THE STUDY THROUGH MODELS OF REINFORCED CONCRETE BEAMS FAILING IN SHEAR

by

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ABSTRACT

John D. Finch

The Study Through Models of Reinforced Concrete Beams Failing in Shear

Seventeen rectangular reinforced concrete model beams at 1/4 scale were tested for correlation in ultimate shear strength with prototype beams fabricated by independent researchers. The various parameters investigated to obtain confidence in modelling techniques were the spacing and diameter of stirrups, the percentage of longitudinal steel and the a/d ratio. Six of the beams were fabricated with stirrups. All model test results were related to their prototypes by using dimensional analysis and similitude relationships.

Having established a reasonable level of confidence in the correlation between prototype and model beams failing in shear, one proposed explanation of this failure is briefly studied by relating research data previously published to predicted values based on concrete strengths, a/d ratios and percentages of longitudinal steel.
ACKNOWLEDGEMENTS

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The Canada Emergency Mesures Organization sponsored the programme under a research grant for investigation of the ultimate strength of concrete structures.
The following notation is used in this thesis:

\[ a = \text{distance from end support to first interior load} \]
\[ A_s = \text{area of tension reinforcement} \]
\[ A_v = \text{area of web reinforcement} \]
\[ b = \text{width of rectangular beam} \]
\[ d = \text{distance from extreme compression fibre to centroid of tension reinforcement} \]
\[ E = \text{modulus of elasticity of longitudinal steel} \]
\[ E_v = \text{modulus of elasticity of web reinforcement} \]
\[ f_c' = \text{compressive strength of concrete} \]
\[ f_t' = \text{tensile strength of concrete} \]
\[ f_y = \text{yield stress of tension reinforcement} \]
\[ f_{yv} = \text{yield stress of web reinforcement} \]
\[ h = \text{overall height of rectangular beam} \]
\[ I = \text{moment of inertia about x-x axis for rectangular section} \]
\[ j_d = \text{internal moment arm} \]
\[ k = \text{correction factor for bi-axial stress condition} \]
\[ l = \text{span length of beam} \]
\[ M = \text{bending moment} \]
\[ M_{fl} = \text{flexural capacity of rectangular beam} \]
\[ M_u = \text{bending moment at failure of the beam} \]
\[ P = \frac{A_s}{bd} \times 100 \]
\[ P = \text{applied load to beam} \]
Pu = applied load to beam at failure
s = spacing of transverse reinforcement
ST = length of a concrete tooth in Kani's derivation
v = nominal shear stress in concrete = \frac{V}{bd}
v_u = shearing stress at ultimate load
v_{cu} = allowable shear stress carried by concrete at ultimate load
V = shearing force
Vu = shearing force at ultimate load
y = distance from center of gravity of section to extreme fiber
\Delta x = width of a concrete tooth in Kani's derivation
S = deflection
\phi = diameter
\lambda = scale factor between prototype and model

Note: m as a subscript refers to the model beam and its properties.
DEFINITIONS

Shear span is the length of beam between a support and the first interior concentrated load.

Transition point is the critical a/d ratio beyond which the full flexural capacity of the beam is reached.

Micro-concrete is a sand and cement mortar used to represent concrete in model studies.

Model beam is a beam which is so related to a prototype that observation on the model beam may be used to predict the performance of the prototype in the desired respect.

Prototype beam is the original or full-sized beam which is to be modelled.
I. INTRODUCTION

At the present time (1968), there are two methods available to the structural engineer for solving design problems.

The first method is based on a mathematical model which represents the structure or the element of the structure acting in either the elastic region or the so-called plastic zone. The conventional approach which considers only the working load conditions is based on the theory of elasticity. Idealized structures and structural materials are commonly assumed in this design concept. The ultimate load approach however, is based on considerations of the collapse of the structure under loading conditions which represent the working load condition multiplied by an appropriate load factor. Also associated with this approach is a consideration of the serviceability (deflection, cracking etc.) of the structure under normal working conditions. The use of computers in either of the two previously mentioned design approaches usually enables the engineer to deal with complex structures in a reasonably easy fashion in the design office.

In general however, for this method, the designer has to make assumptions concerning the behaviour of members and materials which are simplifications or assessments of the actual behaviour. On this basis an appropriate analysis of the structural behaviour may be carried out and then a detailed design prepared. When no rigorous theoretical analysis is available, as in the case of the ultimate shear strength
of a reinforced concrete beam, the structural problem is usually simplified, on a conservative basis backed up by laboratory testing, to produce what is usually termed as a "safe" solution.

The second method available to the structural engineer for design and analysis makes use of model studies. Two model testing methods are used in both design and research, these are referred to as the direct and indirect methods. In the direct method, the distribution of stresses, moments and forces in the prototype structure due to a particular load are determined directly from the model structure when subjected to a load which simulates the load applied to the prototype structure. This method of model testing can be used to investigate the behaviour of structures both in the elastic and inelastic ranges, and even to determine the ultimate strength of the prototype structure. As ultimate shear strength correlations were sought in this thesis, direct modelling was used.

In the indirect method, arbitrary forces or displacements are imposed on the model at various points. The model itself may not be similar either geometrically or material-wise. The deformations of the models are measured, and from these deformations can be obtained the influence lines for the internal forces and moments and external redundant forces and moments acting on the prototype structure. These influence lines can then be used to obtain the distribution of stresses in the prototype structure due to the actual applied loads. Due its nature, this method of model testing can only be used to examine the
behaviour of a structure in the elastic range.

The greatest advantage of model analysis and model testing is that no prior knowledge of the behaviour of the structure or element of the structure is required. In modelling, by using the principles of similitude, a physical model is made at some convenient scale and then tested. From this test then, the behaviour of the prototype, or the structure modelled, can be predicted. It is the purpose of this thesis to show how model studies can be used in predicting ultimate loads on reinforced concrete beams failing in shear.

From the recent increased use of model analysis, it would seem that this technique is of recent origin. In actual fact, it is probably the oldest method of structural analysis and design. Michelangelo, some four centuries ago, specifically recommended this technique to his builders. Like other concepts in engineering however, notably ultimate design, this method, for one reason or another, lapsed into a long period of disuse. Specifically, in reinforced concrete, it has only been in the last 15 to 20 years that appreciable use has been made of this old method. Some of the better known direct modelling recent studies can be summarized as follows:

In 1959, the Cement and Concrete Association of Great Britain, made a study of model prestressed concrete bridges\textsuperscript{(1)} before employing a load distribution theory. This theoretical approach confirmed by many tests showed a good correlation between the behaviour of the model and the actual structure.
The University of Illinois in conjunction with the Portland Cement Association in 1963 tested a 60 foot square, nine panel slab at scales of 3/4, 1/4 and 1/16\(^2\). It was found that deflections, crack patterns, modes of failure and ultimate capacities "were closely in agreement".

A model at 1/32 scale of the Metropolitan Cathedral of Christ the King Church in Liverpool, England was carried out in 1964 by G.D. Base\(^3\). The actual structure in this case was approximately 220 feet high by 345 feet in diameter and yet at the 1/32 scale, the test results again "confirmed the consulting engineer's predictions".

In 1964 at the Massachusetts Institute of Technology, Mr. Samuel T. Lee\(^4\) took the earlier results from the University of Illinois research and using a scale factor of 1/28 tested three such models. Once again good correlation was found.

At McGill University, concrete modelling was first attempted by C.L. Pang in 1965\(^5\). He took three series of prestressed concrete beams at scales of 1, ½ and 1/8 and tested them to failure. All values obtained from his tests were in close agreement with the theoretical values. This present thesis is hopefully a further step towards the perfecting of modelling techniques at McGill.

Further work in model studies at McGill have been performed by F. Berry and M.S. Mirza. Berry\(^{20}\) studied at ¼ scale in 1966 the performance of a concrete floor joist section under a simulated wheel load to determine the feasibility of this configuration as a bridge deck.
The distribution properties of the deck under a simulated wheel load were measured and compared with theoretical values. Mirza, in his Ph.D. thesis (19), investigated, also at quarter scale, concrete beams under pure torsion, combined bending and torsion and combined bending, shear and torsion. Additional work in modelling is now under way at McGill in the field of dynamically applied loads to portal frames and beam to column connections.

At this point it is well to recapitulate the advantages of using models.

Complex structures and phenomena can be reproduced at model scales. The result of this is that a known predictable pattern of failure is available when analysis or design of prototype is attempted. This pattern of failure is thus obtained without knowing the how and the why of the failure mechanism, which in difficult analyses is perhaps impossible to fully understand. This also implies that complete structures can be fully tested instead of simple elements.

Also, by reducing the physical scale many times, it is recognized that a much smaller cost for the study of the same phenomenon can be realized. This cost factor is especially important when limited financial aid is available to either confirm theoretical designs or duplicate failure modes. This further implies that smaller testing machines can be used, a wider range of variables can be investigated in a finite time period and valuable laboratory floor space is reduced to a minimum; all these significant advantages are realized when models are employed.
Based on the preceding paragraphs, it can be readily seen why modelling techniques are so applicable to the study of reinforced concrete beams failing in shear. Surely there have been few greater mysteries in structural engineering than the how and why of this difficult subject.

The factors influencing the behaviour and strength of reinforced concrete beams failing in shear are both numerous and complex\(^{(6)}\). They include the proportions and shape of the beam, the structural restraints and the interaction of the beam with other components in the system, the amount and arrangement of tensile, compressive and transverse reinforcement, the load distribution and loading history, the properties of the concrete and steel, the concrete placement and curing and the environmental history. In spite of seventy years of research, there is still no firm agreement among engineers as to the mechanism of failure in this problem.

It is not within the scope of this thesis to present an historical review of the research done on shear failures in reinforced concrete beams. For the interested, references are available\(^{(6)}\)(\(^{(7)}\)). Needless to say, this subject has been extensively researched (more than 120 published papers on this topic in the last five years, and 5 theses at McGill University alone).

Of all the theories on shear failure mechanisms presently being evaluated by the engineering community, there is much interest in Dr. Gaspar Kani's so-called "concrete tooth analogy". Professor Kani,
Professor of Civil Engineering at the University of Toronto, has proposed an approach to the problem which has further been supported by laboratory tests. It is true that so far his theory has been advanced only for the simplest of conditions i.e. no stirrups, no compressive steel, rectangular cross-sectional beams, simply supported conditions etc., however, it is thought by many that this explanation at least points the way towards a full understanding of this phenomenon. Professor Kani's theory will be outlined in Chapter III of this thesis.

In all the previous comments on the relative merits of model study, it was implied that a correlation between models and prototypes could be found for the different modes of failure, including shear. However, when one examines the complex quantities in a shear failure, it becomes necessary to go beyond the principles of similitude to obtain valid modelling results. It was therefore decided to use not only similitude relationships but also the principles of dimensional analysis.

Dimensional analysis theory is based on two essential facts.

1. Any mathematical equation which describes some aspect must be in a dimensionally homogeneous form. For example, the equation

\[ f = \frac{Mv}{I} \]

is correct regardless of whether the units are in pounds and feet, pounds and inches or grams and yards.

2. Any governing equation of the form

\[ F (x_1, x_2, \ldots, x_n) = 0 \]
must also be expressible in the form \( G(\Pi_1, \Pi_2, \ldots, \Pi_r) = 0 \)
where the \( \Pi \)'s are dimensionless products of the physical variables
\( x_1, x_2, \ldots, x_n \) all written in a non dimensional manner. A more de-
tailed discussion of this theory will follow in Chapter III.

By using these principles of similitude and dimensional
analysis as applied to reinforced concrete beams failing in shear, it
will be shown that model beams can be used to predict prototype failure
loads.
II. SCOPE OF RESEARCH

This thesis forms part of an overall program at McGill University in the study of scaled down members made of reinforced micro-concrete.

Before complex structures subjected to both static and dynamic loadings can be attempted at model proportions, it is necessary to establish confidence in correlations between prototype and model beams which fail in shear or to be more precise, model beams which fail under combined flexure and shear with a predominant diagonal cracking mode.

For a valid investigation, it was necessary to model prototype beams which had been previously tested and extensively documented and which also included a reasonable range of factors which could influence the shear strength of a beam.

For the modeling of beams with and without stirrups, a duplication at \( \frac{1}{3} \) scale of Mohammed A. Faris', 1962, Master of Engineering thesis (10) was attempted. This thesis entitled "Ultimate Shear in Reinforced Concrete Beams With Stirrups", tested seven types of beams under a four point load system and a constant a/d ratio (2.83) but with a varying diameter and spacing of stirrups. Specifically at model size, work involved the fabrication and testing of these seven types of beams each having the same overall dimensions of 1-3/4" x 3-3/4" x 3'-6" with the same compressive and tensile reinforcement consisting of 2 no. 8 gauge (\( \phi = 0.157" \)) and 6 no. 8 gauge wires respectively (see Fig.4 on p.33).
DATA FOR MODEL BEAMS USED IN CORRELATION WITH FARIS' RESULTS

LOADING DIAGRAM

SHEAR FORCE DIAGRAM

BENDING MOMENT DIAGRAM

Fig. 1
One beam was constructed without stirrups while the remaining six had stirrups of either 16 ga. (\( \phi = 0.0625" \)) or 13 ga. (\( \phi = 0.0882" \)). The three different stirrup spacings for the modelled beams were 1\( \frac{1}{8} " \), 1" and 5/8" (see Figs. 4 & 5 on p. 33 & 34). The overall dimensions of the beams and the details of the reinforcing cages were obtained from similitude relationships based on a scale factor of 4. All beams were tested in appropriate loading increments to failure. To further coincide with Faris' work, deflection curves and material properties for all the modelled beams were also obtained.

Next, as most engineers would agree that the a/d ratio influences shear strength, it was decided to model 10 beams at \( \frac{1}{3} \) scale from Professor Gaspar Kani's recent works (11). The beams, which had various a/d ratios from approximately one up to the transition point, were tested under a two point load system with two different percentages of steel (2.73% and 1.88%). Cross section model beam dimensions were kept constant at 1\( \frac{1}{2} " \) x 3" but overall lengths varied from 14" up to 42". Fabrication, testing, and material properties were all handled as per the previous set of beams.

To investigate the validity of the above test results, a general theory concerned with dimensional analysis and similitude relationships was then established. This correlation between actual model beam results and the prototype values is then derived considering the influence of the various parameters involved. The predicted ultimate load value for the prototype was then compared with the actual result and a percentage of deviation was calculated.
DATA FOR MODEL BEAMS USED IN CORRELATION WITH KANI'S RESULTS

LOADING DIAGRAM

SHEAR FORCE DIAGRAM

BENDING MOMENT DIAGRAM

Fig. 2
Having established a reasonable level of confidence in this particular type of modelling, the investigation on shear failure was carried one step further. In the last portion of the thesis, the shear failure mechanism as proposed by Professor Kani is examined and related to other research data previously published. In this portion of the work Kani's failure curves or his "valley of diagonal tension" is superimposed on other researchers data which have been adjusted dimensionally so as to make a compatible system.
III. THEORY

1. Theory of Structural Models Failing in Shear

The purpose of this section is to develop a theory with which it will be possible to use small scale physical models to predict the structural behaviour of prototype reinforced concrete beams which fail in a shear mode. It will become clear that this theory accomplishes two purposes. First, it considers the relationship between the many variables which are believed to cause shear failures at certain ultimate loads and then it provides the means through which experimental results obtained from a model can be related to a prototype.

Consider some phenomenon in which there are n mechanical properties. In the most general case, the values of these properties can be established through the three fundamental dimensional units of force, length and time. Now if $x_1, x_2, \ldots, x_n$ are the different properties making up the phenomenon, then

$$x_1 \equiv D_1 (F, L, T)$$
$$x_2 \equiv D_2 (F, L, T)$$
$$x_n \equiv D_n (F, L, T)$$

where $\equiv$ stands for dimensional equivalence and $F$, $L$ and $T$ represent the three fundamental dimensional units of force, length and time.

It is now required to prove that the functions $D_1$, $D_2$, $\ldots$, $D_n$ are related to each other in a specific manner where the functions are in the form of products of powers of these fundamental units.
Now using a more specific example of two beams subjected to dynamic loading. (By subjecting the beams to dynamic loading we are considering the most general case, if a beam is subjected to static loading there is only two dimensional units involved as time $T$ is no longer a factor.)

Beam 1 = $D(aF, bL, cT)$

Beam 2 = $D(pF, qL, rT)$ where

$a$, $b$, $c$ and $p$, $q$, $r$ are numbers which indicate the magnitude of the $F$, $L$ and $T$ dimensional units.

It is understood that

$$\frac{D(aF, bL, cT)}{D(pF, qL, rT)}$$

has absolute significance regardless of the size of the dimensional units. Thus if the size of the force unit is changed to $x$ times larger, the length unit to $y$ times larger and the time unit to $z$ times larger, then it still must be true that

$$\frac{D(aF, bL, cT)}{D(pF, qL, rT)} = \frac{D(axF, byL, czT)}{D(pxF, qyL, rzT)}$$

or

$$D(axF, byL, czT) = D(pxF, qyL, rzT) \cdot \frac{D(aF, bL, cT)}{D(pF, qL, rT)}$$

Now after applying the chain rule of partial differentiation with respect to $x$ and remembering that $F$, $L$, and $T$ represent dimensional units and are carried along only for completeness

$$aF \frac{\partial D(axF, byL, czT)}{\partial axF} = pF \frac{\partial D(pxF, qyL, rzT)}{\partial pxF} \cdot \frac{D(aF, bL, cT)}{D(pF, qL, rT)}$$
\textbf{Now letting }aF, bL, \text{ and } cT \text{ vary while holding } pF, qL \text{ and } rT \text{ fixed, one obtains when}

\[ x = y = z = 1 \]

\[
\frac{aF}{D(aF, bL, cT)} \frac{\partial D}{\partial aF} = \frac{pF}{D(pF, qL, rT)} \frac{\partial D}{\partial pF} = \text{a constant}
\]

or

\[
\frac{\partial D}{D(aF, bL, cT)} = k_1 \frac{\partial aF}{aF}
\]

This is a special partial differential equation which can be treated as an ordinary differential equation except that the arbitrary constant of the ordinary differential equation becomes an arbitrary function for the partial differential equation.

Solving by integration yields

\[ \ln D(aF, bL, cT) = k_1 \ln aF + \ln G(bL, cT) \]

or

\[ D(aF, bL, cT) = G(bL, cT) [aF]^{k_1} \]

This process could be repeated for differentiating partially with respect to \( y \) and then to \( z \). The final result would then become

\[ D(aF, bL, cT) = \text{Constant} [aF]^{k_1} [bL]^{k_2} [cT]^{k_3} \]

or

\[ D(aF, bL, cT) = \text{Constant} [F]^{k_1} [L]^{k_2} [T]^{k_3} \]

In this way it is seen that the functional form of the dimensions of any physical quantity is necessarily a product.

Now consider the problem of a simply supported reinforced concrete
beam which has only longitudinal steel and is rectangular in cross section. Since the beam is to be statically loaded only two "fundamental" dimensions are required to describe the parameters, these dimensions being force and length.

The parameters chosen in the study of these beams must be selected with care.

For beams of constant depth, the National Building Code of Canada § 4.5.4B.13(1) suggests that the nominal shear stress, at ultimate load, as a measure of diagonal tension be computed as

\[ \nu_u = \frac{V_u}{bd} \]

and the allowable shear stress for members with unreinforced webs subject to flexural shear without axial load be computed as

\[ V_{cu} = \phi \left( 1.9 \sqrt{f'c} + 2500 \frac{PV_u d}{M} \right) \]

Since applied load \( P = f(V_u) \) and area of steel \( A_s = f(P) \) the five parameters selected were

- depth \( d \)
- width \( b \)
- area of steel \( A_s \)
- compressive strength of concrete \( f'c \)
- applied load \( P \)

In addition to the above quantities a sixth, the modulus of elasticity \( E \) of the longitudinal steel was chosen. This was selected to show the influence of the stress-strain characteristics of the steel.
This latter term also reflects the actual elastic behaviour of the steel throughout the load history of the beam since the tension steel in all beams failing in shear does not yield.

The six parameters selected are therefore \( d, b, A_s, E, f'_c \) and \( P \).

It should be mentioned however that these six parameters were also selected on the basis that their values were uniquely defined in the prototype results which were modelled. Unfortunately the modulus of elasticity of the concrete and Poisson's ratio were not detailed in the results for the prototype's studied. It can be assumed that more precise correlations could be obtained the more data affecting shear strength there is available.

A table of the parameters in this study of beams failing in shear with unreinforced webs would be as follows:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 ) depth ( d )</td>
<td>( L )</td>
</tr>
<tr>
<td>( x_2 ) width ( b )</td>
<td>( L )</td>
</tr>
<tr>
<td>( x_3 ) area of steel ( A_s )</td>
<td>( L^2 )</td>
</tr>
<tr>
<td>( x_4 ) modulus of elasticity of steel ( E )</td>
<td>( F L^{-2} )</td>
</tr>
<tr>
<td>( x_5 ) tensile strength of concrete ( f'_t )</td>
<td>( F L^{-2} )</td>
</tr>
<tr>
<td>( x_6 ) applied load ( P )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

It should be noted that item 5 is listed as the tensile strength of concrete. This has been done since in shear problems, the tensile strength of the concrete is the predominant concrete property and since

\[
f'_t = a \text{ constant} \times \sqrt{f'_c}
\]
the parameters \( f'c \) and \( f't \) are directly related and no discrepancy is introduced.

It has been previously shown that the dimensional descriptions occur in a product form. However in any set of physical quantities, there is a limited number of quantities which cannot themselves be combined to form a dimensionless product, but which can be combined with other quantities in the set to yield a dimensionless product. The quantities involved in the limited set are said to be dimensionally independent while the other quantities are dimensionally dependent upon the special limited set. The theory involved in establishing the basis of dimensional dependence and independence is well documented in various texts \((16,17)\).

The important results can be summarized as follows:

1. For statics problems there are two relevant dimensions, force and length. Correspondingly there are two and only two dimensionally independent physical quantities in a set of quantities involved in a statics problem.

2. The limited set of dimensionally independent quantities are not unique. For a statics problem there can be, with the following exceptions, any two quantities which together include force and length among their dimensions.

   Exception No.1: The two quantities cannot have exactly the same dimensions

   Exception No.2: A dimensionless quantity such as strain or Poisson's ratio cannot be included.

3. The dimensionally dependent relationships can be determined
by forming with the previously determined dimensionally independent set a dimensionless product for each of the remaining variables. Thus if there are \( n \) quantities in the phenomenon, there will be \( n-2 \) such relationships.

Now, once again considering our shear problem, since there are two fundamental dimensions involved in the six quantities, there must be two quantities which are dimensionally independent. A necessary and sufficient condition that two physical quantities be independent is that the determinant formed from the exponents of the two fundamental dimensions should not vanish.

In our problem, try quantities \( x_1 \), the depth \( d \) with units of \( F^0L^1 \) and \( x_5 \), the compressive strength of concrete \( f'c \) with units of \( F^1L^{-2} \).

\[
\begin{vmatrix}
0 & 1 \\
1 & -2
\end{vmatrix} \neq 0
\]

Therefore the depth of the beam and the tensile strength of the concrete are selected as the dimensionally independent quantities.

Now since \( L \stackrel{\sim}{\neq} x_1 \) and since \( x_5 \stackrel{\sim}{=} F L^{-2} \) then

\[
F \stackrel{\sim}{=} x_5 x_1^2
\]

and

\[
x_2 \stackrel{\sim}{=} L \stackrel{\sim}{=} x_1 \\
x_3 \stackrel{\sim}{=} L^2 \stackrel{\sim}{=} x_1^2 \\
x_4 \stackrel{\sim}{=} F L^{-2} \stackrel{\sim}{=} \frac{x_5 x_1^2}{x_1^2} = x_5 \\
\]

\[
x_6 \stackrel{\sim}{=} F = x_5 x_1^2
\]
Then rewriting these four or n-2 dimensional relationships in the form of dimensionless products we obtain:

\[
\left( \frac{x_1}{x_2} \right) \times \left( \frac{x_3}{x_4} \right) \times \left( \frac{x_5}{x_6} \right) = 1
\]

which comprises a complete set. Substituting in the actual quantities we obtain the following dimensionless expression:

\[
\left( \frac{d}{b} \right) \left( \frac{x_2^2}{A_s} \right) \left( \frac{f't}{E} \right) \left( \frac{f'y}{d^2} \right) = 1
\]

(1)

The above expression has been derived for simply supported reinforced concrete beams with no transverse reinforcement.

Now examining the case where stirrups are used as web reinforcing, the same parameters for unreinforced webs are used plus parameters showing the influence of the stirrups.

From N.E.C. 4.5.4B.15(1), the area of steel required in stirrups placed perpendicular to the longitudinal reinforcement shall be

\[
A_v = \frac{V'_u}{f_{fyd}}
\]

where \( V'_u \) is the shear carried by the web reinforcement at ultimate load.

Therefore the terms \( A_v \) for area of web reinforcement and \( f_{wy} \) for yield point of web reinforcement have been selected as additional parameters. Once again the term \( f'c \) has been replaced by the more suitable term \( f't \), the two quantities being related by

\[
f't = \text{a constant } \sqrt{f'c}
\]
Therefore a suitable table for the parameter involved in the study of reinforced concrete beams with web reinforcement would be as follows:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$ depth $d$</td>
<td>$L$</td>
</tr>
<tr>
<td>$x_2$ width $b$</td>
<td>$L$</td>
</tr>
<tr>
<td>$x_3$ modulus of elasticity of longitudinal steel $E$</td>
<td>$FL^{-2}$</td>
</tr>
<tr>
<td>$x_4$ modulus of elasticity of transverse steel $E_v$</td>
<td>$FL^{-2}$</td>
</tr>
<tr>
<td>$x_5$ area of longitudinal steel $A_s$</td>
<td>$L^2$</td>
</tr>
<tr>
<td>$x_6$ area of transverse steel $A_v$</td>
<td>$L^2$</td>
</tr>
<tr>
<td>$x_7$ yield point of stirrups $f_{yv}$</td>
<td>$FL^{-2}$</td>
</tr>
<tr>
<td>$x_8$ strength of concrete $f' t$</td>
<td>$FL^{-2}$</td>
</tr>
<tr>
<td>$x_9$ applied load $P$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Using a similar approach with the two fundamental dimensions and the nine quantities, we have $9-2$ or $7$ relationships, which are as follows:

\[
\left( \frac{d}{b} \right) \left( \frac{f't}{E} \right) \left( \frac{f't}{E_v} \right) \left( \frac{d^2}{As} \right) \left( \frac{d^2}{Av} \right) \left( \frac{f't}{fyv} \right) \left( \frac{f't \cdot d^2}{P} \right) = 1 \quad (2)
\]

It should be noted in equations (1) and (2) that the yield point of the longitudinal steel $f_y$ has been left out of the derivations. This has been done since experimental data indicates that in shear failure under the conditions for this research, the yield point of the tensile steel is not reached.

Equations (1) and (2) now give us expressions relating the various parameters in our study of shear with and without stirrups.

It is now necessary to look at the principles of similitude
as they apply to this thesis. These principles were used in scaling down the prototype beams selected since the non-dimensional terms are true for the model and the prototype. If we assume that the physical dimensions are scaled by some factor \( \lambda \) where \( \lambda \) is taken as a whole number, we can say that

\[
\frac{L}{L_m} = \lambda
\]

where the subscript \( m \) refers to the model. If no subscript appears, it is understood that reference is made to the prototype.

Then \( b \), \( d \), and \( h \) are also scaled by \( \lambda \) for dimensions of the model since

\[
\frac{b}{b_m} = \lambda \quad ; \quad \frac{d}{d_m} = \lambda \quad ; \quad \text{and} \quad \frac{h}{h_m} = \lambda
\]

Now considering the ratios of areas of steel in the prototype and model

\[
\frac{A_s}{A_{sm}} = \frac{\int_{D/4} \pi D^2/4}{\int_{D_m/4} \pi D_m^2/4} = \frac{\int_{D_m^2/4} \pi D_m^2 \lambda^2/4}{\int_{D_m^2}/4} = \lambda^2
\]

for the moments of inertia

\[
I = \int y^2 dA
\]

\[
= \int_{y_m \lambda^2} y_m^2 dA_m \lambda^2
\]

\[
= \lambda^4 \int_{y_m} y_m^2 dA_m
\]

and

\[
I_m = \int_{y_m} y_m^2 dA_m
\]

\[
\therefore \frac{I}{I_m} = \lambda^4
\]
for deflection, considering simply supported beams

\[
\delta = \text{a constant } K \times \frac{P L^3}{E I_m}
\]

\[
= \frac{K P_m \lambda^2 L_m^3 \lambda^3}{E I_m \lambda^4} = \frac{K P_m L_m^3}{E I_m}
\]

and \[\delta_m = \text{a constant } K \times \frac{P_m L_m^3}{E I_m}\]

\[\therefore \frac{\delta}{\delta_m} = \lambda\]

Now, recalling the basic assumption of the absolute significance of the dimensional units, it is possible to write equations relating model and prototype reinforced concrete beams failing in shear.

CASE 1 - No transverse reinforcement

\[
\left(\frac{d}{b_m}\right)\left(\frac{d^2}{b_m}\right)\left(\frac{f't_m}{E_m}\right)\left(\frac{f't_m d_m^2}{P_m}\right) = \left(\frac{d}{b}\right)\left(\frac{d^2}{A_s}\right)\left(\frac{f't}{E}\right)\left(\frac{f't d^2}{P}\right)
\]

(3)

CASE 2 - With transverse reinforcement

\[
\left(\frac{d}{b_m}\right)\left(\frac{f't_m}{E_m}\right)\left(\frac{f't_m}{E_v m}\right)\left(\frac{d^2}{A_s m}\right)\left(\frac{d^2}{A_v m}\right)\left(\frac{f't_m}{f y v_m}\right)\left(\frac{f't_m d_m^2}{P_m}\right)
\]

\[
= \left(\frac{d}{b}\right)\left(\frac{f't}{E}\right)\left(\frac{f't}{E_v}\right)\left(\frac{d^2}{A_s}\right)\left(\frac{d^2}{A_v}\right)\left(\frac{f't}{f y v}\right)\left(\frac{f't d^2}{P}\right)
\]

(4)

in both equations (3) and (4) it is assumed that

\[\text{a constant } \sqrt{f'c} = f't\]
In equations (3) and (4) all quantities except \( P \) are treated as known or measured values. \( P \) (which is the theoretical ultimate load for the prototype) is then easily solved for and compared with the actual \( P \) from the prototype test results. The comparison is made more meaningful by further calculating a percentage deviation.

2. Outline of Dr. Gaspar Kani's Shear Mechanism Theory

As mentioned earlier in the introduction, the last portion of the thesis is an attempt to investigate the relationship of Dr. Gaspar Kani's proposed shear failure mechanism theory to previously published research data in this field. These relationships will be shown in the Chapter on results.

It is beyond the scope of this thesis to describe in detail, Professor Kani's explanation of this complex phenomenon. This explanation is already well documented in available references\(^{(8,9,11)}\). A short summary of his theory however is required so that the diagrams in the next Chapter may be understood.

**Summary of Dr. Kani's Shear Mechanism Theory**

Under increasing load a reinforced concrete beam transforms into a comb-like structure. In the tensile zone of the beam, the cracks take on the appearance of concrete teeth or vertical cantilevers, while the compressive zone represents the backbone of the concrete comb. Dr. Kani's analysis of this structural system reveals that there are two rather different possible mechanisms.
1. As long as the capacity of the concrete teeth is not exceeded the beam-like behaviour governs.

2. After the resistance of the concrete teeth has been destroyed or exceeded, the beam transforms into a tied arch mechanism.

For both mechanisms, Professor Kani has developed analytical expressions. These expressions have been related to a diagram (Fig. 3) which plots $\frac{M_u}{M_{fl}}$ on the ordinate axis and the $a/d$ ratio on the abscissa. Dr. Kani defines $M_u$ as the actual moment at failure while $M_{fl}$ is defined to be the full flexural capacity of the member or $A_s f_y jd$. It is shown that the treatment of shear failure is considered as a measure of the reduction of flexural capacity. He therefore plots an interaction diagram in Fig. 3.

Several points of interest should be noted on Dr. Kani's figure.

1. $a/d_{TR}$ is the $a/d$ ratio which limits the region of diagonal failure. Beyond it, only flexural failure can be expected. Excluding over-reinforced sections, Dr. Kani states that

$$a/d_{TR} = 6 p \frac{f_y}{ft} \frac{S_T}{\Delta x}$$

where $S_T$ is the average length of the concrete teeth, and $\Delta x$ is the average spacing between the teeth.

2. In the region of low $a/d$ ratios, the capacity of the structure is determined by the strength of the remaining arch, and

$$M_u = \frac{M_{fl}}{k} \cdot \frac{d}{a}$$

where $k$ is a constant.
PLOT OF PROF. KANI'S ANALYTICAL EXPRESSIONS
FOR HIS SHEAR MECHANISM THEORY

$M_f = $ FLEXURAL CAPACITY OF BEAM

CAPACITY OF REMAINING ARCH

CAPACITY OF CONCRETE TEETH

Fig. 3
which is a correction factor for the bi-axial stress condition at the point of applied load.

3. In the region of medium a/d ratios, the capacity of the concrete teeth determines the strength of the structure

\[ \mu = \frac{Mf}{a/d \cdot TR} \cdot \frac{a}{d} \]

4. The common boundary point of the two regions is given by Dr. Kani as

\[ \frac{a}{d \min} = \sqrt{\frac{a/d \cdot TR}{K}} \]

NOTE: Although Dr. Kani's theory shows merit and has further increased the knowledge of why and how a shear failure propagates, it is not fully accepted by the profession. The purpose of including it in this thesis is to indicate the credibility of the theory in relation to other published research data in this field.
IV. SPECIMENS AND TEST PROCEDURES

1. Reinforcing Steel

It has been noted in the theory portion of this thesis that all the scale factors used to obtain model quantities from prototype quantities are expressible in terms of the length scale factor $\lambda$. It is further assumed that if the principles of similitude are adhered to and the moduli of elasticity between model and prototype specimens are equal, then the stresses will be theoretically equal between the large and small scale members. Although it is possible theoretically to adjust for variations in elastic moduli, it is much simpler to use materials which exhibit the same stress-strain characteristics, as such, prior to the fabrication of the reinforcing cages, simple tensile tests were carried out on commercially available wires so that the similarity of stress-strain curves could be established in advance. It was found that a galvanized soft steel wire as manufactured by the Steel Company of Canada was quite suitable for modelling requirements so that no drastic adjustment of stress-strain ratios was necessary.

The small gauge steel wires ($\phi = 0.157"$ and $\phi = 0.125"$) selected and used for the longitudinal reinforcement in the model beams were obtained in lengths of approximately 30 feet from local suppliers. Although ordered as straight pieces, most of the bars had a slight wavy characteristic which made it difficult to fabricate the cages perfectly. Upon arrival from the supplier, the smooth bars were cleaned and then cut to the desired lengths.
16 GA. WIRE (MODEL)

E = 27,800,000 psi
fyt = 67,000 psi

NO. 2 BAR (PROTOTYPE)

E = 30,000,000 psi
fyt = 71,000 psi

TENSILE STRESS IN KIPS PER SQUARE INCH

1000 2000

STRAIN IN MICRO INCHES PER INCH

1000 2000
13 GA. WIRE (MODEL)
E = 28,300,000 psi
f_yt = 53,000 psi

NO. 3 BAR (PROTOTYPE)
E = 30,000,000 psi
f_yt = 63,500 psi
8 GA. WIRE (MODEL)

\[ E = 30,000,000 \text{ psi} \]
\[ f_y = 62,000 \text{ psi} \]

NO. 5 BAR (PROTOTYPE)

\[ E = 30,000,000 \text{ psi} \]
\[ f_y = 55,000 \text{ psi} \]
REINFORCEMENT FOR MODEL BEAMS IN FARIS' CORRELATION

-2-No. 8 GA. - 40"
-2-No. 8 GA. - 12.75"
-2-No. 8 GA. - 11.25"

DIAGRAM FOR LONGITUDINAL STEEL (ALL BEAMS)

Stirrups sym. about C

6 @ 1.25"   10 @ 1.25"

DIAGRAM FOR 1.25" STIRRUP SPACING

Fig. 4
REINFORCEMENT FOR MODEL BEAMS IN FARIS' CORRELATION.

Stirrups sym. about

7 @ 1"  12 @ 1"

DIAGRAM FOR 1" STIRRUP SPACING

11 @ .625"  19 @ .625"

DIAGRAM FOR .625" STIRRUP SPACING

TYPICAL SECTION
AT MID-SPAN

Fig. 5
The steel for the stirrups (ο = 0.0882" and θ = 0.0625") was supplied by Stelco and came in coils. After straightening the wires, the stirrups were cut to the required lengths and then bent by a Di-acro bending machine so that the correct rectangular shape was obtained. In accordance with code requirements (N.B.C. 4.5.3.66(5) (d)) the length of the lapped splices was 60 bar diameters for the smooth bars used.

With the longitudinal steel cut to the correct lengths, and the stirrups bent to the correct shape, the next step was the fabrication of the reinforcement cages. Two different methods of fabrication were attempted.

The first method consisted of attaching the stirrups to the longitudinal steel by an epoxy. This epoxy manufactured under the trade name of "Duro Plastic Epoxy" is actually a two part mixture of epoxy resin and polyamide resin which when combined in equal proportions produces a satisfactory bond between metallic parts.

By using a series of plywood wedges, the stirrups were rigidly held in place at the correct spacing along the tensile steel. After the liquid epoxy was applied at points of contact between the stirrups and the longitudinal steel, the whole assembly was allowed to set for 24 hours. At the end of this hardening period, the wooden wedges were removed to be used on another specimen and the completed cage was quite stable enough to be handled freely without risk of damage. Although this manner of fabrication has been successfully used by other researchers, after several trials, it was decided to use an electric spot welder.
The "Miller lectro-spot" 230 volt, 60 cycle portable spot welder, after another trial period, appeared to give a neater final product in a shorter time period than did the epoxy method. Consequently, all the steel used in the fabrication of the cages was rigidly connected using the electric spot welder. The only exception to the above was that small rectangular anchorage plates used for modelling Professor Kani's tests were epoxied to the tensile steel.

Two points of interest on the use of the electric spot-welder should be noted however. First, by trial and error, it was found necessary to set the regulator to only 2 to 3 cycles on the "Miller lectro-spot". This low value insured against the stirrup being burned through and yet gave sufficient adhesion between transverse and longitudinal reinforcement. The second point to mention is that all stirrups were attached from the center of the cage working towards its ends. This allowed each stirrup to be secured at its four corners and then closely examined before moving outwards to the next stirrup.

Inverted U-shaped wire chairs were also spot-welded to the longitudinal reinforcement so that the effective depth, "d" of the beam was maintained throughout the length of the model.

As a final step, 1½" x 3" rectangular steel plates were epoxied to the ends of the longitudinal bars where it was deemed necessary. This additional precaution against anchorage failure was used for all of the model beams without transverse reinforcement. It should be further noted that since smooth bars were used in the model beams as opposed to the
normal deformed bars in the prototypes, it was found necessary to check against premature bond failure, in the small scale members. When a smooth bar is embedded in concrete, the bond stresses are developed largely through adhesion and friction. Deformed bars rely on the interlocking effect between the ribs of the bar and the concrete. It was therefore decided to limit the bond stresses to 250 p.s.i. in accordance with the code (N.B.C. § 4.5.4B 22.(4)(d)).

2. Concrete

Ideally, the different particles of a concrete used in a model (coarse aggregate, sand and cement) should be reduced in size according to the length factor selected. This is possible, within limits, in the case of coarse aggregate but it is hardly practicable to reduce the size of the finer sand and cement grains. When the size of the coarse aggregate is reduced by scale, the dimensions correspond to those of the sand particles. It is therefore usual to use an appropriately graded sand to simulate the coarse and fine aggregates i.e. a sand and cement mortar is used to represent concrete in the model, this is sometimes referred to as "micro-concrete".

It was decided to use a "micro-concrete" mix previously developed at McGill\(^{(19)}\) for these model beam studies. This mix which consisted of a blend of graded crushed quartz and sand was as follows:

\[
\begin{align*}
\text{# 10 crushed quartz sand} & \quad 3 \text{ lbs} \\
\text{# 16 } & \quad \text{" } \\
\text{# 24 } & \quad \text{" }
\end{align*}
\]

\[
\begin{align*}
\text{" } & \quad 3 \quad \text{" } \\
\text{" } & \quad 3.75 \quad \text{" } 
\end{align*}
\]
-38-

# 35 crushed quartz sand 3.75 lbs
# 70 " " " 1.5 "
High Early Strength Cement 4.62 "
Water 101.7 cu. inches

The sand was mixed together in a small portable mixer before adding the XXX Cement. After another thorough blending, the water was added slowly and another thorough mixing was carried out prior to placing the mortar in the formwork.

3. Formwork

As with all other items in this project, the formwork was built in the Strength of Materials Laboratory of McGill University. The forms themselves were made entirely out of 3/4 inch plywood finished on both sides. To prevent lateral bulging of any of the beams before their curing, cross ties of 1" x 2" plywood strips were fastened transversally across the top of the beam separators and either small steel angles or 2" x 2" plywood blocks were fastened snuggly against the exterior beam formwork on the plywood platform.

All forms were soaked using a brush with form oil (light bodied petroleum oil) two days before pouring to prevent the concrete from sticking to the forms. Since the plywood forms absorbed the form oil quite readily, a further light application of form oil was found necessary just before the insertion of steel cages into the formwork on the day of pouring.
4. Placing, Curing and Preparation for Testing

The mix as previously described was found to make approximately 250 cubic inches of concrete. With the forms and the portable mixing unit side by side, the concrete from the bucket on the portable mixer was placed directly into the forms with a large aluminum spoon. The pouring itself proceeded systematically with one beam at a time, starting from one end of the beam and proceeding to the other. The concrete was vibrated externally with a small portable air-driven vibrator applied directly to the formwork. Corners and locations of dense reinforcement were further compacted by hand tamping with a smooth ½" steel rod. To find the compressive strength of the concrete, test cylinders of two sizes, 2" x 4" and 3" x 6", to coincide with the relative model beam dimensions, were also filled and tamped concurrently with the placing of concrete into the forms. Although it was expected that the smaller test cylinders would yield higher relative compressive strengths this was not the case in this study, and as a result, a statistical mean was taken to be the f'c or the compressive strength of each beam based on the cylinder results regardless of the cylinder size.

Initial curing of the beams consisted of thoroughly soaking a burlap material and entirely covering the beams and test cylinders with this burlap. To ensure sufficient moisture prior to the stripping of the forms, additional water was added four or five times a day for two consecutive days until it was deemed advisable to remove the forms. Immediately upon form removal, the beams and the concrete cylinders were fully immersed in a tank of water.
Two days prior to the testing of the beams and cylinders, they were removed to air dry for a period of 24 hours. The test cylinders were then capped in a vertical capping jig using a compound which was a mixture of sulphur flour, fine sand (50 mesh), limestone dust, bentonite (325 mesh) and fly ash. Previous test results have shown the strength of this capping compound to be between 6000 - 7000 p.s.i. Also, the same day prior to testing, the beams were painted in their shear span regions with a dilute solution of plaster of paris in water. In the early experiments some shear spans were purposely left without the whitening solution so as to judge the relative merits of the mixture in crack detection. After a few trials, all subsequent beams were whitened in their shear spans.

5. Test Procedures

All tests were carried out on the 60,000 lb. capacity Riehle Testing Machine located in the McGill University Strength of Materials Laboratory.

After the points of support and loading were marked on the model specimens, the test assemblies were arranged. At this stage of the project, another advantage of working in models became evident. Whereas the prototype beams had to be manoeuvred into place by overhead cranes and hydraulic jacks, the model beams and the loading beams were small enough to be manually carried with ease and quickly moved into position on the testing machine.
TESTING ARRANGEMENT FOR THE

MODELLING OF FARIS* BEAMS
TESTING ARRANGEMENT FOR THE MODELLING OF KANI’S BEAMS
After the test assemblies had been constructed and checked for alignment, Starrett dial gauges (0.001" readings) were attached at required locations. Besides a correlation of failure loads, it had been previously decided to examine the deflection correlation as well. An initial load of 50 lbs was applied to the beam, then all dial gauges were set at zero. Test loads were then applied in various increments depending upon the predicted ultimate failure loads. As the loads increased, the deflections were noted and the crack patterns on the beams were marked. At failure, the maximum load, maximum deflection and the mode of failure were all recorded.
V. RESULTS

For clarity and conciseness, all results are presented in tables or in diagrams.

1. Table 1.

Table 1 shows the results of the correlation study between Mohammed A. Faris' beams (10) and the authors model beams. The numbers referred to in the beam designation column for the prototype are the same as Faris used in his thesis. The results varied from a low of 66.2 kips at failure for the beam with no stirrups to 151 kips for the beam with the largest diameter of stirrup and the shortest spacing between the stirrups.

All the beams in this table failed with a diagonal cracking mode. Beam 1 without any stirrups failed quite suddenly without any warning. All the remaining beams with stirrups gave indication of failure by showing various amounts of cracking prior to failure. As a rule, the closer the spacing and the larger the diameter of the stirrups, the more diagonal cracking prior to failure. The usual crack pattern started with flexural tensile cracks opening up vertically in the shear span zone, followed by flexural tensile cracks over the support in the region of negative moment. Then with increasing load the cracks on the bottom of the beam increased in number, width and length while bending over towards the central portion of the beam. The cracking and deflection increased until the diagonal cracks reached the
## TABLE 1: Test Results from the Modelling of Faris' Beams

<table>
<thead>
<tr>
<th>Beam</th>
<th>As sq.in.</th>
<th>Av sq.in.</th>
<th>S in.</th>
<th>b in.</th>
<th>d in.</th>
<th>h in.</th>
<th>fy ksi</th>
<th>fyv ksi</th>
<th>f'c ksi</th>
<th>E ksi x10^3</th>
<th>Ev ksi x10^3</th>
<th>Pu kips</th>
<th>Pu(theor) kips</th>
<th>% difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>I (prot.)</td>
<td>1.86</td>
<td>-</td>
<td>-</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>-</td>
<td>3090</td>
<td>30</td>
<td>-</td>
<td>57.9</td>
<td>66.2</td>
<td>+ 14.3</td>
</tr>
<tr>
<td>1 (model)</td>
<td>0.1164</td>
<td>-</td>
<td>-</td>
<td>1.76</td>
<td>3</td>
<td>3.75</td>
<td>62</td>
<td>-</td>
<td>3950</td>
<td>30</td>
<td>-</td>
<td>5.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III-B (prot)</td>
<td>1.86</td>
<td>0.1</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>71</td>
<td>3230</td>
<td>30</td>
<td>30</td>
<td>94.2</td>
<td>86.2</td>
<td>- 8.5</td>
</tr>
<tr>
<td>2 (model)</td>
<td>0.1164</td>
<td>0.00614</td>
<td>1</td>
<td>1.75</td>
<td>3</td>
<td>3.75</td>
<td>62</td>
<td>67</td>
<td>3580</td>
<td>30</td>
<td>27.8</td>
<td>7.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III-A (prot)</td>
<td>1.86</td>
<td>0.1</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>71</td>
<td>3110</td>
<td>30</td>
<td>30</td>
<td>92.6</td>
<td>99.4</td>
<td>+ 7.3</td>
</tr>
<tr>
<td>3 (model)</td>
<td>0.1164</td>
<td>0.00614</td>
<td>1.25</td>
<td>1.78</td>
<td>2.9</td>
<td>3.65</td>
<td>62</td>
<td>67</td>
<td>3510</td>
<td>30</td>
<td>27.8</td>
<td>7.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II-A (prot)</td>
<td>1.86</td>
<td>0.22</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>63.5</td>
<td>3070</td>
<td>30</td>
<td>30</td>
<td>121.4</td>
<td>102</td>
<td>- 15.9</td>
</tr>
<tr>
<td>4 (model)</td>
<td>0.1164</td>
<td>0.01222</td>
<td>1.25</td>
<td>1.79</td>
<td>2.9</td>
<td>3.65</td>
<td>62</td>
<td>53</td>
<td>2900</td>
<td>30</td>
<td>28.3</td>
<td>6.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II-C (prot)</td>
<td>1.86</td>
<td>0.22</td>
<td>2.5</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>63.5</td>
<td>3090</td>
<td>30</td>
<td>30</td>
<td>153.7</td>
<td>151</td>
<td>- 1.75</td>
</tr>
<tr>
<td>5 (model)</td>
<td>0.1164</td>
<td>0.01222</td>
<td>0.625</td>
<td>1.79</td>
<td>2.87</td>
<td>3.62</td>
<td>62</td>
<td>53</td>
<td>2780</td>
<td>30</td>
<td>28.3</td>
<td>7.66</td>
<td></td>
<td></td>
</tr>
<tr>
<td>II-B (prot)</td>
<td>1.86</td>
<td>0.22</td>
<td>4</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>63.5</td>
<td>3090</td>
<td>30</td>
<td>30</td>
<td>143</td>
<td>132</td>
<td>- 7.7</td>
</tr>
<tr>
<td>6 (model)</td>
<td>0.1164</td>
<td>0.01222</td>
<td>1</td>
<td>1.79</td>
<td>2.87</td>
<td>3.62</td>
<td>62</td>
<td>53</td>
<td>2710</td>
<td>30</td>
<td>28.3</td>
<td>6.5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>III-C (prot)</td>
<td>1.86</td>
<td>0.1</td>
<td>2.5</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>55</td>
<td>71</td>
<td>3140</td>
<td>30</td>
<td>30</td>
<td>115.9</td>
<td>129</td>
<td>+ 11.3</td>
</tr>
</tbody>
</table>
point of applied load or support by splitting longitudinally.

Overall cracking similitude between the model beams and their prototypes was in reasonable agreement, however it was noticed that the number of cracks in the prototype beams was approximately 4 times the number of cracks in the model beams.

This was to be expected for two reasons: 1. From similitude relationships, the strain in the shear spans of the prototype and model beams is approximately the same. 2. The shear span over which this cracking occurs is $\lambda$ or 4 times as large for the prototype than for the model and therefore 4 times as many cracks should occur in the prototype for ideal correlation.

2. Tables 2 and 3.

Tables 2 and 3 show the correlation results from the modelling of 10 of Professor Kani's beams fabricated at the University of Toronto. The beam designation numbers for the prototypes are identical to those used in his publications (8,9).

All beams failed in a diagonal mode as expected however beams with a/d ratios larger than 2.5 tended to fail suddenly, usually with a very loud crunching sound. Those beams with an a/d ratio less than 2.5 tended to fail more slowly and without the sudden unexpected jarring noise.

The cracking history of all ten beams in this series was strikingly similar. After the familiar flexural cracks had formed in the central
TABLE 2: Test Results from the Modelling of Prof. Kani's Beams (p = 1.88%)

<table>
<thead>
<tr>
<th>Beam</th>
<th>a/d</th>
<th>As sq.in.</th>
<th>b in.</th>
<th>d in.</th>
<th>h in.</th>
<th>fy ksi</th>
<th>f'c psi</th>
<th>E ksi x10^3</th>
<th>Pu kips</th>
<th>Pu(theor) kips</th>
<th>% difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>#24 (prot)</td>
<td>1.5</td>
<td>1.2</td>
<td>6</td>
<td>10.68</td>
<td>12</td>
<td>57.4</td>
<td>4040</td>
<td>30</td>
<td>81.8</td>
<td>89.4</td>
<td>+ 7.6</td>
</tr>
<tr>
<td>6 (model)</td>
<td>1.5</td>
<td>0.0776</td>
<td>1.49</td>
<td>2.65</td>
<td>2.97</td>
<td>62</td>
<td>3930</td>
<td>30</td>
<td>5.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#27 (prot)</td>
<td>2.5</td>
<td>1.2</td>
<td>6</td>
<td>10.68</td>
<td>12</td>
<td>57.4</td>
<td>4320</td>
<td>30</td>
<td>23.1</td>
<td>19.75</td>
<td>- 14.5</td>
</tr>
<tr>
<td>5 (model)</td>
<td>2.5</td>
<td>0.0776</td>
<td>1.54</td>
<td>2.68</td>
<td>3</td>
<td>62</td>
<td>3710</td>
<td>30</td>
<td>1.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#35 (prot)</td>
<td>3.53</td>
<td>1.18</td>
<td>6.11</td>
<td>10.61</td>
<td>12</td>
<td>71.2</td>
<td>3780</td>
<td>30</td>
<td>20.2</td>
<td>18.4</td>
<td>- 8.9</td>
</tr>
<tr>
<td>3 (model)</td>
<td>3.53</td>
<td>0.0776</td>
<td>1.48</td>
<td>2.64</td>
<td>2.96</td>
<td>62</td>
<td>3650</td>
<td>30</td>
<td>1.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#29 (prot)</td>
<td>4.5</td>
<td>1.2</td>
<td>6</td>
<td>10.68</td>
<td>12</td>
<td>50.8</td>
<td>3560</td>
<td>30</td>
<td>19.3</td>
<td>21.9</td>
<td>+ 11.9</td>
</tr>
<tr>
<td>1 (model)</td>
<td>4.5</td>
<td>0.0776</td>
<td>1.5</td>
<td>2.68</td>
<td>3</td>
<td>62</td>
<td>3290</td>
<td>30</td>
<td>1.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#32 (prot)</td>
<td>6</td>
<td>1.2</td>
<td>6</td>
<td>10.68</td>
<td>12</td>
<td>50.8</td>
<td>3530</td>
<td>30</td>
<td>17.8</td>
<td>18</td>
<td>+ 1.13</td>
</tr>
<tr>
<td>8 (model)</td>
<td>6</td>
<td>0.0776</td>
<td>1.5</td>
<td>2.68</td>
<td>3</td>
<td>62</td>
<td>3200</td>
<td>30</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Beam</td>
<td>a/d</td>
<td>As sq.in.</td>
<td>h in.</td>
<td>d in.</td>
<td>h in.</td>
<td>fy ksi</td>
<td>f'c psi</td>
<td>E ksi x10^3</td>
<td>Pu kips</td>
<td>Pu (theor) kips</td>
<td>% difference</td>
</tr>
<tr>
<td>------</td>
<td>-----</td>
<td>-----------</td>
<td>------</td>
<td>------</td>
<td>------</td>
<td>-------</td>
<td>--------</td>
<td>-------------</td>
<td>--------</td>
<td>----------------</td>
<td>--------------</td>
</tr>
<tr>
<td>#87 (prot)</td>
<td>1.01</td>
<td>1.75</td>
<td>6.06</td>
<td>10.61</td>
<td>12</td>
<td>53.1</td>
<td>3950</td>
<td>30</td>
<td>107.7</td>
<td>111.0</td>
<td>+ 3.1</td>
</tr>
<tr>
<td>7 (model)</td>
<td>1.01</td>
<td>0.11</td>
<td>1.5</td>
<td>2.65</td>
<td>2.97</td>
<td>58</td>
<td>3560</td>
<td>30</td>
<td>6.21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#100 (prot)</td>
<td>2.02</td>
<td>1.76</td>
<td>6.03</td>
<td>10.62</td>
<td>12</td>
<td>53.1</td>
<td>3950</td>
<td>30</td>
<td>50.3</td>
<td>54.7</td>
<td>+ 8.75</td>
</tr>
<tr>
<td>10 (model)</td>
<td>2.02</td>
<td>0.11</td>
<td>1.51</td>
<td>2.68</td>
<td>3</td>
<td>58</td>
<td>3810</td>
<td>30</td>
<td>3.45</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#95 (prot)</td>
<td>2.47</td>
<td>1.80</td>
<td>6.04</td>
<td>10.83</td>
<td>12</td>
<td>49</td>
<td>3670</td>
<td>30</td>
<td>32.7</td>
<td>35.5</td>
<td>+ 8.55</td>
</tr>
<tr>
<td>4 (model)</td>
<td>2.47</td>
<td>0.11</td>
<td>1.5</td>
<td>2.69</td>
<td>3.01</td>
<td>58</td>
<td>3680</td>
<td>30</td>
<td>2.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#84 (prot)</td>
<td>4.01</td>
<td>1.80</td>
<td>5.95</td>
<td>10.67</td>
<td>12</td>
<td>49.6</td>
<td>3980</td>
<td>30</td>
<td>24.9</td>
<td>27.4</td>
<td>+ 10.1</td>
</tr>
<tr>
<td>2 (model)</td>
<td>4.01</td>
<td>0.11</td>
<td>1.48</td>
<td>2.69</td>
<td>3.01</td>
<td>58</td>
<td>3790</td>
<td>30</td>
<td>1.72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#91 (prot)</td>
<td>6.05</td>
<td>1.74</td>
<td>6.08</td>
<td>10.58</td>
<td>12</td>
<td>52.8</td>
<td>3980</td>
<td>30</td>
<td>22.91</td>
<td>19.75</td>
<td>- 13.8</td>
</tr>
<tr>
<td>9 (model)</td>
<td>6.05</td>
<td>0.11</td>
<td>1.51</td>
<td>2.67</td>
<td>2.99</td>
<td>58</td>
<td>3890</td>
<td>30</td>
<td>1.26</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 3: Test Results from the Modelling of Prof. Kani's Beams (p = 2.73%)
BEAM NO. 2 (FARIS' CORRELATION)

BEAM NO. 3 (FARIS' CORRELATION)
BEAM NO. 1 (KANI'S CORRELATION)

BEAM NO. 10 (KANI'S CORRELATION)
BEAM NO. 8 (KANI’S CORRELATION)
portion of the beam where the bending moment was the greatest, the first cracks in the shear spans developed. While the cracks in the central portion were mostly vertical, the cracks appearing in the shear span started vertically and then tended to follow orthogonal directions to the tensile stress trajectories by bending over in the general direction of the applied loads. Failure occurred when the crack carried over to the point of loading or splitting along the tensile steel to the reaction.

In the shortest beam manufactured (No. 7) the beam failed due to anchorage slipping; this was discovered when upon examining the end section of the beam, the longitudinal steel had been found to split away from the 1\(\frac{1}{2}\)" x 3" rectangular steel plates causing much cracking and spalling in this end region. The result of this can be seen in Fig. 14 where the result with the a/d ratio = 1 is lower than expected.

Correlation of ultimate loads between model and prototype beams varied between + 11.9% for beam No.1 (p = 1.88% and a/d = 4.5) to -14.5% for beam No.5 (p = 1.88% and a/d = 2.5).

3. Load Deflection Curves.

Figures 6 to 12 show the correlation between the center deflection of the beams in the modelling of Faris' results.

To make the results more meaningful, the load deflection curves for the prototype and its model have been plotted on the same graph. In each case for the model beam deflection results, the applied loads have been multiplied by \(\lambda^2\) and the deflection by \(\lambda\) for each
LOAD DEFLECTION CURVES

PROTOTYPE I

MODEL I

Pu (Theor.) from Model calculations

Fig. 6

LOAD IN KIPS

DEFLECTION IN INCHES
LOAD DEFLECTION CURVES

PROTOTYPE III-8

MODEL 2

Fig. 7

Pu (Theor.) from Model calculations

LOAD IN KIPS

DEFLECTION IN INCHES
LOAD DEFLECTION CURVES

PROTOTYPE II-A
MODEL 3

Pu (Theor.) from Model calculations

Fig. 8
LOAD DEFLECTION CURVES

PROTOTYPE II-A

MODEL 4

Pu (Theor.) from Model calculations

Fig. 9

LOAD IN KIPS

DEFLECTION IN INCHES
LOAD DEFLECTION CURVES

PROTOTYPE II-C

MODEL 5

LOAD IN KIPS

0.1 0.2 0.3 0.4
DEFLECTION IN INCHES

Pu (Theor.) from Model calculations

Fig. 10
LOAD DEFLECTION CURVES

PROTOTYPE \( \Pi - \delta \)

MODEL \( \delta \)

Pu (Theor.) from Model calculations

Fig. 11
LOAD DEFLECTION CURVES

PROTOTYPE III - C

MODEL 7

Pu (Theor.) from Model calculations

Fig. 12
reading during the incremental loading. The load deflection data for the prototype beams have been copied from similar graphs in Mr. Faris' thesis(10).

4. Plot of Test Results on a/d Ratios.

Figures 13 and 14 show the actual test results from Tables 2 and 3 as plotted on an Mu/Mfl ordinate axis and an a/d abscissa axis.

As shown all beams failed below their full flexural capacity and all showed the peculiar "valley of diagonal tension" which is typical of the beams tested over this a/d range. The effect of the premature anchorage failure can be seen readily from Fig. 14 where the beam with the a/d ratio of 1 instead of failing at a much higher load, failed at only 59% of its full flexural capacity.

5. Correlation of Published Data With Kani's Theory.

The remainder of this chapter on results is an attempt to superimpose independent test results from various researchers on the predicted values based on Professor Gaspar Kani's shear mechanism theory. In each case, the lines from the capacity of the concrete teeth and the capacity line from the remaining arch in Kani's explanation have been drawn to outline "the valley of diagonal tension." This "valley" is caused by the apparent change in the moment at failure of the beam as various beams which increasing a/d ratios are tested.

Plot of T.W. Lee's Results:
PLOT OF $\mu/M_f$ VALUES FOR VARIOUS $a/d$ RATIOS

FOR $p = 1.88\%$

$M_f = \text{FLEXURAL CAPACITY OF BEAM}$

$\mu = \text{LOCAL AXIAL STRESSES}$

KANI'S VALLEY OF DIAGONAL TENSION

Fig. 13
PLOT OF $\frac{\mu}{M_f l}$ VALUES FOR VARIOUS $a/d$ RATIOS

FOR $p = 2.73\%$

$M_f l$ = FLEXURAL CAPACITY OF BEAM

KANI'S VALLEY OF DIAGONAL TENSION

Fig. 14
In 1965 at McGill University, T.W. Lee published a master's thesis (12) entitled "Shear Strength of Reinforced Concrete Beams". In this thesis, 8 rectangular beams with an $f'_c$ of 3,080 psi were tested with $a/d$ ratios ranging from 2.48 to 6.2. Two different percentages of longitudinal steel, 1.35% and 2.65% respectively. The overall dimensions of the beams were 7" x 12" x 14'.

Sample calculations for the series of beams with $p = 2.65\%$ are given in the appendix based on the actual data as recorded in the thesis.

It can be seen that two of the beams failed at an $a/d$ ratio higher than the transition point and are therefore classed as flexural failures.

All results have been shown in Figs. 15 and 16.

**Plot of Moody, Viest, Elstner and Hognestad Results:**

A series of 16 rectangular beams with a constant percentage of longitudinal steel ($p = 1.89\%$) were tested by the above researchers in 1964 (13). Cross section dimensions were kept constant at 6" x 12" and a similar $a/d$ ratio of 3.41 was used for all tests. The significant part of these experiments was that the compressive strength of the concrete was varied from 1770 psi to 5970 psi. The results are plotted in Figures 17 and 18.

**Plot of Morrow and Viest Results:**

At the University of Illinois in 1957 a paper (14) entitled
"Shear Strength of Reinforced Concrete Frame Members Without Web Reinforcement" was published. In this of experiments 16" x 12" beams with a constant p of 1.85%, a series of beams were tested with varying a/d ratios of 1.0 up to 7.8. A concrete strength of 4000 psi was used throughout the study. Test results are plotted in Fig. 19.

Plot of Results from the University of Stuttgart:

Leonhardt and Walther (15) in 1961 published their "Contribution to the Treatment of Shear Problems in Reinforced Concrete". These tests covered a range of a/d values from 1.0 to 8.0 with a constant compressive strength of 4300 psi. All 12.6" x 7.5" beams had a constant p of 2.07%.
PLOT OF T.W. LEE'S RESULTS (p = 2.65%) SUPERIMPOSED ON KANI'S
DIAGONAL TENSION ENVELOPES

Mf1 = FLEXURAL CAPACITY OF BEAM

CAPACITY LINE OF REMAINING ARCH

CAPACITY LINE OF CONCRETE TEETH

LEE'S RESULTS

Fig. 15
PLOT OF T.W. LEE'S RESULTS ($p = 1.35\%$) SUPERIMPOSED ON KANI'S DIAGONAL TENSION ENVELOPES

$M_{fl} = \text{FLEXURAL CAPACITY OF BEAM}$

$\text{CAPACITY LINE OF CONCRETE TEETH}$

$\text{CAPACITY LINE OF REMAINING ARCH}$

Lee's Results

Fig. 16
PLOT OF MOODY, VIEST, ELSTNER AND HOGNESTAD RESULTS
SUPERIMPOSED ON KANI'S DIAGONAL TENSION ENVELOPES

Mf1 = FLEXURAL CAPACITY OF BEAM

CAPACITY LINES OF CONCRETE TEETH
FOR f'c = 5310 psi
f'c = 4790 psi

CAPACITY LINES OF REMAINING ARCH

Fig. 17
PLOT OF MOODY, VIEST, ELSTNER AND HOGNESTAD RESULTS
SUPERIMPOSED ON KANI'S DIAGONAL TENSION ENVELOPES

Mf1 = FLEXURAL CAPACITY OF BEAM

CAPACITY LINES OF REMAINING ARCH

CAPACITY LINES OF CONCRETE TEETH

FOR f'c = 2860 psi
f'c = 2330 psi

Fig. 18
PLOT OF MORROW AND VIEST RESULTS

Mfl = FLEXURAL CAPACITY

Fig. 19
PLOT OF UNIVERSITY OF STUTTGART RESULTS

$M_{fl} = \text{FLEXURAL CAPACITY OF BEAM}$
VI. DISCUSSION OF RESULTS AND CONCLUSIONS

1. Testing of Model Beams.

From Tables 1, 2 and 3, it is seen that all beams tested at the ½ scale failed between a percentage deviation of +14.3% and -15.9%. These deviations were both obtained from the correlation of Mohammed A. Faris' thesis, however, a general scatter of results between the separate modelling studies was noted.

As may be expected, the series based on Kani's results proved to be somewhat more accurate in correlation studies. Only two beams out of ten in Kani's study had larger deviations than 12%, while two out of only seven of Faris' results showed the same deviation. This was probably due to the fact that Kani's beams involved fewer variables i.e. no compressive steel, no web reinforcement and simpler loading system. One other factor could also be involved in the larger deviations. All of Faris' beams were reported as being exactly 7" x 12" in cross-section for the b and d dimensions. It is doubtful if all beams were fabricated precisely to these dimensions, and as both terms appear in the correlation equations (3) and (4), a probable error was introduced here.

All model beams failed as planned, i.e. with a diagonal cracking mode. As expected, the presence of stirrups materially increased the resistance of the beams to shear failure. This increase amounted to 228% when considering beam 1 (without stirrups) and beam 5 (smallest spacing and largest diameter of stirrups). The fact that the beam without web reinforcing in Faris' study (a/d = 2.83) failed very
suddenly, while the beams with stirrups failed more gradually is also significant from a designer's point of view. It is obvious that sudden failures in reinforced concrete structures must be avoided.

The correlation of Kani's work again demonstrated that the shear span to depth ratio has a direct bearing on the ultimate shear capacity of the beam. In the practical sense, this means that the shear stress at failure for the same beam varies considerably instead of being a constant as was once believed. The author believes that this variation in shear stress as calculated by the nominal formula for members of constant depth

\[ V_u = \frac{Vu}{bd} \]

is much too simplified and as seen from the curves (Fig. 13 and 14) leads to very inconsistent safety factors. It is to be hoped that with further research, the American Concrete Institute and the National Building Code of Canada can adopt a more rational approach to shear in their future editions.

2. Correlation Equations.

Equations (1), (2), (3) and (4) must be recognized as attempts to provide satisfactory correlations between prototype and model beams failing in shear.

It was pointed out previously that the variables involved in these equations are, in the author's view, the most important factors in this complex mechanism. It is conceivable that more factors, i.e.
dowel action of longitudinal steel, relative values of actual bond stress, support conditions and applied loading techniques should also be considered. This could be the subject of future research. Nevertheless, the author believes that the dimensional analysis approach to model study, where many factors are involved in a complex phenomenon, is the correct method to use for valid correlation results.

The simpler method of proposing that applied loads vary only as \( \lambda^2 \) is not valid for this type of analysis since direct modelling i.e. the same \( f' \) t, the exact model dimensions as given by

\[
\frac{L}{L_m} = \lambda, \quad \frac{b}{b_m} = \lambda, \quad \frac{d}{d_m} = \lambda
\]

and the exact steel relationship of \( \frac{A_s}{A_{sm}} = \lambda^2 \) are virtually impossible to attain exactly. It is suggested that valid results from this direct approach are only obtained when the number of factors considered in the phenomenon are few and/or compensating errors result in a chance cancelling of the variables contributions to the final result.

As a demonstration of the above argument, Table 4 was prepared. This table shows the comparison between the simplified approach \( (P = P_m \lambda^2) \) and the author's dimensional analysis approach using equations 3 and 4.

It is apparent that the simplified method for the analysis of ultimate shear in model beam correlation is not as accurate as the method used using the derived equations. This is especially true for the study of Faris' beams which had a total of nine variables for the case where the beams had web reinforcement and compressive steel. Percentage difference
TABLE 4: Table Showing Comparison Between Simplified Approach (\(P = P_{mu} \leq 2\)) and Author's Dimensional Analysis Approach (Equations 3 and 4)

<table>
<thead>
<tr>
<th>Beam</th>
<th>(P_{u}) (simplified)</th>
<th>(P_{u}(Eqs. 3 or 4))</th>
<th>% diff. (simplified)</th>
<th>% diff. (Eqs. 3 or 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I (Faris)</td>
<td>57.9</td>
<td>66.2</td>
<td>+43.5</td>
<td>+14.3</td>
</tr>
<tr>
<td>III-B (Faris)</td>
<td>94.2</td>
<td>86.2</td>
<td>+32.7</td>
<td>-8.5</td>
</tr>
<tr>
<td>III-A (Faris)</td>
<td>92.6</td>
<td>99.4</td>
<td>+24.2</td>
<td>+7.3</td>
</tr>
<tr>
<td>II-A (Faris)</td>
<td>121.4</td>
<td>102</td>
<td>-15.6</td>
<td>-15.9</td>
</tr>
<tr>
<td>II-C (Faris)</td>
<td>153.7</td>
<td>151</td>
<td>-20.6</td>
<td>-1.75</td>
</tr>
<tr>
<td>II-B (Faris)</td>
<td>143</td>
<td>132</td>
<td>-27.3</td>
<td>-7.7</td>
</tr>
<tr>
<td>III-C (Faris)</td>
<td>115.9</td>
<td>129</td>
<td>+17.3</td>
<td>+11.9</td>
</tr>
<tr>
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<td>81.8</td>
<td>89.4</td>
<td>-0.4</td>
<td>+7.6</td>
</tr>
<tr>
<td>#27 (Kani)</td>
<td>23.1</td>
<td>19.75</td>
<td>-29.4</td>
<td>-14.5</td>
</tr>
<tr>
<td>#35 (Kani)</td>
<td>20.2</td>
<td>18.4</td>
<td>-16.8</td>
<td>-8.9</td>
</tr>
<tr>
<td>#29 (Kani)</td>
<td>19.3</td>
<td>21.9</td>
<td>+3.6</td>
<td>+11.9</td>
</tr>
<tr>
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<td>17.8</td>
<td>18</td>
<td>-10.0</td>
<td>+1.13</td>
</tr>
<tr>
<td>#87 (Kani)</td>
<td>107.7</td>
<td>111</td>
<td>-7.8</td>
<td>+3.1</td>
</tr>
<tr>
<td>#100 (Kani)</td>
<td>50.3</td>
<td>54.7</td>
<td>+9.94</td>
<td>+8.75</td>
</tr>
<tr>
<td>#95 (Kani)</td>
<td>32.7</td>
<td>35.5</td>
<td>+7.96</td>
<td>+8.55</td>
</tr>
<tr>
<td>#84 (Kani)</td>
<td>24.9</td>
<td>27.4</td>
<td>+15.7</td>
<td>+10.1</td>
</tr>
<tr>
<td>#91 (Kani)</td>
<td>22.91</td>
<td>19.75</td>
<td>-16.6</td>
<td>-13.8</td>
</tr>
</tbody>
</table>
using the simplified method varied from + 43.5% to - 27.3% while the
dimensional analysis results varied from only + 14.3% to - 15.9%. In
the modelling of Kani's beams the direct method gave more reasonable
answers varying from + 15.7% to - 29.4%. This was due to the fact
that only six variables were involved. Even so, the dimensional
analysis gave better results with ultimate shear values varying only
between + 11.9% and - 14.5%.

The conclusion is that the dimensional analysis approach
yields more accurate results when distorted models are considered for
correlation. It would also yield more accurate results the greater
the distortion and/or the more variables that are considered in the
phenomenon.

It should be noted that in the author's equations 3 and 4,
that should all the quantities considered in the ultimate shear rela-
tions between model and prototype beams be faithfully modelled i.e.
precisely, then these equations reduce to the familiar relationship.

\[ P = P_m \lambda^2 \]

As a final study of the results of the experiments, it would
be interesting to examine the distortion caused by the individual non-
dimensional terms used in correlating the model beam results. Consider
equation 3 (no web reinforcement)

\[ \left( \frac{d m}{b m} \right) \left( \frac{d^2 m}{A_s m} \right) \left( \frac{f'_t m}{E m} \right) \left( \frac{f'_t m d^2 m}{P m} \right) = \left( \frac{d}{b} \right) \left( \frac{d^2}{A_s} \right) \left( \frac{f'_t}{E} \right) \left( \frac{f'_t d^2}{P} \right) \]

or
(3a)

\[
\left( \frac{d}{b} \right) \frac{d^2}{bm} \left( \frac{A_s}{2m} \right) \left( \frac{f't}{Em} \right) \left( \frac{f't - d^2}{P} \right) = 1
\]

Now for exact or non-distorted modelling each of the four bracketed terms above would be equal to unity. The amount that each term varies from unity can therefore be called its amount of distortion.

A similar approach can be used on equation 4 (with web reinforcement) which yields

(4a)

\[
\left( \frac{d}{b} \right) \frac{f't}{Em} \left( \frac{f't}{Em} \right) \left( \frac{d^2}{A_s} \right) \left( \frac{d^2}{2m} \right) \left( \frac{f't}{f'y} \right) \left( \frac{f't - d^2}{f'y} \right) = 1
\]

By calculating the individual terms in equations 3a and 4a for the prototype and model beams involved, Tables 5 and 6 were tabulated giving all the distortion values for all the beams.

A total of 86 distortion values were then considered and it is seen that:

1. For beams with no web reinforcement (Table 5), the values ranged from 0.885 to 1.115 with only 2 values falling below 0.90 and one value being above 1.10.

2. For beams with web reinforcement (Table 6), the values ranged from 0.856 to 1.35 with only 3 results being less than 0.88 and 4 results being above 1.10.

It is seen from these tables that the factor causing the most
distortion was \( f't \). The \( f't \) factor or tensile strength of concrete factor was based primarily on the results from the cylinder compressive tests and as such it points out the effect of a variance in concrete strengths between models and prototypes.

Nevertheless, if distortion values can be considered as a measure of control in the experiments, it is concluded that careful planning, fabrication and testing in this project resulted in a reasonable level of control throughout the experiments.

3. Load Deflection Curves.

For perfect correlation between model and prototype beams made of the same material, the relationship for deflection should be

\[
\delta = \lambda \delta_m \quad \text{where the} \ \delta's
\]

are the respective deflections of prototype and model beams.

However, in most cases perfect similarity in beam geometrics, support conditions, and mechanical properties cannot be completely obtained, so, as in the comparison of ultimate shear loads between model and prototype beams, some discrepancies in the relative deflections must also be expected.

However, all the results from the comparison of the seven beams (see Figs. 6 to 12.) generally follow the same deflection pattern. In the region of small loads (no cracking), the load deflection curve is almost a straight line, however as the loading increases and cracking occurs, the curve starts bending over more and more exhibiting a ductile
TABLE 5: Distortion Values for Beams with no Web Reinforcement

<table>
<thead>
<tr>
<th>Beam</th>
<th>Term</th>
<th>$\frac{d}{b_{\text{dm}}}$</th>
<th>$\frac{d^2}{A_{\text{sm}}}$</th>
<th>$\frac{f'\cdot t}{E}$</th>
<th>$\frac{f'\cdot t \cdot d^2}{P_{\text{m}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (Faris)</td>
<td>1.005</td>
<td>1.002</td>
<td>0.885</td>
<td>1.115</td>
<td></td>
</tr>
<tr>
<td>6 (Kani)</td>
<td>1.001</td>
<td>1.045</td>
<td>1.012</td>
<td>0.941</td>
<td></td>
</tr>
<tr>
<td>5 (Kani)</td>
<td>1.021</td>
<td>1.025</td>
<td>1.075</td>
<td>0.889</td>
<td></td>
</tr>
<tr>
<td>3 (Kani)</td>
<td>0.976</td>
<td>1.068</td>
<td>1.018</td>
<td>0.945</td>
<td></td>
</tr>
<tr>
<td>1 (Kani)</td>
<td>0.995</td>
<td>1.025</td>
<td>1.04</td>
<td>0.942</td>
<td></td>
</tr>
<tr>
<td>8 (Kani)</td>
<td>0.994</td>
<td>1.028</td>
<td>1.051</td>
<td>0.931</td>
<td></td>
</tr>
<tr>
<td>7 (Kani)</td>
<td>0.992</td>
<td>1.012</td>
<td>1.05</td>
<td>0.945</td>
<td></td>
</tr>
<tr>
<td>10 (Kani)</td>
<td>0.991</td>
<td>1.019</td>
<td>1.017</td>
<td>0.975</td>
<td></td>
</tr>
<tr>
<td>4 (Kani)</td>
<td>1.002</td>
<td>0.996</td>
<td>0.997</td>
<td>1.002</td>
<td></td>
</tr>
<tr>
<td>2 (Kani)</td>
<td>0.986</td>
<td>0.964</td>
<td>1.026</td>
<td>1.027</td>
<td></td>
</tr>
<tr>
<td>9 (Kani)</td>
<td>0.981</td>
<td>0.989</td>
<td>1.012</td>
<td>1.019</td>
<td></td>
</tr>
</tbody>
</table>
TABLE 6: Distortion Values for Beams with Web Reinforcement

<table>
<thead>
<tr>
<th>Beam</th>
<th>Term</th>
<th>( \frac{d}{d} )</th>
<th>( \frac{f't}{E} )</th>
<th>( \frac{f't}{Em} )</th>
<th>( \frac{f't}{Ev} )</th>
<th>( \frac{d^2}{As} )</th>
<th>( \frac{d^2}{Avm} )</th>
<th>( \frac{f't}{fvy} )</th>
<th>( \frac{f't}{fyv} )</th>
<th>( \frac{f't}{P} )</th>
<th>( f'td^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 (Faris)</td>
<td>1.00</td>
<td>0.95</td>
<td>0.882</td>
<td>1.002</td>
<td>0.981</td>
<td>0.897</td>
<td>1.35</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (Faris)</td>
<td>1.052</td>
<td>0.942</td>
<td>0.871</td>
<td>1.072</td>
<td>1.05</td>
<td>0.89</td>
<td>1.155</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 (Faris)</td>
<td>1.06</td>
<td>1.028</td>
<td>0.954</td>
<td>1.072</td>
<td>0.952</td>
<td>0.856</td>
<td>1.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5 (Faris)</td>
<td>1.07</td>
<td>1.055</td>
<td>0.995</td>
<td>1.098</td>
<td>0.975</td>
<td>0.882</td>
<td>1.058</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 (Faris)</td>
<td>1.07</td>
<td>1.067</td>
<td>1.005</td>
<td>1.095</td>
<td>0.972</td>
<td>0.89</td>
<td>0.877</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7 (Faris)</td>
<td>1.00</td>
<td>1.018</td>
<td>0.942</td>
<td>1.002</td>
<td>0.98</td>
<td>0.96</td>
<td>1.105</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
characteristic until the ultimate load is reached.

It is seen that for the mid-span deflections measured, a reasonable agreement can be expected in load-deflection characteristics. This is especially so in the linear deformation region. After cracking or when the curve becomes non-linear greater discrepancies occur between the model and prototype beams.

Model beam No.1 (Fig. 4.) which had no web reinforcement failed suddenly just after the linear portion of the load-deflection curve changed to the non-linear portion. Model beam No.5 (Fig. 5.) which had the closest spacing and the largest diameter of stirrups exhibited a marked non-linear portion approaching a ductile characteristic at ultimate load.

It can be concluded that stirrups in reinforced concrete beams besides the obvious purpose of providing additional beam strength in shear, also give the beam more ductility before failure which is desirable to avoid sudden catastrophes.

4. Conclusions From Model Study.

As seen from the results obtained and within the limits of the tests performed, it would indicate that beams failing in shear can be modelled successfully within very reasonable limits (+14.3% to -15.9%).

5. Kani's Theory and Its Relation to Other Published Data.

Professor Gaspar Kani has made a valid contribution to the
understanding of the shear mechanism within a reinforced concrete beam. His theory is based on observations that most researchers have long observed in loading histories of hundreds of beams, i.e. gradual formation of concrete teeth, propagation of inclined cracks and then failure either suddenly if above the critical a/d ratio or more slowly if below. Unfortunately certain aspects of his explanation of the internal mechanism are not completely accepted by all structural engineers. It could be that in Kani's attempt to rationalize and simplify the mechanism involved he inadvertently either leaves out certain aspects of the phenomenon or he bases his mathematical analysis on certain assumptions that are not necessarily valid.

Nevertheless, Professor Kani's laboratory results bear out his theory and from the limited examination of other data available by independent researchers, their results also generally follow Kani's proposals. As can be seen from Figs. 15 and 20 in this limited study, it can be concluded that Dr. Kani's theory is in general agreement with each researcher's results. It is shown from the graphs that the lower the compressive strength, the larger and deeper the "valley of diagonal tension" becomes. Also, by increasing the percentage of longitudinal steel, the a/d min. point is shifted to a higher value and the a/d TR. point is also subsequently extended to a higher value.
APPENDIX

Sample Calculations

1. Beams Without Stirrups
   i.e. Prototype Beam I
   Model Beam 1

From (3)

\[
\left( \frac{d}{b} \right) \left( \frac{d^2}{A_s} \right) \left( \frac{\sqrt{f'c_m}}{E_m} \right) \left( \frac{\sqrt{f'c_m} \cdot d^2}{P_m} \right) = \left( \frac{d}{b} \right) \left( \frac{d^2}{A_s} \right) \left( \frac{\sqrt{f'c}}{E} \right) \left( \frac{\sqrt{f'c} \cdot d^2}{P \text{ (theor)}} \right)
\]

\[P \text{ (theor.)} = \left( \frac{d}{b} \right) \left( \frac{d^2}{A_s} \right) \left( \frac{\sqrt{f'c}}{E} \right) \left( \sqrt{f'c} \cdot d^2 \right) \left( \frac{b_m}{d_m} \right) \left( \frac{A_s}{A_s} \right) \left( \frac{E_m}{E_m} \right) \left( \frac{P_m}{P_m} \right)
\]

\[= \left( \frac{12}{7} \right) \left( \frac{144}{1.86} \right) \left( \frac{\sqrt{3090}}{30 \times 10^6} \right) \left( \frac{3090}{30 \times 10^6} \right) 144 \left( \frac{1.76}{3.00} \right) \left( \frac{0.1164}{9.0} \right) \left( \frac{30 \times 10^6}{3950} \right) \left( \frac{5200}{3950 \times 9} \right)
\]

\[= 66.2 \text{kips}
\]

\[% \text{ difference} = \frac{66.2 - 57.9}{57.9} \times 100 = +14.3\%
\]

2. Beams With Stirrups
   i.e. Prototype Beam III-B
   Model Beam 2

From (4)

\[
\left( \frac{d}{b} \right) \left( \frac{\sqrt{f'c_m}}{E_m} \right) \left( \frac{\sqrt{f'c_m} \cdot d^2}{E \cdot A_s} \right) \left( \frac{d^2_m}{A_s} \right) \left( \frac{\sqrt{f'c_m} \cdot d^2_m}{P_m} \right) = \left( \frac{d}{b} \right) \left( \frac{\sqrt{f'c_m} \cdot d^2_m}{P_m} \right) \left( \frac{\sqrt{f'c_m} \cdot d^2_m}{P_m} \right)
\]
3. Calculations For Beams With Varying a/d Ratios

Full Flexural Capacity = MFL = As fy jd

Maximum Bending Moment at Failure = Mu = Ultimate Moment in

Midspan Cross-Section at Failure = \( \frac{P_u}{2} \times a \)

\[
P = 1.88\%
\]
\[
As = 0.0776 \text{ sq.in.}
\]
\[
fy = 62,000 \text{ psi}
\]
\[
MFL = 0.0776 \times 62,000 \times 0.875 \times 2.68 = 11,280 \text{ kip inches}
\]
\[
a/d = 1.5
\]
\[
Mu = 5.1/2 \times 1000 \times 3.98 = 10,150 \text{ kip inches}
\]
\[ \frac{\text{Mu}}{\text{MFL}} = \frac{10,150}{11,280} \times 100 = 90\% \]
\[ \frac{a/d}{= 2.5} \]
\[ \text{Mu} = \frac{1.02}{2} \times 1,000 \times 6.7 = 3,420 \text{ kip inches} \]
\[ \frac{\text{Mu}}{\text{MFL}} = \frac{3,420}{11,280} \times 100 = 30.3\% \]
\[ \frac{a/d}{= 3.53} \]
\[ \text{Mu} = \frac{1.05}{2} \times 1,000 \times 9.32 = 4,900 \text{ kip inches} \]
\[ \frac{\text{Mu}}{\text{MFL}} = \frac{4,900}{11,280} \times 100 = 43.5\% \]
\[ \frac{a/d}{= 4.5} \]
\[ \text{Mu} = \frac{1.25}{2} \times 1,000 \times 12.5 = 7,820 \text{ kip inches} \]
\[ \frac{\text{Mu}}{\text{MFL}} = \frac{7,820}{11,280} \times 100 = 69.4\% \]
\[ \frac{a/d}{= 6.0} \]
\[ \text{Mu} = \frac{1.0}{2} \times 1,000 \times 16.1 = 8,050 \text{ kip inches} \]
\[ \frac{\text{Mu}}{\text{MFL}} = \frac{8,050}{11,280} \times 100 = 71.5\% \]

4. Plot of T.W. Lee's Results (P = 2.65%) Superimposed on Kani's Diagonal Tension Envelopes

From Kani's theory:
\[ (a/d) TR = \frac{6p fy s_T}{ft'(\Delta x)} \]

where
\[ p = \frac{As}{bd} \]
\[ fy = \text{yield point of longitudinal steel} \]
\[ ft' = \text{tensile strength of concrete} \]
\[ s_T = \text{average length of crack} \]
\[ \Delta x = \text{distance between cracks} \]
\[ \Delta x = \frac{36 + 24 + 18}{3 + 3 + 2} = 9.75 \]
\[ s_T = 4 \]
\[ \frac{a/d}{TR} = \frac{6 \times 2.65 \times 49,200 \times 4}{100 \times 462 \times 9.75} = 6.95 \]

and \( (a/d)_{\text{min}} = \sqrt{\frac{(a/d)_{TR}}{0.9}} = \sqrt{\frac{6.95}{0.9}} = 2.78 \)

For portion of curve where \( a/d \ll (a/d)_{\text{min}} \).

\[ \text{MCR} = \frac{MFL}{0.9} \cdot \frac{d}{a} \]

at \( a/d = 2.0 \) \( \text{MCR} = \frac{752}{0.9} \times \frac{1}{2} = 418 \text{ kip inches} \)
BIBLIOGRAPHY


