APPLICATION OF DUAL INTEGRAL EQUATIONS 
TO DIFFRACTION PROBLEMS 

by 
Man Fong Yan, B.Eng. 

Department of Electrical Engineering
APPLICATION OF DUAL INTEGRAL EQUATIONS
TO DIFFRACTION PROBLEMS

by

Man Fong Yan, B. Eng.

A thesis submitted to the Faculty of Graduate Studies
and Research in partial fulfillment of the
requirements for the degree of
Master of Engineering.

Department of Electrical Engineering,
McGill University,
Montreal, Quebec,
ABSTRACT

A rigorous technique is proposed for the solution of electromagnetic wave diffraction by circular aperture, multiple annular apertures, single strip and multiple strips in terms of an electric Hertz vector and a magnetic Hertz vector. It is shown that the mixed boundary conditions lead to multiple integral equations, which can be solved by a Fredholm integral equation of the second kind by using the modified operator of the Hankel transform and the Erdelyi-Kober fractional integration operator. Calculations are presented and field maps shown for a circular aperture 12 wavelengths in diameter.

Man-Fong YAN

Electrical Engineering

Master of Engineering
ACKNOWLEDGEMENTS

The author wishes to express his sincere gratitude to Dr. T. J. F. Pavlasek for his continued guidance and encouragement throughout the course of this research.

The author is deeply grateful to Dr. Lan Jen Chu for giving the original impetus for this subject.

Special thanks are also due to Mr. J. P. Legendre and Mr. I. A. Cermak for their many helpful discussions and excellent graph plotting subroutines.

Grateful acknowledgement is made to the National Research Council of Canada for financial support of the research.
CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>GENERAL TREATMENT OF MULTIPLE INTEGRAL EQUATIONS WITH ARBITRARY WEIGHTING</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FUNCTION AND HANKEL KERNEL</td>
<td>6</td>
</tr>
<tr>
<td>2.1</td>
<td>Solution Of Dual Integral Equations</td>
<td>6</td>
</tr>
<tr>
<td>2.2</td>
<td>Solution Of Multiple Integral Equations</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>DIFRACTION BY STRIPS</td>
<td>10</td>
</tr>
<tr>
<td>3.1</td>
<td>Diffraction By A Strip</td>
<td>11</td>
</tr>
<tr>
<td>3.2</td>
<td>Two Identical Strips</td>
<td>12</td>
</tr>
<tr>
<td>3.3</td>
<td>An Array Of Strips</td>
<td>14</td>
</tr>
<tr>
<td>3.4</td>
<td>Discussion</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>DIFRACTION BY AXIALLY SYMMETRIC PLANAR APERTURE</td>
<td>17</td>
</tr>
<tr>
<td>4.1</td>
<td>Dual Integral Equations For The Circular Aperture</td>
<td>17</td>
</tr>
<tr>
<td>4.2</td>
<td>Diffraction By A Circular Aperture—Normal Incidence</td>
<td>23</td>
</tr>
<tr>
<td>4.3</td>
<td>The Annular Aperture—Normal Incidence</td>
<td>26</td>
</tr>
</tbody>
</table>
4.4 The Boundary Conditions On The Rim Of The Aperture 30

CHAPTER 5 NUMERICAL ANALYSIS 33

5.1 Asymptotic Evaluation Of The Field 33
5.2 Numerical Calculation 36

CHAPTER 6 CONCLUSION 46

APPENDIX A 50

A.1 The Hankel Inversion Theorem 50
A.2 The Modified Operator Of Hankel Transform 50
A.3 The Erdelyi-Kober Operator 51

APPENDIX B 53

B.1 The Approximate Solution Of The Fredholm Integral Equation Of The Second Kind : Replacement By A Finite System 53

REFERENCES 55
CHAPTER 1

INTRODUCTION

Planar diffraction problems are treated extensively in the literature. Excellent bibliographies can be found in a series of papers by J.P. Vasseur and in Wu's thesis. The case of diffraction of a plane wave incident normally on an infinitely extended slit has been solved rigorously by Morse and Rubenstei, while the circular aperture has been treated rigorously by Meixner and Andrejewski et al. The solution for the slit is in the form of an infinite series of ellipsoidal functions, Meixner's circular aperture solution is in the form of a series of oblate spheroidal functions. Also, Copson derived a set of integro-differential equations for the electric field components. In Copson's solution, singularities at the edge introduce difficulties and the set of equations generally admits a number of solutions. Bouwkamp studied diffraction of a plane-polarized electromagnetic wave by a small conducting circular disk. A set of integro-differential equations for the currents in the disk was set up. The method was confined to the case of normal incidence. The dual integral equations approach was used by Hün and Zimmer as well as
Tranter\textsuperscript{11} for the strip problem. It seems quite attractive to extend the same scheme to axially symmetric diffracting apertures. Usually, in the treatment of half planes and strips, pure TM and TE waves are assumed\textsuperscript{12,13} in order to make substantial simplifications when reducing the complex formulations, but in general, both TM and TE waves should be taken into account. Actually, electromagnetic fields are described insufficiently by a single scalar wave function, as is frequently attempted and should be determined by a set of such functions interrelated by Maxwell's equations.

This thesis presents a general survey of the advantages of making use of the concept of TM and TE waves for axially symmetric apertures. It is found that dual integral equations, triple integral equations and, in general, multiple integral equations with the Hankel kernel can be formulated. By means of recently developed theory of this branch of classical mathematics the problem can be solved formally. It is believed that this is the first time this method is applied to the three dimensional diffraction problem.

The theory presented here is based on the concept that any electromagnetic field, in a homogeneous, isotropic medium, free from charges and currents, can be expressed in terms of an electric Hertz vector $\mathbf{W}$ and a
magnetic Hertz vector $\mathbf{N}$ and $\mathbf{M}$ both being along an arbitrarily chosen direction. Specifically, the $\mathbf{N}$ and $\mathbf{M}$ vectors are written as

$$\mathbf{N} = N \hat{k}, \quad \mathbf{M} = M \hat{k}$$

(1)

where $\hat{k}$ is a unit vector along the z-axis. Then

$$\mathbf{E}^{\text{TM}} = \nabla \times \nabla \times \mathbf{N}, \quad \mathbf{H}^{\text{TM}} = \mathbf{E} \frac{\partial}{\partial t} \nabla \times \mathbf{N}$$

(2)

$$\mathbf{E}^{\text{TE}} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{M}, \quad \mathbf{H}^{\text{TE}} = \nabla \nabla \times \mathbf{M}$$

(3)

or explicitly in terms of cylindrical coordinates $(u^1, u^2, z)$

$$\begin{bmatrix} E_1 \\ E_2 \\ E_z \end{bmatrix} = \begin{bmatrix} \frac{1}{h_1} \frac{\partial^2 N}{\partial u^2 \partial z} - \frac{\mu}{h_2} \frac{\partial^2 M}{\partial u^2 \partial t} \\ \frac{1}{h_2} \frac{\partial^2 N}{\partial u^2 \partial z} + \frac{\mu}{h_1} \frac{\partial^2 M}{\partial u^2 \partial t} \\ \frac{\partial^2 N}{\partial z^2} - \mu \varepsilon \frac{\partial^2 N}{\partial t^2} \end{bmatrix}$$

(4)

$$\begin{bmatrix} H_1 \\ H_2 \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{\varepsilon}{h_2} \frac{\partial^2 N}{\partial u^1 \partial t} + \frac{1}{h_1} \frac{\partial^2 M}{\partial u^1 \partial z} \\ -\frac{\varepsilon}{h_1} \frac{\partial^2 N}{\partial u^1 \partial t} + \frac{1}{h_2} \frac{\partial^2 M}{\partial u^1 \partial z} \\ \frac{\partial^2 M}{\partial z^2} - \mu \varepsilon \frac{\partial^2 M}{\partial t^2} \end{bmatrix}$$

(5)

where $\mathbf{N}$ and $\mathbf{M}$ must satisfy the vector wave equations
\[ \nabla^2 \mathbf{N} - \mu \varepsilon \frac{\partial^2 \mathbf{N}}{\partial t^2} = 0 \quad (6) \]

\[ \nabla^2 \mathbf{M} - \mu \varepsilon \frac{\partial^2 \mathbf{M}}{\partial t^2} = 0 \quad (7) \]

Since cylindrical waves are of interest in this work, a solution is sought of the type

\[ \psi = e^{-j\omega t} \sum_{n=-\infty}^{\infty} e^{jn\theta} \int_0^\infty u A_n(u) J_n(u) e^{\pm j\sqrt{k^2-u^2}z} du \quad (8) \]

by the principle of superposition. It is easy to see that particular solutions of the Helmholtz's equations (6) and (7) which are periodic in both the azimuthal angle \( \theta \) and the time \( t \), can be constructed from elementary waves of the type

\[ \psi_n = e^{-j\omega t+jn\theta} J_n(\sqrt{k^2-\chi^2}) e^{\mp j\chi z} \]

General speaking, the propagation constant \( \chi \) assumes any complex value. Explicit expression for \( \chi \) in terms of the frequency \( \omega \) and the constant of the medium is determined by prescribed boundary conditions on a plane \( z=\text{constant} \) or \( r=\text{constant} \). For the present case the term \( \pm j\sqrt{k^2-u^2}z \) in the expression (8) must be so chosen that the field vanishes as \( z \) tends to \( \pm \) infinity.
The next chapter deals with the mathematical treatment of the appropriate integral equations. In subsequent chapters, it will be shown that by means of (4), (5) and (8) a set of multiple integral equations can be formulated which can be solved for the unknown weighting functions and hence the diffracted fields. Detailed discussion of the boundary condition at the rim of the aperture is given and a comparison is made with the result given by Bouwkamp\textsuperscript{9,16,17}.

The method of steepest descent is used in chapter 5 to evaluate asymptotic expansions for the E and H field. The standard theory developed by Kantorovich and Krylov\textsuperscript{18}, and Milklin\textsuperscript{19} for the approximate solution of the Fredholm integral equation of the second kind is invoked. Gauss-Legendre quadrature of order n=20 is used to approximate the integration numerically. A summary of the method is given in Appendix B.
CHAPTER 2

GENERAL TREATMENT OF MULTIPLE INTEGRAL EQUATIONS
WITH ARBITRARY WEIGHTING FUNCTION AND HANKEL KERNEL

In this chapter are presented the mathematical techniques essential to the solution of the dual, triple and sets of multiple integral equations of interest to this work. A complete collection of significant texts and papers is given in the book by Sneddon\textsuperscript{20,21}. All the operators involved here are defined in Appendix A.

2.1 Solution Of Dual Integral Equations

A useful solution is sought to the dual integral equations of the type

\begin{align}
S_{\frac{1}{2} \nu_{1},2 \nu} \left[1 + w(\psi)\right] \psi(u); r \right] &= \theta_1(r), \quad r \in D_1 \quad (1) \\
S_{\frac{1}{2} \nu,0} \psi(r) &= \phi_2(r), \quad r \in D_2 \quad (2)
\end{align}

where \( w(r), \theta_1(r), \phi_2(r) \) = known functions
\( \psi(u) \) = unknown function
\( D_1 = \{r: 0 \leq r < 1\} \)
\( D_2 = \{r: r > 1\} \)
The Method described below is due to Erdélyi and Sneddon\textsuperscript{22}.

Put
\[ \psi(u) = S_{\frac{1}{2}\nu,-\alpha} h(u) \]  
(3)
where \( h(u) \) is a new unknown function defined in the range \( D_1 \cup D_2 \). From (1) and (2) there is obtained
\[ S_{\frac{1}{2}\nu,-\alpha} h(r) + S_{\frac{1}{2}\nu,-\alpha} \left\{ w(r) S_{\frac{1}{2}\nu,-\alpha} h(r) \right\} = \theta_1(r), \quad r \in D_1 \]
\[ S_{\frac{1}{2}\nu,0} S_{\frac{1}{2}\nu,-\alpha} h(r) = \phi_2(r), \quad r \in D_2 \]

After some manipulation it is found that
\[ h_2(r) = h(r) H(r-1) = K_{\frac{1}{2}\nu,-\alpha,\alpha} \phi(r) \]  
(4)
and
\[ h_1(r) + \int_0^1 h_1(u) G(r,u) du = F(r), \quad r \in D_1 \]  
(5)
where
\[ h_1(r) = h(r) H(1-r) \]
\[ H(u) = \text{the Heaviside's unit function} \]
\[ G(r,u) = r^{-\alpha} u^{1+\alpha} \int_0^\infty tw(t) J_{\nu,-\alpha}(rt) J_{\nu,\alpha}(ut) dt \]  
(6)
\[ F(r) = \int_0^1 r K_{\frac{1}{2}\nu,-\alpha,\alpha} \theta_1(r) - S_{\frac{1}{2}\nu,-\alpha} w(r) S_{\frac{1}{2}\nu,0} \phi(r) \]  
(7)
and
\[ u \psi(u) = 2^{-\alpha} u^{1+\alpha} \int_0^1 t^{1+\alpha} J_{\nu,\alpha}(ut) h_1(t) dt + \]
\[ + 2^{-\alpha} u^{1+\alpha} \int_1^\infty t^{1+\alpha} J_{\nu,\alpha}(ut) K_{\frac{1}{2}\nu,-\alpha,\alpha} \phi(t) dt \]  
(8)

It may be shown that the above method is valid for a convergent kernel (6).
2.2 Solution Of Multiple Integral Equations

The set of multiple integral equations of particular interest are defined as

\[ S_{\frac{1}{2},0} \psi(r) = 0, \quad r \in D_1 \cup D_3 \cup \ldots \cup D_{2n+1} \quad (9) \]

\[ S_{\frac{1}{2},0} \left[ (1 + w(u)) \psi(u) ; r \right] = \theta(r), \quad r \in D_2 \cup D_4 \cup \ldots \cup D_{2n} \quad (10) \]

where \( w(r) \) = known function

\[ \theta_1(r) = \theta(r) \left[ H(r - a_{i-1}) - H(a_i - r) \right] \]

\[ D_1 = \{ r : 0 \leq r < a_1 \} \]

\[ D_2 = \{ r : a_1 \leq r < a_2 \} \]

\[ \ldots \ldots \ldots \ldots \]

\[ D_{2n+1} = \{ r : a_{2n} < r \} \]

It can be verified that (9) and (10) may be put in a more convenient single equation by use of the Heaviside's unit function, (the idea is similar to that of Noble\textsuperscript{23}), namely,

\[ S_{\frac{1}{2},0} \psi(r) = \sum_{m=1}^{n} \theta_{2m}(r) - S_{\frac{1}{2},0} w(r) \psi(r) \left[ H(r - a_{2m-1}) - H(r - a_{2m}) \right], \quad r \in L \quad (11) \]
where \( L = D_1 \cup D_2 \cup \ldots \cup D_{2n+1} \)

Applying \( S^{-1}_{\varepsilon \lambda_0} \) to equation (11) it may be found that the system reduces to a Fredholm integral equation of the second kind, viz,

\[
\psi(r) = F(r) - \int_0^\infty G(r,s) (s) ds
\]

where \( F(r) = \sum_{m=1}^n \int_{a_{2m-1}}^{a_{2m}} \theta_{2m}(t) J(rt) dt \)

\[
G(r,s) = \frac{w(s)L(s,r;a_1,a_2,\ldots,a_{2n})}{s^2 - u^2}
\]

\[
L(s,r;a_1,a_2,\ldots,a_{2n}) = \sum_{m=1}^{2n} (-)^m I_\nu(a_m s, a_m r)
\]

\[
I_\nu(x,y) = xJ_{\nu+}(x) J_\nu(y) - yJ_\nu+(x) J_{\nu+}(y)
\]

In principle the Fredholm equation of the second kind (12) can be solved by various approximate techniques \(^{18,19}\) such as iteration methods, replacement by suitable quadrature formulae, the method of moments, the method of Galerkin, the method of least squares, and Ritz's method. Practically, as in many other cases, the method of Galerkin is usually used. Due to the infinite range of integration in equation (12) there is difficulty in numerical integration.
CHAPTER 3

DIFFRACTION BY STRIPS

The first rigorous solution of the diffraction of a plane electromagnetic wave by an infinitely long slit in a plane screen was presented by Schwarzschild\textsuperscript{24} who used a scheme of successive approximations based on the Sommerfeld half-plane theory. Sieger\textsuperscript{25} obtained a solution by separation of variables in the wave equation using elliptical coordinates. An extensive numerical computation was performed by Morse and Rubenstein\textsuperscript{3}. An integral equation treatment has been given by Levine\textsuperscript{26}, while Millar\textsuperscript{27} derived an asymptotic expansion based on Schwarzschild's approach.

In the following are presented some known but interesting problems illustrating the application of the above developed theory. Diffraction by a single strip was considered early by Rayleigh in 1897\textsuperscript{28}. It is shown below how to apply the dual integral equations technique to two dimensional problems and extend it to the case of multiple strips. Note that the dual integral equations used here are essentially those derived in the book by Born and Wolf\textsuperscript{12}. The same approach has been used by Tranter\textsuperscript{11}, Hünl and Zimmer\textsuperscript{10}, and Groschwitz and Hünl\textsuperscript{29}.
3.1 Diffraction By A Strip

Consider a strip occupying \( y=0, \forall x < a \), and a normally incident \( H \)-polarized plane wave. The dual integral equations are then found to be

\[
\int_0^\infty P(s)\cos(kxs)\,ds = 1/2, \quad |x| \leq a \tag{1}
\]

\[
\int_0^\infty \frac{P(s)\cos(kxs)}{\sqrt{1-s^2}}\,ds = 0, \quad |x| > a \tag{2}
\]

Let \( r = \frac{|x|}{a}, \ u = kax \)

Dual equations with a Hankel kernel are then obtained,

\[
\int_0^\infty \sqrt{(ka)^2 - u^2} \frac{Y(u) J_{-1/2}(ru)}{r} \,du = \frac{(ka)^2}{2} r^{-1/2}, \quad r \in D_1
\]

\[
\int_0^\infty Y(u) J_{-1/2}(ru) \,du = 0, \quad r \in D_2
\]

where

\[
Y(u) = \frac{\left(\frac{1}{2} \pi u\right)^{1/2} P(u)}{\sqrt{1 - \left(\frac{u}{ka}\right)^2}}
\]

\[
D_1 = \left\{ r : 0 \leq r < 1 \right\}
\]

\[
D_2 = \left\{ r : r > 1 \right\}
\]

Making use of the result of the last chapter, it is found that

\[
\nu = -\frac{1}{2}, \quad \alpha = -\frac{1}{2}
\]

and

\[
F(r) = \sqrt{\frac{\pi}{4}} r^{1/2} (ka)^2
\]

Hence the zero order solution for \( Y(u) \) is

\[
Y_0(u) = (2u)^{1/2} \int_0^1 \frac{1}{t^2} J_0(ut) h_1(t) \,dt
\]
It follows that
\[ P(s) = \frac{\text{ka}}{\sqrt{1 - s^2}} \frac{J_1(\text{kas})}{s} \]
which agrees with the result given in the above mentioned text. Once the weighting function \( P(s) \) is obtained the problem is completely determined since the induced current can be evaluated from the following formula:
\[ J_x(t) = \frac{c}{2\pi} \int \frac{P(s) e^{jkst}}{1 - s^2} \, ds \]
Substituting \( J_x \) into Helmholtz formulas the electromagnetic field can then be calculated by a suitable quadrature formula.

3.2 Two Identical Strips

Consider a more complicated diffracting system of conducting strips occupying \( y = 0 \), \( a_1 < |x| < a_2 \) as shown in Fig. 2. By an argument similar to that above, a set of integral equations is obtained.
\[ \int_{-\infty}^{\infty} \frac{P(s)}{\sqrt{1 - s^2}} \cos(kxs) \, ds = 0 \quad , \quad |x| \in D_1 \cup D_3 \]  
\[ \int_{-\infty}^{\infty} P(s) \cos(kxs) \, ds = \frac{1}{2} \quad , \quad |x| \in D_2 \]  
where
\[ D_1 = \{ x : 0 \leq |x| < a_1 \} \]
\[ D_2 = \{ x : a_1 \leq |x| < a_2 \} \]
\[ D_3 = \{ x : |x| > a_2 \} \]
It may be put in form
\[ \int_0^\infty \frac{P'(u)}{\sqrt{1 - (u/k)^2}} J_{-1/2}(xu) du = 0, \quad |x| \notin D_1 \cup D_3 \]

\[ \int_0^\infty P'(u) J_{-1/2}(xu) du = \frac{k}{2} x^{-1/2}, \quad |x| \in D_2 \]

where \( P'(u) = P(s) \left( \frac{1}{2} \pi ks \right)^{1/2} \)

\( u = ks \)

If zero order approximation is sought, it can be found that by equation 2.2(13)

\[ P(u) = \frac{\pi}{\sqrt{\pi}} k^2 u^{-3/2} \sin \left( \frac{1}{2} (a_2 - a_1) u \right) \cos \left( \frac{1}{2} (a_2 + a_1) u \right) \]

Hence, to this approximation

\[ \frac{P(s)}{\sqrt{1 - s^2}} = \frac{2}{\pi k} \frac{1}{s} \sin \left( \frac{k}{2} (a_2 - a_1) s \right) \cos \left( \frac{k}{2} (a_2 + a_1) s \right) \]

The current density in either strip is thus given by \( J(x) = \frac{c}{2\pi} \int P(s) e^{jkxs} ds \)

\[ = \frac{2kc}{\pi^2} \int_0^\infty \sin \left( \frac{k}{2} (a_2 - a_1) s \right) \cos \left( \frac{k}{2} (a_2 + a_1) s \right) \cos kxs ds \]

which is evaluated as in Reference 30, then

\[ J(x) = \frac{kc}{4\pi} \quad a_1 < |x| < a_2 \]

It is interesting to note that this zero order induced current is identically equal to zero for \( |x| \leq a_1 \) and \( |x| \geq a_2 \).
Although $J_{x}^{(0)}(x)$ is constant in the strip, yet the sharp edge condition as established by Meixner and Andrejewski, Bouwkamp, and Jones is satisfied, viz,

$$J_n = 0$$

at the edge.

3.3 An Array Of Strips

Assume an array of strips occupying $y=0, |x| \in D_2 \cup D_4 \cup \ldots \cup D_{2n}$, where

$$D_2 = \{|x| : a_1 < |x| < a_2\}$$
$$D_4 = \{|x| : a_3 < |x| < a_4\}$$

$$\ldots \ldots \ldots \ldots \ldots$$
$$D_{2n} = \{|x| : a_{2n-1} < |x| < a_{2n}\}$$

Following the same reasoning as above, a set of multiple integral equations is obtained, namely,

$$\int_{0}^{\infty} \frac{P'(u)}{\sqrt{1-(u/k)^2}} J_{-1/2}(xu) du = 0, \ |x| \in D_1 \cup D_3 \ldots \cup D_{2n-1} \quad (5)$$

$$\int_{0}^{\infty} P'(u) J_{-1/2}(xu) du = \frac{k}{2} x^{1/2}, \ |x| \in D_2 \cup D_4 \ldots \cup D_{2n} \quad (6)$$

This set of integral equations can be solved by the scheme described in 2.2. Therefore the whole problem is reduced to solving a Fredholm integral equation of the second kind.
It can be verified that the zero order induced currents are mere constants, and that the edge condition mentioned above is obeyed.

3.4 Discussion

In conclusion, the basic ideas involved in the formulation of planar strips diffraction problems developed above may be summarized. It will be shown later the same ideas can be applied to the circular aperture as well. To start with, it is required that the boundary conditions on the strips and on the rest of the plane (free space) be satisfied by a set of "discrete" integral equations, namely, each integral equation is defined in a particular interval only. Then, the specific problem is reduced to that of solving a Fredholm integral equation of the second kind for the weighting function, after suitable transformations. Once the weighting functions is solved, the induced current distribution in each strip can be evaluated. It follows that the electric and magnetic field can then be calculated from the Helmholtz formula in straightforward manner. However, in practice it is not a simple matter to solve equation (12) in chapter 2 numerically. It may be that Galerkin's method will be useful, provided that an economic algorithm can be found.
Fig. 1
Single Strip

Fig. 2
Two Identical Strips
CHAPTER 4

DIFFRACTION BY AXIALLY SYMMETRIC PLANAR APERTURES

In the case of the circular disk, it is a straightforward matter to derive a set of integro-differential equations

$$\vec{E}_0^t + ( \nabla \nabla \cdot + k^2 ) \int J \psi ds = 0$$

by requiring that the tangential components of the electric field vanish on the disk. The detail of solving (1) is given in the book by Jones. It will be of intrinsic interest to attack the problem by splitting the diffracted field into an electric wave-(TM), (transverse magnetic), and a magnetic wave-(TE), (transverse electric). This chapter shows how it is possible to formulate the dual integral equations appropriate to the circular aperture, and hence extends the idea to multiple annular apertures.

4.1 Dual Integral Equations For The Circular Aperture

Consider a perfectly conducting infinite screen with a circular aperture of unit radius (without loss of generality the radius is normalized to unity) with centre at the origin of circular cylindrical coordinates \((r, \theta, z)\). It will be assumed that the time dependence is of the form
\[ e^{-j\omega t} \] in the subsequent derivation, thus allowing \( \frac{e^{+jkr}}{r} \) to describe an out-going spherical wave according to Stratton's\textsuperscript{14} and Sommerfeld's\textsuperscript{15} notation.

It is convenient to suppose that the aperture gives rise to a perturbation field \( \vec{E}^d, \vec{H}^d \) which is superposed on the field that exists when the screen is completely closed\textsuperscript{31}. Consequently then results:

\[
\begin{align*}
\vec{E}_{total} &= \vec{E}^i + \vec{E}^r + \vec{E}^d, \quad \text{for } z < 0 \\
\vec{E}^d, \quad \text{for } z > 0 \\
\vec{H}_{total} &= \vec{H}^i + \vec{H}^r + \vec{H}^d, \quad \text{for } z < 0 \\
\vec{H}^d, \quad \text{for } z > 0
\end{align*}
\]

where the superscripts \( i, r, \) and \( d \) denote incidence, reflection, and diffraction respectively. By means of the Green's function it has been shown that the tangential components of the magnetic field and the normal component of the electric field in the aperture proper are the same as those of the incident wave, provided that the condition of integrability developed by Meixner\textsuperscript{6} is satisfied, viz,

\[
\int_{\text{surface}} |\vec{E}^2| \, ds \quad \text{is finite}
\]

Hence the appropriate boundary conditions for the problem are,

\[
\vec{H}^d_{tg}(r, \theta, 0) = \vec{H}^i_{tg}(r, \theta, 0), \quad r \in \mathcal{D}_1
\]

\[ d \]
\[
\overline{E}_n^d(r, \theta, 0) = \overline{E}_n^i(r, \theta, 0), \quad r \in D_1
\] (4)

and
\[
\overline{E}_{tg}^d(r, \theta, 0) = 0, \quad r \in D_2
\] (5)

\[
\overline{H}_n^d(r, \theta, 0) = 0, \quad r \in D_2
\] (6)

or alternatively\(^{14}\)
\[
\frac{\partial \overline{H}_n^d(r, \theta, 0)}{\partial z} = 0, \quad r \in D_2
\] (7)

\[
\frac{\partial \overline{rH}_n^d(r, \theta, 0)}{\partial z} = 0, \quad r \in D_2
\] (8)

where \(D_1\) and \(D_2\) are defined as
\[
D_1 = \left\{ x : 0 \leq r < 1 \right\}
\] (9)

\[
D_2 = \left\{ x : r > 1 \right\}
\] (10)

Making use of the linear property of the Maxwell's equations
the perturbation field can be written in terms of TE and
TM wave components, namely,
\[
\overline{E}^d = \overline{E}^d_{TE} + \overline{E}^d_{TM}
\] (11)

\[
\overline{H}^d = \overline{H}^d_{TE} + \overline{H}^d_{TM}
\] (12)

If we make the assumption that the incident wave at \(z=0\) can be expressed in the following form
\[
f^i(r, \theta, 0) = \sum_{n=-\infty}^{\infty} f_n(r) e^{jn\theta}
\] (13)

and that from Chapter 1 equations (8) and (5), it can be
assumed that

\[ N_z = e^{-jwt} \sum_{n=-\infty}^{\infty} e^{j\theta} \int_{0}^{\infty} u P_n(u) J_n(ru) e^{j\sqrt{k^2-u^2} z} \, du , \quad (14) \]

\[ H_z = e^{-jwt} \sum_{n=-\infty}^{\infty} e^{j\theta} \int_{0}^{\infty} u Q_n(u) J_n(ru) e^{j\sqrt{k^2-u^2} z} \, du , \quad (15) \]

There is then obtained

\[ H_r = \frac{\omega \epsilon n}{r} e^{-jwt} \sum_{n=-\infty}^{\infty} e^{j\theta} \int_{0}^{\infty} u P_n(u) J_n(ru) e^{j\sqrt{k^2-u^2} z} \, du \]

\[ + je^{-jwt} \sum_{n=-\infty}^{\infty} e^{j\theta} \int_{0}^{\infty} u^2 \sqrt{k^2-u^2} Q_n(u) J_n'(ru) e^{j\sqrt{k^2-u^2} z} \, du \]

\[ (16) \]

\[ H_\theta = j\omega e^{-jwt} \sum_{n=-\infty}^{\infty} e^{j\theta} \int_{0}^{\infty} u^2 P_n(u) J_n'(ru) e^{j\sqrt{k^2-u^2} z} \, du \]

\[ - \frac{n}{r} e^{-jwt} \sum_{n=-\infty}^{\infty} e^{j\theta} \int_{0}^{\infty} u \sqrt{k^2-u^2} Q_n(u) J_n(ru) e^{j\sqrt{k^2-u^2} z} \, du , \quad (17) \]

Where \( z \) is supposed to be positive. Let \( z=0 \) in the above expressions and make use of the recursion relations of Bessel function of the first kind, viz,

\[ J_{n+1}(x) = \frac{x}{nJ_n(x)} - J'_n(x) \]

\[ J_{n-1}(x) = \frac{x}{nJ_n(x)} + J'_n(x) \]

It is a simple matter to show that
\[ jH_{r+\theta} = e^{-j\omega t} \sum_{n=-\infty}^{\infty} e^{jn\theta} \int_{0}^{\infty} u^2 Y(u) J_{n-1}(ru) \, du \quad , \quad (18) \]

\[ -jH_{r+\theta} = e^{-j\omega t} \sum_{n=-\infty}^{\infty} e^{jn\theta} \int_{0}^{\infty} u^2 X(u) J_{n+1}(ru) \, du \quad , \quad (19) \]

Where \[ Y(u) = j\omega \mathcal{E} P_n(u) - \sqrt{(k^2 - u^2)} \, Q_n(u) \quad , \quad (20) \]

\[ X(u) = j\omega \mathcal{E} P_n(u) + \sqrt{(k^2 - u^2)} \, Q_n(u) \quad , \quad (21) \]

Henceforth by (3), (7) and (8) two pairs of dual integral equations are obtained, namely,

\[ \int_{0}^{\infty} u^2 Y(u) J_{n-1}(ru) \, du = H^1_{\theta n} + jH^1_{r n} \quad , \quad r \in D_1 \quad (22) \]

\[ \int_{0}^{\infty} u^2 \sqrt{(k^2 - u^2)} \, Y(u) J_{n-1}(ru) \, du = 0 \quad , \quad r \in D_2 \quad (23) \]

And

\[ \int_{0}^{\infty} u^2 X(u) J_{n+1}(ru) \, du = -H^1_{\theta n} + jH^1_{r n} \quad , \quad r \in D_1 \quad (24) \]

\[ \int_{0}^{\infty} u^2 \sqrt{(k^2 - u^2)} \, X(u) J_{n+1}(ru) \, du = 0 \quad , \quad r \in D_2 \quad (25) \]

Let

\[ Y'(u) = u^2 \sqrt{(k^2 - u^2)} \, Y(u) \quad (26) \]

\[ X'(u) = u^2 \sqrt{(k^2 - u^2)} \, X(u) \quad (27) \]

The above equations can be put in the form

\[ \int_{0}^{\infty} \frac{Y'(u)}{u^2 \sqrt{k^2 - u^2}} J_{n-1}(ru) \, du = H^1_{\theta n} + jH^1_{r n} \quad , \quad r \in D_1 \quad (28) \]

\[ \int_{0}^{\infty} Y'(u) J_{n-1}(ru) \, du = 0 \quad , \quad r \in D_2 \quad (29) \]
and
\[ \int_{0}^{\infty} \frac{X'(u)J_{n+1}(ru)\,du}{\sqrt{k^2-u^2}} = -H_0^1 + jH_r^1, \quad r \in D_1 \quad (30) \]
\[ \int_{0}^{\infty} X'(u)J_{n+1}(ru)\,du = 0, \quad r \in D_2 \quad (31) \]
These two pairs of dual integral equations will be used later.

However, it is important to remark that the two pairs of dual integral equations derived above are by no means sufficient to determine the weighting functions \( P_n(u) \) and \( Q_n(u) \). Since the boundary conditions (5) and (6) on the screen with infinite conductivity imply that (7) and (8) are satisfied. On the contrary, the mere conditions (7) and (8), namely, that the normal derivatives of the covariant components of magnetic field tangential to the conductor surface vanishes, will not always provide sufficient information from which the former can be inferred. Hence in the present case it cannot be asserted that \( P_n(u) \) and \( Q_n(u) \) are completely determined without resort to equations (4), (5) and (6). It is clear that in order to avoid ambiguity either equations (28) and (29) or (30) and (31) should be supplemented by equations (44) and (45) established in the next section. In short, boundary conditions (7) and (8) are convenient in certain cases, but in the present case care should be taken in using them.
4.2 Diffraction By A Circular Aperture—Normal Incidence

Let a monochromatic plane-polarized electromagnetic wave, \( \mathbf{E}^i = e^{jkz} \hat{z} \)

\( \mathbf{H}^i = \frac{1}{\mu} e^{jkz} \hat{y} \)

travelling in the direction from \( z = -\infty \) to \( z = \infty \), impinge upon a circular aperture of unit radius (The rationalized MKS system of units is adopted throughout). Then the magnetic field distribution in the aperture proper, can be expressed in circular cylindrical coordinates as

\[
\mathbf{H}_1^i = \left( \frac{1}{2j} \sqrt{\frac{k}{\mu}}, \frac{k}{2}, 0 \right) e^{jkz+j\theta} \tag{34}
\]

\[
\mathbf{H}_{-1}^i = \left( -\frac{1}{2j} \sqrt{\frac{k}{\mu}}, \frac{k}{2}, 0 \right) e^{jkz-j\theta} \tag{35}
\]

Whence by (28), (29), (30), (31) the following sets of dual integral equations are obtained, namely,

(i) For the case \( n = 1 \)

\[
\int_0^\infty \frac{Y'(u)}{\sqrt{k^2-u^2}} J_0(ru) du = \frac{\sqrt{\frac{k}{\mu}}}{M}, \quad r \in D_1 \tag{36}
\]

\[
\int_0^\infty Y'(u) J_0(ru) du = 0, \quad r \in D_2 \tag{37}
\]

and

\[
\int_0^\infty \frac{X'(u)}{\sqrt{k^2-u^2}} J_2(ru) du = 0, \quad r \in D_1 \tag{38}
\]

\[
\int_0^\infty X'(u) J_2(ru) du = 0, \quad r \in D_2 \tag{39}
\]
(ii) For the case n = -1

\[
\int_0^\infty \frac{Y'(u)J_2(ru)du}{\sqrt{k^2 - u^2}} = 0 \quad , \quad r \in D_1 \tag{40}
\]

\[
\int_0^\infty Y'(u)J_2(ru)du = 0 \quad , \quad r \in D_2 \tag{41}
\]

and

\[
\int_0^\infty \frac{X'(u)J_0(ru)du}{\sqrt{k^2 - u^2}} = -\sqrt{\frac{\pi}{2}} \quad , \quad r \in D_1 \tag{42}
\]

\[
\int_0^\infty X'(u)J_0(ru)du = 0 \quad , \quad r \in D_2 \tag{43}
\]

The above equations for both cases should be supplemented by the equations as mentioned in last section, viz,

\[
\int_0^\infty u^3 P_n(u)J_1(ru)du = E_n^1 \quad , \quad r \in D_1 \tag{44}
\]

\[
\int_0^\infty u^3 Q_n(u)J_1(ru)du = 0 \quad , \quad r \in D_2 \tag{45}
\]

(i) for \( n = 1 \), is obtained

\[
\int_0^\infty u^3 \frac{1}{\sqrt{k^2 - u^2}} Q_1(u)J_1(ru)du = \int_0^\infty \frac{uY'(u)J_1(ru)du}{\sqrt{k^2 - u^2}} \quad , \quad r \in D_1 \tag{46}
\]

\[
\int_0^\infty u^3 Q_1(u)J_1(ru)du = 0 \quad , \quad r \in D_2 \tag{47}
\]

(ii) for \( n = -1 \), it can be shown that the same set of dual integral equations (46) and (47) are obtained except that \( Q_1(u) \) is replaced by \( Q_{-1}(u) \).
By means of the method described in 2.1, the
dual integral equations (36) and (37) are reduced to the
problem of solving the following Fredholm integral equation
of the second kind,
\[ ch_{\frac{1}{2}}(r) + \int_{0}^{1} h_{\frac{1}{2}}(u) G(r,u) \, du = F(r) \quad , \quad r \in D_{1} \]  \hspace{1cm} (48)
where
\[ G(r,u) = r^{-\alpha} u^{1+\alpha} \int_{0}^{\infty} tw(t) J_{\nu-\alpha}(rt) J_{\nu-\alpha}(ut) \, dt \]  \hspace{1cm} (49)
\[ F(r) = 2^{2\alpha} r^{-2\alpha} \frac{1}{\Gamma(\frac{1}{2})} r^{\frac{1}{2}} \]  \hspace{1cm} (50)
and
\[ Y'(u) = 2^{-\alpha} u^{1+\alpha} \int_{0}^{1} t^{1+\alpha} J_{\nu-\alpha}(ut) h_{\frac{1}{2}}(t) \, dt \]  \hspace{1cm} (51)
If \( \alpha, c \) and \( w(t) \) are chosen so that
\( \alpha = \frac{1}{2}, \)  
\( c = -j, \)  
and  
\( w(t) = \frac{j}{\sqrt{(k/t)^{2} - 1}}, \)
then they correspond to a similar case which has been treated
by Lebedev and Uflyand\(^{32}\) in the discussion of certain problems
in elasticity. It is easily verified that the kernel can be
put in the form
\[ G(r,u) = \frac{u}{r^{\frac{1}{2}}} \left\{ G_{c}(|r-u|) + G_{c}(|r+u|) \right\} \]  \hspace{1cm} (51)
where \( G_{c}(u) \) is the Fourier cosine transform of \( w(t) \) and the
free term is given by
\[ F(r) = \frac{2}{\sqrt{\pi}} \lambda(\epsilon \mu) r^{-1} \]  \hspace{1cm} (52)
Similarly, for (46) and (47), it is convenient to choose
\[ \alpha = - \frac{1}{2}, \quad c = j \]
\[ w(t) = \sqrt{(k/t)^2 - 1} - j \]

Whence \( G(r,u) \) is given by
\[
G(r,u) = \sqrt{xu} \int_{0}^{x} tw(t) J_{3/2}(xt) J_{3/2}(ut) dt,
\]
\[
F(r) = \frac{1}{\Gamma(1/2)} x \int_{0}^{x} \frac{u^2 f(u) du}{\sqrt{x^2 - u^2}},
\]
\[
u^3 Q_1(u) = \sqrt{2} u^{1/2} \int_{0}^{1} t^{1/2} J_{3/2}(ut) g_1(t) dt
\]
where
\[
f(u) = - \int_{0}^{\infty} \frac{cY'(s)}{\sqrt{s^2 - \alpha^2}} J_1(us) ds \]

and \( g_1(t) \) is the solution of the Fredholm equation (48).

The numerical evaluation of the field components is presented in the next chapter.

4.3 The Annular Aperture — Normal Incidence

Topics on the diffraction by annular apertures have received considerable attention in the past years. Despite this, the rigorous solution of the problem still remains largely unsolved. In this section a set of triple equations for the annular aperture is given, which can be
solved by a method shown by Sneddon.  

A perfectly conducting infinite screen with an annular aperture specified by the radius $a_1$ and $a_2$, with centre at the origin of circular cylindrical coordinates $(r, \theta, z)$ is assumed (see Fig. 2). Following the same reasoning as in the previous section, the triple equations are obtained for normal incidence as follows

(i) for the case $n=1$

\[
\int_0^\infty \frac{Y'(u) J_0(ru)}{\sqrt{k^2-u^2}} \, du = \frac{\sqrt{\xi}}{k}, \quad r \in D_2 \tag{53}
\]

\[
\int_0^\infty Y'(u) J_0(ru) \, du = 0, \quad r \in D_1 \cup D_3 \tag{54}
\]

and

\[
\int_0^\infty \frac{X'(u) J_2(ru)}{\sqrt{k^2-u^2}} \, du = 0, \quad r \in D_2 \tag{55}
\]

\[
\int_0^\infty X'(u) J_2(ru) \, du = 0, \quad r \in D_1 \cup D_3 \tag{56}
\]

where $Y'(u)$ and $X'(u)$ are as before, and,

\[
D_1 = \{ r : 0 \leq r < a_1 \}
\]

\[
D_2 = \{ r : a_1 < r < a_2 \}
\]

\[
D_3 = \{ r : r > a_2 \}
\]

(ii) for the case $n=-1$
\[
\int_{0}^{\infty} \frac{X'(u)}{\sqrt{r^2 - u^2}} J_2(ru) \, du = 0, \quad r \in D_2 \quad (57)
\]

\[
\int_{0}^{\infty} Y'(u) J_2(ru) \, du = 0, \quad r \in D_1 \cup D_3 \quad (58)
\]

and

\[
\int_{0}^{\infty} \frac{X'(u)}{\sqrt{r^2 - u^2}} J_0(ru) \, du = -\sqrt{\frac{c}{\pi}}, \quad r \in D_2 \quad (59)
\]

\[
\int_{0}^{\infty} X'(u) J_0(ru) \, du = 0, \quad r \in D_1 \cup D_3 \quad (60)
\]

It follows that the main concern is to find a way to solve the triple equations (53) and (54). Fortunately the same type of equations have been investigated by Cooke\textsuperscript{33}. The set of triple equations with the kernel \(J_0(xu)\) arises in the discussion of problems in electrostatics, elasticity, and the conduction of heat. It is interesting to encounter equations of this type in diffraction problems.

Detailed derivation is available in the above reference and therefore only the final result is given below. Cooke has shown that the problem can be reduced to solving a Fredholm integral equation of the second kind.

Namely,

\[
G_2(u) + \int_{a_1}^{a_2} \left\{ N(u,t) + M(u,t) \right\} G_2(t) \, dt = F(u) \quad (61)
\]

where

\[
N(u,t) = \frac{2}{\pi(u-t^2)^{1/2}} \left\{ \ln \frac{u+a}{t+a} - \ln \frac{u-a}{t-a} \right\}.
\]
where \( Y'(u) \) is given by the formula

\[
Y'(u) = \int_{a_1}^{a_2} \frac{2}{s} \frac{d}{ds} \int_{a_1}^{a_2} \frac{yG_2(y)}{\sqrt{y^2 - s^2}} dy
\]

In addition to the set of equations mentioned, the following set of triple equations must be considered, which for the case \( n=1 \) has the form

\[
\int_{0}^{\infty} u^3 \sqrt{k^2 - u^2} Q_1(u) J_1(\rho u) du = \int_{\rho}^{\infty} u^3 \frac{Y'(u)}{\sqrt{k^2 - u^2}} J_1(\rho u) du, \quad \rho \in D_2
\]

\[
\int_{0}^{\infty} u^3 Q_1(u) J_1(\rho u) du = 0, \quad \rho \in D_1 \cup D_3
\]

If these can be solved numerically for \( Q_1(u) \) then equations
(53) and (54) can be solved explicitly for \( P_1(u) \). The task of solving the annular aperture problem thus depends on the ability to evaluate \( Q_1(u) \).

4.4 The Boundary Conditions On The Rim Of The Aperture

The behaviour of the electromagnetic fields in the neighbourhood of a sharp edge has been investigated by Meixner and Andrejewski\(^5\), Bouwkamp\(^9,16,17\), and by Jones\(^13\). It has been shown that the admissible singularities of \( \mathbf{E} \) and \( \mathbf{H} \) at a sharp edge are at most of order \( d^{-1/2} \), since such singularities cannot manifest themselves as real sources of true electromagnetic energy. "\( d \)" measures the distance from the observation point to the edge. Therefore in most cases in addition to writing the correct equations governing the system additional care must be taken to choose those solutions which are physically appropriate. For instance the integro-differential equations described in the book of Jones\(^13\), and in the paper of Bouwkamp\(^17\) and Copson\(^7,8\) admits several solutions.

It is interesting to find that in the present method no such ambiguity arises. The set of multiple integral equations will "automatically" provide the acceptable
conditions on the rim of the aperture. This fact can be seen from the rigorous result for the charge distribution of a disk charged to unit potential as depicted in reference 21. For the case of diffraction by a small hole i.e. \( \kappa a \ll 1 \), the \( E_z \) component and the \( H_z \) component in the plane \( z=0 \) for \( r > 1 \) and for \( r < 1 \) respectively are given by Bouwkamp as follows:

\[
E_z = \frac{4}{3} \int \frac{k}{\pi} \frac{\cos \theta}{r \sqrt{r^2 - 1}} \, , \, z = 0^+, \, r > 1
\]

\[
H_z = -\frac{4}{\pi} \frac{r \sin \theta}{\sqrt{1 - \frac{r^2}{R^2}}} \, , \, z = 0 \, , \, r < 1
\]

From equations (36), (37), (48), (50) and (46) , (47), it can be shown that \( (\kappa a \ll 1) \)

\[
u^3 Q_1(u) = \sqrt{2} u^{1/2} \int_0^1 t^{1/2} J_{3/2}(ut) g_1(t) \, dt
\]

\[
u^3 P_1(u) \approx c_1 \sin u
\]

\[
Q_1(u) = -Q_{-1}(u)
\]

\[
P_1(u) = P_{-1}(u)
\]

where \( c_1 \) is a constant. Hence it is easily verified that \( 30,34 \)

\[
H_z \approx \begin{cases} 0 & r > 1 \\ \frac{r \sin \theta}{\sqrt{1 - \frac{r^2}{R^2}}} \int_0^1 t^{1/2} g_1(t) \, dt & r = 1 \, , \, z = 0 \\ 0 \, & r < 1 
\end{cases}
\]
The above results are in agreement with those given by Meixner and Andrejewski, Bouwkamp, and by Sommerfeld's half-plane solution.
CHAPTER 5

NUMERICAL ANALYSIS

As in all diffraction problems, further substantial difficulties arise when reducing the formal solution to a state which permits numerical computation. By use of the standard method of steepest descent \(^{13,34,35}\) asymptotic expansions for the magnetic and electric field components are obtained but the formulae are then valid only when the point of observation is several wavelengths away from the screen and from the axis.

5.1 Asymptotic Evaluation Of The Field

From Chapter 4 it can be shown that

\[
N_z = 2jsin \theta \int_{0}^{\infty} uP_1(u)J_1(ru)e^{-z\sqrt{u^2-k^2}} du
\]

\[
= jsin \theta \int_{0}^{\infty} uP_1(u)H_1^{(1)}(ru)e^{-z\sqrt{u^2-k^2}} du \quad (1)
\]

\[
M_z = 2cos \theta \int_{0}^{\infty} uQ_1(u)J_1(ru)e^{-z\sqrt{u^2-k^2}} du
\]

\[
= cos \theta \int_{-\infty}^{\infty} uQ_1(u)H_1^{(1)}(ru)e^{-z\sqrt{u^2-k^2}} du \quad (2)
\]
\[ H_x = \frac{\sin 2\theta}{2} \int_{-\infty}^{\infty} u^2 \left[ jw\xi P_1(u) + jk^2 - u^2 \right] Q_1(u) \right] H_2^{(1)}(ru) e^{-z\sqrt{u^2 - k^2}} \, du \]

\[ H_y = \int_{-\infty}^{\infty} u^2 \frac{H_2^{(1)}(ru)}{ru} \left[ jw\xi P_1(u) - jk^2 - u^2 \right] Q_1(u) \right] e^{-z\sqrt{u^2 - k^2}} \, du \]

\[ + \int_{-\infty}^{\infty} \left[ jw\xi P_1(u) \cos^2 \theta + jk^2 - u^2 \right] Q_1(u) \sin^2 \theta \right] u^2 H_2^{(1)}(ru) e^{-z\sqrt{u^2 - k^2}} \, du \]

\[ H_z = \cos \theta \int_{-\infty}^{\infty} u^3 Q_1(u) H_1^{(1)}(ru) e^{-z\sqrt{u^2 - k^2}} \, du \]

and similar expressions are obtained for the electric field components. In fact, once \( N \) and \( M \) are given all the \( E \) and \( H \) field may be derived from equations (2) and (3) in Chapter 1. Whence the main concern here is to evaluate an infinite integral of the type

\[ I = \int_{-\infty}^{\infty} f(u) H_n^{(1)}(ru) e^{-z\sqrt{u^2 - k^2}} \, du \]

Notice that in order to facilitate the calculation in the complex domain, the Bessel function in the integral has been tactically transformed into a Hankel function of the first kind which is the only pertinent form for an outgoing wave in this case. Furthermore, the lower limit is extended to \(-\infty\) so that the contour may be closed at infinity.

For sufficiently large arguments the following approximation can be employed, but it is this approximation
which limits the validity of the region in which the field can be calculated,

\[ H_n^{(1)}(x) \approx \frac{2}{\pi x} e^{i \left[ x - \frac{n+1/2}{2} \right]} \]  \hspace{1cm} (7)

Then (6) becomes

\[ I \approx \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{f(u) e^{ijr \left( u-(n+1/2) \pi \right)}}{\sqrt{ru-(n+1/2) \pi}} - z \sqrt{u^2-k^2} \; du \]

By means of the method of steepest descent 13,34,36, it may be shown that

\[ I \approx 2e^{-\frac{j(n-1)\pi}{2}} \frac{f(k \sin \alpha)}{\sin \alpha} \frac{e^{jkR}}{kR} \]  \hspace{1cm} (8)

where \( \sin \alpha = \frac{r}{R} \),

\[ R = \sqrt{r^2 + z^2} \]

It follows then that

\[ N_z \approx 2jsin\theta \ 1(k \sin \alpha) \frac{e^{jkR}}{R} \]  \hspace{1cm} (9)

\[ M_z \approx 2\cos\theta \ 1(k \sin \alpha) \frac{e^{jkR}}{R} \]  \hspace{1cm} (10)

and that

\[ H_x \approx \sin2\theta \ e^{-\frac{j\pi}{2}} \ k \sin\alpha \left[ jw \ 1(k \sin \alpha) + k^2 - u^2 \ 1(k \sin \alpha) \right] \frac{e^{jkR}}{R} \]  \hspace{1cm} (11)

\[ H_y \approx \frac{2}{k \sin \alpha} \left[ jw \ 1(k \sin \alpha) - \sqrt{k^2 - u^2} \ 1(k \sin \alpha) \right] \frac{e^{jkR}}{R} \]
\[
+2e^{-j\pi/2}k\sin\alpha \left[ j\omega \varepsilon P_1(k\sin\alpha)\cos^2\theta + \sqrt{k^2 - u^2} Q_1(k\sin\alpha)\sin^2\theta \right] e^{jkR} \\
H_z = 2\cos\theta \ k^2\sin^2\alpha \ Q_1(k\sin\alpha) \frac{e^{jkR}}{R}
\]

From the above derivation, it is obvious that expressions (9) to (13) will be valid in the region such that equation (7) is satisfied. Thus, for points about a few wavelengths away from the origin and those slowly approaching to the z-axis as z increases the above formulae are valid\(^\text{37}\). For a moderate size aperture Bekefi's method\(^\text{38}\) or Miyamoto-Wolf\(^\text{39}\) formula could be used to calculate the field distribution in the axial region.

5.2 Numerical Calculation

Throughout this work Gaussian quadrature of order 4 and 20 are used. The Fredholm integral equation of the second kind encountered in Chapter 4 is solved by the method of replacement by a finite system of equations. For convenience twenty points are used, that is, twenty discrete points are calculated (see Table 1). Detailed formulation of the method is given in appendix B. An interesting example can be found in reference 18.
Figure 2 is the photographic reduction (15 times) of the phase map produced on the IBM 1403 line printer by a program written by Cermak, Legendre and Silvester. The map consists of 10 submaps joined together. Each submap (outlined rectangle) is a contour plot of a matrix containing 21x81 equally spaced values, i.e. 2 wavelengths by 8 wavelengths.

Figures 1 to 6 are the calculated field maps of the three magnetic field components from formulas (11) to (13) using an IBM 360/75 computer. The x and z components are purely diffracted waves. Expressions (11) and (12) indicate that these two purely diffracted waves will be zero along the z-axis which is in agreement with the result calculated from the Helmholtz representations as described in reference 41. It is interesting to point out that the phase distribution for the H_y component (Fig. 1) near the axis and approximately a distance equal to the aperture radius from the aperture is essentially plane. A similar phase distribution can be found in Tan's work. Essentially the amplitude patterns are similar to the case of diffraction of a plane wave by a circular mirror or by a focusing lens.

The above comparisons confirm the validity of the
above asymptotic expansions and also suggests that the choice of twenty discrete point values for the Helmholtz integral equation (48) is good enough for this case.
FIGURE 1. THE PHASE STRUCTURE OF THE CALCULATED $H_y$ COMPONENT.

E-PLANE, $a = 6\lambda$, $\lambda = 3.203$ cm.
FIGURE 2. A PHOTOGRAPHIC REDUCTION OF THE PHASE MAP (H COMPONENT, H-PLANE) AS PRODUCED ON THE IBM 1403 LINE PRINTER. THE MAP CONSISTS OF 10 SUB-MAPS JOINED TOGETHER. EACH SUB-MAP (OUTLINED RECTANGLE) IS A CONTOUR PLOT OF A MATRIX CONTAINING 21x81 EQUALLY SPACED VALUES.
FIGURE 3. THE CALCULATED $|H_y|$ COMPONENT.

1 = 0.1433E-01  2 = 0.2341E-01  3 = 0.3248E-01  4 = 0.4156E-01
5 = 0.5063E-01  6 = 0.5971E-01  7 = 0.6878E-01  8 = 0.7786E-01
9 = 0.8693E-01  0 = 0.9601E-01  A = 0.1051E 00  B = 0.1142E 00

E-PLANE, a = 6  , $\lambda$ = 3.203cm.
FIGURE 4. THE PHASE STRUCTURE OF THE CALCULATED $H_z$ COMPONENT.

H-Plane, $a = 6 \lambda$, $\lambda = 3.203\text{cm.}$
FIGURE 5. THE CALCULATED $|H_z|$ COMPONENT.

$1 = 0.4985 \times 10^{-2}$  $2 = 0.2138 \times 10^{-1}$  $3 = 0.3778 \times 10^{-1}$  $4 = 0.5418 \times 10^{-1}$  
$5 = 0.7058 \times 10^{-1}$  $6 = 0.8698 \times 10^{-1}$  $7 = 1.034 \times 10^{0}$

H-PLANE, $a = 6$, $\lambda = 3.203 \text{ cm}$.
The phase structure of the calculated $H_x$ component.

$\theta = 45^\circ$, $a = 6 \lambda$, $\lambda = 3.203 \text{cm}$. 
TABLE 1

The calculated 20 discrete point values of \( h_1(x_m) \) and \( g_1(x_m) \).

\[ x_i = \frac{(1 + t_i)}{2}, \quad x_{i+10} = \frac{(1 - t_i)}{2} \]

<table>
<thead>
<tr>
<th>( t_i )</th>
<th>( m )</th>
<th>( h_1(x_m) )</th>
<th>( g_1(x_m) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>.07652</td>
<td>1</td>
<td>-.209034</td>
<td>-.175019</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>-.181902</td>
<td>-.166529</td>
</tr>
<tr>
<td>.22778</td>
<td>2</td>
<td>-.284151</td>
<td>+.210429</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>-.214110</td>
<td>+.718288</td>
</tr>
<tr>
<td>.37371</td>
<td>3</td>
<td>-.985697</td>
<td>+.398065</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>-.213516</td>
<td>-.973806</td>
</tr>
<tr>
<td>.51087</td>
<td>4</td>
<td>-.251889</td>
<td>-.310033</td>
</tr>
<tr>
<td></td>
<td>14</td>
<td>-.231810</td>
<td>+.819069</td>
</tr>
<tr>
<td>.63605</td>
<td>5</td>
<td>-.196778</td>
<td>-.166027</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>-.248249</td>
<td>-.203050</td>
</tr>
<tr>
<td>.74633</td>
<td>6</td>
<td>+.840582</td>
<td>+.184890</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>-.283317</td>
<td>+.340470</td>
</tr>
<tr>
<td>.83911</td>
<td>7</td>
<td>-.341231</td>
<td>+.610218</td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>-.336396</td>
<td>-.469123</td>
</tr>
<tr>
<td>.91223</td>
<td>8</td>
<td>-.377140</td>
<td>-.473578</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>-.438247</td>
<td>+.864816</td>
</tr>
<tr>
<td>.96397</td>
<td>9</td>
<td>-.901439</td>
<td>+.165690</td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>-.632628</td>
<td>-.646038</td>
</tr>
<tr>
<td>.99312</td>
<td>10</td>
<td>+.879941</td>
<td>+.993700</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>-.154431</td>
<td>-.486233</td>
</tr>
</tbody>
</table>
CHAPTER 6

CONCLUSION

Dual integral equations with Hankel kernels have been successfully applied to mixed boundary value problems in potential theory. In the realm of electromagnetic diffraction problems only the two dimensional single strip (or slit) case has been examined by this technique. The present work shows that the method can be extended to the three dimensional case as well, namely, diffraction by a circular disc (or aperture). As far as is known to the author this is the first instance of applying the method to three dimensional diffraction problems. It is also pointed out in Chapter 4 that by means of this method it is possible to obtain a rigorous solution of the hitherto unsolved problem of diffraction by an annular aperture. The fundamental formulation for the problem is expounded clearly in this work. The problem involves solving two sets of triple integral equations. One set can be solved by the theory given by Cooke while the other set is difficult to solve. The complete theory of multi-integral equations with Hankel kernel is still not
fully developed and is beyond the scope of this thesis.

However, realizing that an arbitrarily shaped non-planar aperture is completely unmanageable by rigorous theory, an adequate general formulation to solve this diffraction problem is required. From past investigations, there is no evidence that any particular formulation exists. In fact the general objective of the future work would be to give a systematic way of calculating the mean square error in any proposed approximation formula. Moreover, if one intends to work with approximate scalar or vector formulas a plan for improved approximations is needed. For instance, the rigorous solution for the case of diffraction by a circular aperture given in Wu's thesis can be used as a criterion for the calculation of the mean square errors. The formulas to be considered would be

(1) the Helmholtz-Kirchhoff formula,
(2) the edge-current method,
(3) Miyamoto-Wolf\textsuperscript{39} formula,
(4) Millar's formula,
(5) Braunbek's method,
(6) Bekefi's method\textsuperscript{38},
(7) the Rayleigh's first and second formulas.
The next step is the study of the behaviour of the mean square errors from various formulas and search for a clue to improve the approximation.

A different approach in pursuing the subject is via a vector integral equation (or integro-differential equation). The main concern in this approach is to seek an efficient numerical technique to solve for the induced current. Two dimensional cases have been investigated by Harrington. The method of moments has been found to give fruitful results. Three dimensional diffraction problems do not seem as yet to have been systematically attempted by this scheme. Following the same line of reasoning as for the case of two dimensional problems, the integral equations obtained from a specific boundary value problem may be treated by the method of moments in exactly the same manner as before, or, alternatively, it may be cast in such a form that the problem can be reduced to minimize a suitable functional. An interesting and important way is to make use of the method of finite triangular elements, since an obstacle of any configuration can be approximated to sufficient accuracy by a number of such elements. The method is especially elegant in the sense that once the convergence
of the various orders of approximation and an economic algorithm have been worked out, all kinds of diffraction problems can be synthesized from a single element, irrespective of the nature of the problems. As yet the only non-planar problem that has been solved has been the slit investigated by Tan^{42}. It is hoped that the above approach will prove useful for this class of problems.
APPENDIX A

1. The Hankel Inversion Theorem

If the function $f(x)$ is integrable over $[0, \infty)$ and its variation is bounded near the point $x$, then, for

$$\nu \geq -\frac{1}{2}$$

$$\int_{0}^{\infty} (xu)^{1/2} J_{\nu}(xu) \, du \int_{0}^{\infty} (ut)^{1/2} J_{\nu}(ut) \, f(t) \, dt = \frac{1}{2} \{f(x+0)+f(x-0)\}$$

(1)

If $f(x)$ is continuous at the point $x$ then (1) may be restated in the following manner. Let

$$g(y; \nu) = \mathcal{H}_\nu[f(x); y] = \int_{0}^{\infty} f(x) \, J_{\nu}(xy) \, (xy)^{1/2} \, dx$$

$$y \geq 0$$

(2)

then

$$f(x) = \mathcal{H}_\nu[g(y; \nu); x]$$

(3)

Where $\mathcal{H}_\nu[f(x); y]$ is the Hankel transform of order $\nu$ of $f(x)$. It is obvious that $\mathcal{H}_\nu^2 = \mathcal{H}_\nu$.

2. The Modified Operator Of Hankel Transforms
In certain occasions it is convenient to use the modified operator of Hankel transforms, \( S_{\eta, \alpha} \), introduced by Erdelyi and Kober (1940) which is defined as follows:

\[
S_{\eta, \alpha} f(x) = 2^{\alpha} x^{-\alpha} \int_0^\infty t^{1-\alpha} f(t) J_{2\eta+\alpha} (xt) dt \tag{4}
\]

so that

\[
S_{\eta, \alpha} f(x) = 2^{\alpha} x^{-\alpha-\frac{\gamma}{2}} \int_{\gamma} \left\{ t^{-\alpha} f(t); x \right\} \tag{5}
\]

From (1) it can be shown that

\[
f(t) = S_{\eta, \alpha, -\alpha} \left\{ S_{\eta, \alpha} f(x); t \right\} \tag{6}
\]

hence

\[
S_{\eta, \alpha}^{-1} = S_{\eta, \alpha+\alpha, -\alpha} \tag{7}
\]

3. The Erdelyi-Kober Operator

The operator \( I_{\eta, \alpha}, K_{\eta, \alpha} \) are defined by the equations:

\[
I_{\eta, \alpha} f(x) = 2^{\frac{x-\eta}{\Gamma(\alpha)}} \int_0^x (x^2 - u^2)^{\frac{2\eta-1}{2}} u^{2\eta+1} f(u) du \tag{8}
\]

\[
K_{\eta, \alpha} f(x) = 2^{\frac{2\eta}{\Gamma(\alpha)}} \int_x^\infty (u^2 - x^2)^{\frac{2\eta-1}{2}} u^{2\eta+1} f(u) du \tag{9}
\]

\[ \text{Re} \ \alpha > 0 \]

\[ \text{Re} \ \eta > -1/2 \]
It can be shown that

\[ I_{\eta,0} = I \]  
(10)

\[ K_{\eta,0} = I \]  
(11)

\[ I_{\eta,\alpha} I_{\eta,\alpha+\beta} = I_{\eta,\alpha+\beta} \]  
(12)

\[ K_{\eta,\alpha} K_{\eta,\alpha+\beta} = K_{\eta,\alpha+\beta} \]  
(13)

\[ I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha} \]  
(14)

\[ K^{-1}_{\eta,\alpha} = K_{\eta+\alpha,-\alpha} \]  
(15)
1. The Approximate Solution Of The Fredholm Integral Equation Of The Second Kind: Replacement By A Finite System

Consider a Fredholm integral equation of the second kind

\[ Y(x) - \lambda \int_a^b K(x,u)Y(u)\,du = f(x) \]  

(1)

Let it be assumed that it is justified to replace the integral involved in (1) by an approximate linear formula of the form

\[ \int_a^b g(x)\,dx = \sum_{i=1}^{n} A_i g(x_i) + E \]  

(2)

where \( A_i \) and \( x_k \) are the weights and abscissas for the particular quadrature, respectively, and \( E \) is the error.

Equation (1), then becomes

\[ Y(x) - \lambda \sum_{i=1}^{n} A_i K(x,x_i)Y(x_i) \simeq f(x) \]  

(3)

It is then possible to solve for the unknown function \( Y(x) \) at discrete points; for convenience, it is required
that
\[ Y(x_j) - \lambda \sum_{i=1}^{n} A_i K(x_j, x_i) Y(x_i) = f(x_j) \] (4)

\[ (j = 1, 2, \ldots n) \]

Once this system of equations has been solved, it is obvious that the unknown function \( Y(x) \) can be approximated by the following formula:

\[ Y(x) = f(x) + \lambda \sum_{i=1}^{n} A_i K(x, x_i) Y(x_i) \] (5)

In order to ensure good accuracy Gauss's quadrature is employed in this work. The error incurred in using Gauss's quadrature is shown in reference [8]. The estimate reads

\[ \text{Error} \leq \frac{(b-a)^{2n+1}}{2n+1} \sum_{1,2,3\ldots n}^{2} \frac{T(2n)}{(n+1)\ldots 2n} \]

where

\[ T(s) = H(0)_{u} + C_{s}^{1} H(1)_{u} + \ldots + H(s)_{u} \]

and

\[ H(s) = \left| Y(s)(x) \right| \]

\[ M_{u}^{(s)} = \left| \frac{d^{s}}{du^{s}} K(x, u) \right| \]
REFERENCES


8. Copson, E. T., "Diffraction By A Plane Screen ",


