} |\lambda_{t_r}| < 1 \) and

\[
f_{t_r} \in C(K_p), \quad \| f_{t_r} \| < 1, \quad y_{t_r} \in \mathbb{L}
\]

and

\[
\| y_{t_r} \| = \sup_{m} \| y_{t_r}(m) \| < 1.
\]

Substituting (2) into (1) for each \( x_{\beta_2}, \ldots, x_{\beta_n} \) but not \( x_{\beta_1} \),

we get

\[
\sum_{t_2=1}^{s} \cdots \sum_{t_n=1}^{s} \lambda_{t_2} \cdots \lambda_{t_n} \left( \sum_{m=1}^{\infty} b_m x(\beta_1, m) y_{t_2}(m) \cdots y_{t_n}(m) \right)
\]

\[
f_{t_2}(m) \cdots f_{t_n}(m)
\]

Let

\[
g_{t_2} \cdots t_n(\beta_1) = \sum_{m=1}^{\infty} b_m x(\beta_1, m) y_{t_2}(m) \cdots y_{t_n}(m)
\]

Then

\[
a(\beta) = \sum_{1 \leq t_2, \ldots, t_n \leq s} \lambda_{t_2} \cdots \lambda_{t_n} g_{t_2} \cdots t_n(\beta_1) f_{t_2}(\beta_2) \cdots f_{t_n}(\beta_n)
\]
where \( f_{t_i} \) is constant under \( t_2, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n \).

For each \( (t_2, \ldots, t_n) \)
\[
\sum_{m=1}^{\infty} |b_m x(\beta_1, m) y_{t_2}(m) \cdots y_{t_n}(m)|
\]
\[
\leq \left( \sum_{m=1}^{\infty} |x(\beta_1, m)| \right) \|b_m\|_{\infty} \|y_{t_2}\|_{\infty} \cdots \|y_{t_n}\|_{\infty}
\]
\[
\leq 1
\]

That is, \( \|g_{t_2 \ldots t_n}\| \leq 1 \) \( \forall (t_2, \ldots, t_n), \ 1 \leq t_i \leq s \).

And of course \( \|f_{t_i}\| \leq 1 \) \( \forall t_i \).

Finally
\[
\sum_{t_2=1}^{s} \cdots \sum_{t_n=1}^{s} |\lambda_{t_2} \cdots \lambda_{t_n}| \leq 1
\]

Therefore we have \( \|a_{\bar{\Theta}}\| < 1 \)

Q.E.D.

Varopoulos \[10\] did not prove directly that \( \mathcal{L} \) is a Q-algebra, instead he proved that \( \mathcal{L} \) is an injective algebra. But then by Thm.(4.4) since \( \mathcal{L} \) is commutative, \( \mathcal{L} \) is a Q-algebra. The following is Varopoulos' proof.
(6.6) **Theorem**

\(\ell^1\) is an injective algebra.

**Proof:**

\(\ell^1 \otimes \ell^1\) can be identified with a space of double sequences \(a = \{a_{nm} \colon n, m > 0\}\). To prove our theorem it suffices to show that for all \(a\) in the unit ball of \(\ell^1 \otimes \ell^1\), \(\sum_{n=0}^{\infty} |a_{nn}| \leq 1\) (by the definition of injectivity).

Let \(G\) be any finite subset of non-negative integers with \(M\) elements. We put a \(Z(M)\)-group structure on \(G\). Let \(a_M\) be the restriction of \(a = \{a_{nm}\}\) on \(G \times G\). We shall construct \(F_M \in C(G) \hat{\otimes} C(G) = (\ell^1 \otimes \ell^1_M)\) such that \(\|F_M\| \leq 1\) and \(<F_M, a_M> = \sum_{m \in G} |a_{mm}|\).

Recall the characteristic function \(Kd\) of the diagonal of \(G \times G\). We had \(Kd(n, m) = \frac{1}{M} \sum_{r=0}^{M-1} \chi_r(n-m) = \frac{1}{M} \sum_{r=0}^{M-1} \chi_r(n) \overline{\chi_r(m)}\) where \(\chi_r\)'s are the characters on \(G = Z(M)\) and \(Kd \in C(G) \hat{\otimes} C(G)\) with \(\|Kd\| = 1\).

Define \(F_M\) on \(C(G) \hat{\otimes} C(G)\) by

\[F_M(n, m) = (\overline{a_{nn}/|a_{nn}|})Kd(n, m)\]

Then \(\|F_M\| \leq 1\) and we have
\[ \langle F_M, \alpha_M \rangle = \sum_{m \in G} |\alpha_{nm}| \leq \|F_M\| \|\alpha_M\| \leq 1 \]

Thus for any finite set \( G \subseteq \mathbb{Z} \) we have
\[
\sum_{m \in G} |\alpha_{nm}| \leq 1
\]
Therefore
\[
\sum_{m=0}^{\infty} |\alpha_{nm}| \leq 1
\]

Q.E.D.

The next class of \( \mathcal{Q} \)-algebras is \( \mathcal{C}^r(I) \), the Banach algebra of all functions on the closed interval \([0, 1] \subseteq \mathbb{R}\) with continuous derivatives of order \( r \) with the norm
\[
\|f\| = \sum_{j=0}^{r} \sup_{t \in I} |f^{(j)}(t)|
\]

Davie [3] proved that \( \mathcal{C}^r(I) \) is a \( \mathcal{Q} \)-algebra for \( r \geq 0 \), but as noted by Varopoulos [10] even more is true.
\( \mathcal{C}^r(I) \) is an injective algebra for each positive integer \( r \geq 0 \).

(6.7) **Theorem**

\( \mathcal{C}^r(I) \) is an injective algebra for each positive integer \( r \geq 0 \).

**Proof:**

We verify the condition (ii) of Prop. (4.2) with constant
\[
C = 2^{r+1}
\]
Given \( f \in \mathcal{C}^r(I) \) there are \( t_j \in I, j = 0, 1, \ldots, r \) such that
\[
\|f\| = \sum_{j=0}^{r} |f^{(j)}(t_j)|. \text{ We choose one such n-tuples.}
\]
Define \( s \in (C^r(I))^\prime \) by
\[
\langle s, g \rangle = \sum_{j=0}^{r} b(j) g(j)(t_j) \quad \forall g \in C^r(I)
\]
where \( b(j) = \frac{f(j)(t_j)}{r(j)(t_j)!} \).

Then \( \langle s, f \rangle = \|f\| \) and
\[
\langle s, gh \rangle = \sum_{j=0}^{r} b(j) (gh)(j)(t_j) = \sum_{j=0}^{r} b(j) \sum_{n=0}^{j} \binom{j}{n} g^{(n)}(t_j) h^{(j-n)}(t_j)
\]
where \( \binom{j}{n} \) is the \( n \)-th binomial coefficients of \( j \).

Define \( P(n,m), Q(n,m) \in (C^r(I))^\prime \) for \( n, m \geq 0 \) by
\[
\langle P(n,m), g \rangle = \binom{n+m}{n} b(n+m) g^{(n)}(t_{n+m}) \quad \forall g \in C^r(I)
\]
\[
\langle Q(n,m), g \rangle = g^{(m)}(t_{n+m}) \quad \forall g \in C^r(I)
\]

Then clearly
\[
\langle s, gh \rangle = \sum_{j=0}^{r} b(j) \sum_{n=0}^{j} \langle P(n,j-n), g \rangle \langle Q(n,j-n), h \rangle.
\]

Also,
\[
\sum_{j=0}^{r} \sum_{n=0}^{j} \binom{j}{n} (1+1)^j \leq 2^{r+1}.
\]
Q.E.D.
(6.8) Corollary

$C^p(I)$ is a $Q$-algebra.

We remark here that we can prove the above corollary directly by verifying the condition (ii) of Thm.(3.9). The proof is very similar to that of Thm.(6.7) except that we cannot use binomial coefficients anymore. Instead we use $k(a)$ where $a = (a_1, ..., a_n) \in \mathbb{Z}^n$, $a_i \geq 0$

$$|a| = \sum_{i=1}^{n} a_i$$ and $k(a)$ is defined by the coefficients of the polynomial as follows.

$$(\sum_{i=1}^{n} x_i)^m = \sum_{|a|=m} k(a) \, x_1^{a_1} \cdots x_n^{a_n}$$

where the sum is taken over all $a \in \mathbb{Z}^n$ such that $a_i > 0 \quad i = 1, ..., n, \quad \sum_{i=1}^{n} a_i = m$.

We note that

$$\sum_{|a|=m+1} k(a) = \sum_{|a|=m} k(a)$$

By Lipschitz space of order $\alpha$ ($0 < \alpha < 1$) we mean all the complex functions on $[0,1]$ with the finite Lipschitz norm $\|f\|_\alpha$ where

$$\|f\|_\alpha = |f(0)| + \sup_{x, y \in [0,1]} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}}$$

The Lipschitz space of order $\alpha$ is denoted by Lip $\alpha$. 
(6.9) **THEOREM**

For all $a$, $0 < a \leq 1$, $\text{Lip}_a$ is a $Q$-algebra.

**Proof:**

We verify the condition (ii) of Thm.(3.9). So let $f_1, \ldots, f_p \in \text{Lip}_a$, $\|f_i\|_a \leq 1 \ \forall \ i$ and $a \in \mathbb{C}^n$. Then

$$\| \sum_{i=1}^{p} a_i f_i \|_a = \sum_{i=1}^{p} |a_i f_i(0)| + \sup_{x,y \in I} \frac{|f(x)-f(y)|}{|x-y|^a}$$

But we have $|f(x)| \leq |f(0)| + \frac{|f(x)-f(0)|}{|x|^a} \leq \|f\|_a$

hence $\|f\|_a \leq \|f\|_a$.

Thus

$$\| \sum_{i=1}^{p} a_i f_i \|_a \leq \| \sum_{i=1}^{p} a_i f_i \|_a$$

$$\leq \| \sum_{i=1}^{p} \sup_{x,y \in I} \frac{|a_i f_i(y) - f_i(y)|}{|x-y|^a} \leq \| \sum_{i=1}^{p} a_i f_i \|_a$$

$$\leq \| \sum_{i=1}^{p} a_i f_i \|_a + n \| \sum_{i=1}^{p} a_i f_i \|_a$$

$$\leq 2^n \| \sum_{i=1}^{p} a_i f_i \|_a$$

Q.E.D.
We remark here that Davie [3] noted that Lip, is a Q-algebra and Varopoulos [10] noted that it is also an injective algebra. To prove the injectivity we need to approximate \( \| \cdot \|_\alpha \) and the proof is very technical so we omitted the proof.

Next is an example of a non Q-algebra. By the Wiener algebra we mean the algebra of all absolutely convergent Fourier series with the following norm.

\[
\| f \|_F = \sum_{n=-\infty}^{\infty} |f_n| < \infty ; \quad f = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}
\]

We denote the Wiener algebra by \( F \). The next theorem is due to Wermer [15].

(6.10) THEOREM

\( F \) is not a Q-algebra.

Proof:

Assume there exist a uniform algebra \( A \) and a closed ideal \( I \) and a Banach algebra isomorphism (a bounded algebraic isomorphism) of \( F \) onto \( A/I \).

Now \( e^{i\theta} \in F \) and let \( x = \tau(e^{i\theta}) \in A/I \). Then

\[
\| x^n \| = \| \tau(e^{in\theta}) \| < \| \tau \| \| e^{in\theta} \| = \| \tau \| \quad \forall n.
\]

By Thm. (1.6) for all polynomials \( P \)

\[
\| P(x) \|_{A/I} \leq C^2 \max_{|z|=1} |P(z)| \quad \text{where} \quad C = \| \tau \|.
\]

Now by the closed graph theorem \( \tau^{-1} \) is bounded hence

\[
\| P(e^{i\theta}) \|_F = \| \tau^{-1}(P(x)) \| \leq \tau^{-1} C^2 \max_{|z|=1} |P(z)|
\]
But this cannot hold for all polynomials $P$. Hence $F$ is not a $Q$-algebra. Q.E.D.

Actually Wermer proved a little more. He proved that $F$ is not algebraically isomorphic to any $Q$-algebra. This is a consequence of the above theorem and the following:

(6.1) Suppose $T : A \to B$ is an algebraic homomorphism from a Banach algebra $A$ onto a Banach algebra $B$. Suppose also that $B$ is semi-simple. Then $T$ is continuous.

Proof:

Let $m_A$ and $m_B$ be the maximal ideal space of $A$ and $B$ respectively. We define $T^* : m_B \to m_A$ by $T^*(\beta) = T \circ \beta$, where $T \circ \beta$ is the composition of $T$ and $\beta$.

Now if $x_n \to x$ in $A$ and $T x_n \to y$ in $B$ we want to prove that $T x = y$. Then by the closed graph theorem $T$ is continuous.

For each $z \in A$, $z(T^* \beta) = (T z)(\beta)$ hence $x_n(T^* \beta) = (T x_n)(\beta)$. But this implies $x(T^* \beta) = y(\beta)$ for all $\beta \in m_B$.

Now by the semi-simplicity of $B$, $T x = y$. Q.E.D.
Before we end this chapter we give one more set of examples, due to Varopoulos. Let $A$ be the algebra of functions on $T = \mathbb{R} \ (\text{mod} \ 2\pi)$ of the form

$$f(\theta) = \sum_{v=-\infty}^{\infty} a_v e^{iv\theta}, \quad ||f|| = \sum_{v=-\infty}^{\infty} |a_v| (1+|v|^\alpha) < \infty$$

Varopoulos [10] proves that $A_\alpha$ is an injective algebra for all $\alpha > \frac{1}{2}$ and it is not an injective algebra for all $0 \leq \alpha \leq \frac{1}{2}$. We omit the proof.
CHAPTER 7

In this final chapter we briefly discuss operator algebras (for the definition see Chapter 2).

It is clear that every closed subalgebra of an operator algebra is also an operator algebra. Lumer [7] extends Cole's result (Chapter 1) to show that every quotient algebra of an operator algebra (B/I where B is an operator algebra, I a closed ideal) is also an operator algebra.

Operator algebras also have a criterion that is extended from the criterion for injective algebras, this in turn is an extension of the criterions for Q-algebras.

(7.1) CRITERION

Suppose B is a Banach algebra then B is an operator algebra if and only if there exists a constant $K > 0$ with the following property:

For $S \in B'_1$, $m \geq 1$ and $F$ a finite dimensional subspace of $B$ we can choose $L_j : F \rightarrow B(H)$ $j = 1, \ldots, m$, $m$ linear mappings from $F$ into $B(H)$ for some Hilbert space $H$ and we can also choose $h, k \in H$ two vectors such that $\|h\|, \|k\| < 1$

\[
\|L_j(x)\|_{B(H)} \leq K\|x\|_B \quad x \in F, \quad j = 1, \ldots, m
\]

\[
\left\langle x_1, \ldots, x_m, S \right\rangle = \left\langle L_1(x_1), \ldots, L_m(x_m)h, k \right\rangle_H
\]

for all $x_j \in F$, $j = 1, \ldots, m$ where $\langle \cdot, \cdot \rangle$ indicates the scalar product between $B$ and $B'$, the $\cdot$ inside the bracket $\langle \cdot, \cdot \rangle$. 
indicates the multiplication in $B$, the bracket $\langle,\rangle_H$
indicates the scalar product in $H$ and the $\cdot$ inside it indi-
cates the operator product.

The above is due to Varopoulos who used it to prove
the following theorem ([12]).

(7.3) **THEOREM**

Let $B$ be a Banach algebra and suppose $B$ is isomorphic
as a Banach space to $C(X)$ for some compact space $X$, then $B$
is an operator algebra.

We earlier remarked that subalgebras and quotient
algebras of operator algebras are again operator algebras.
Now with the theorem above we have every $Q$-algebra is an
operator algebra.

Today research is being carried out to see just which
operator algebras are $Q$-algebras, since operator algebras
are easy to come by but the verification that one Banach
algebra is a $Q$-algebra is not easily made (as we saw in
Chapter 5, especially in $\ell^p$, $1 \leq p \leq \infty$).

There are other directions in which the studies of
$Q$-algebras proceeded, one of which is a question of boundedness
of polynomials acting on Banach algebras. Just which ones
are equivalent on $Q$-algebras or whether other boundedness
conditions hold for $Q$-algebras are the questions being asked.
One such example is a paper by Charpentier [2].
But the most significant part of Q-algebras is how the quantitative property brings so many seemingly qualitatively different Banach algebras together, hence illuminating a very different aspect of Banach algebras.
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